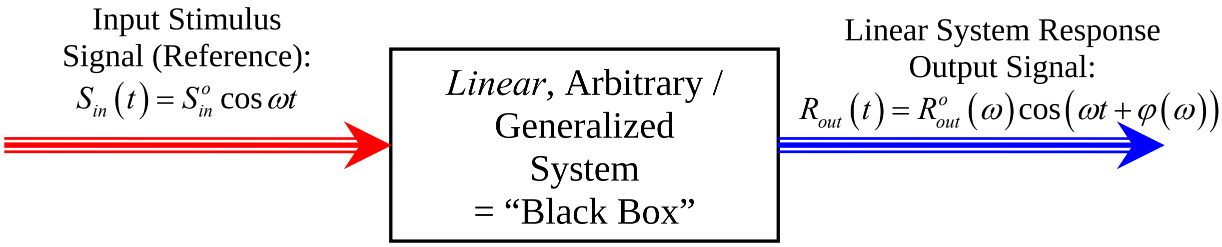
Complex Sound Fields

What is a *Complex* Quantity?

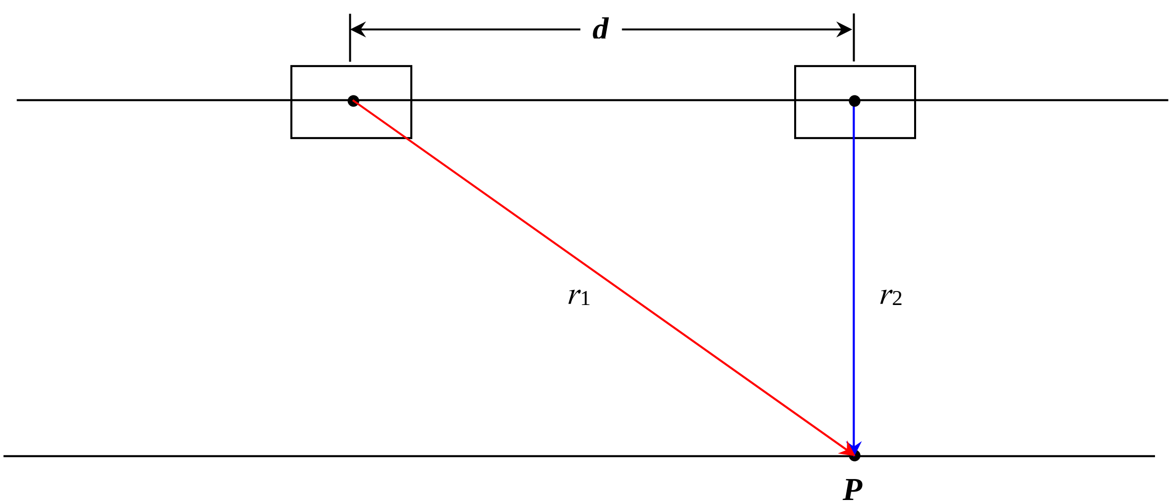
In *any* situation involving wave phenomena, if interference effects are manifest, *e.g*. two (or more) waves {*n.b*. diffraction – a *scattering* process – is also a type of wave interference – wave *self*-interference}, then a well-defined phase relation between waves associated with such phenomena exists, which in general is time-dependent, but could also be stationary in time in certain situations.

There are also {many} situations in which a periodic/harmonic (*i.e*. single frequency) ***input*** stimulus – *i.e*. a harmonic ***reference*** signal $S_{in}(t) = S_{in}^{o}\cos\omega t$ is ***input*** to a ***system*** { = a "black box"} which in turn outputs a {***linear***} ***response*** signal which, in general may have a ***non-trivial*** (*e.g*. frequency-dependent) ***amplitude*** **.and.** ***phase*** relation ***relative*** to the input reference signal $R_{out}(t) = R_{out}^{o}(\omega)\cos(\omega t + \varphi(\omega))$, which we show schematically in the figure below:



Mathematically, we can use ***complex variables*** as a convenient way to describe the underlying physics associated with such phenomena. We don't *have to* use complex variables/complex notation to do this, but it turns out that in many situations it is very convenient/handy to do so!

In acoustics, since we have already talked about/discussed various situations exhibiting wave interference, we're thus already familiar with many examples of complex sounds – we simply haven't discussed them using complex variables/complex notation. One simple acoustics example is the situation where a sine-wave generator at frequency f is used to drive an identical pair of loudspeakers situated a lateral distance d away from each other – destructive/constructive interference effects between the sine-wave sounds coming from the two loudspeakers can clearly be heard *e.g*. walking on a line parallel to the line joining the two loudspeakers, as shown in the figure below:



We don't *need* to use complex variables/complex notation to realize that whenever the path length difference $\Delta r \equiv r_2 - r_1 = n\lambda$, where *n* = 0, ±1, ±2, ±3, ±4, … and $\lambda = v/f$ where *v* = speed of sound in air (~ 343 *m/s* @ NTP), constructive interference will occur – the two individual sound waves are precisely in-phase with each other at the observation point *P*, thus sound intensity maxima will be heard at such locations, whereas whenever the path length difference $\Delta r \equiv r_2 - r_1 = n'\lambda/2$, where *n′* = ±1, ±3, ±5, ±7, … destructive interference will occur – the two individual sound waves are precisely 180^o out-of-phase with each other at the observation point *P* – thus intensity minima will be heard at such locations.

Acoustical Interference Phenomena

Whenever two (or more) periodic sine-wave type signals are linearly superposed (*i.e.* added together), the resultant/overall waveform depends on the ***amplitude***, ***frequency*** **.and.** ***phase*** associated with the ***individual*** signals. Mathematically, this is often most easily and transparently described using complex notation.

Basics of / A Primer on Complex Variables and Complex Notation:

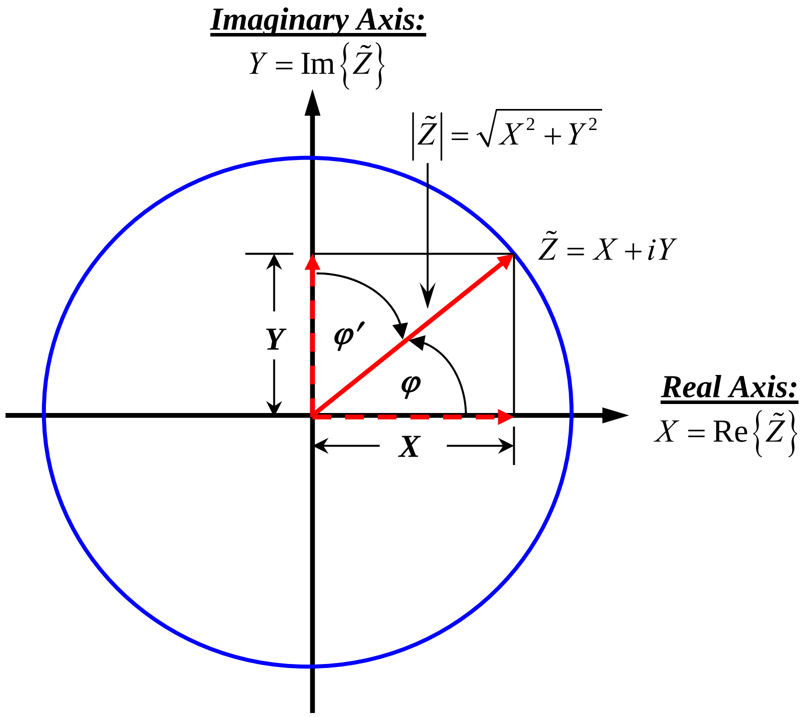
We can use complex variables/complex notation to describe physics situations whenever *relative* phase information is important. A complex quantity, denoted as $\tilde{Z}$ consists of two components: $\tilde{Z} = X + iY$. X is the known as the "*real*" part of $\tilde{Z}$, denoted $X = \text{Re}\{\tilde{Z}\}$ and Y is the known as the "*imaginary*" part of $\tilde{Z}$, denoted $Y = \text{Im}\{\tilde{Z}\}$. If a *reference* signal is present, the real component $X = \text{Re}\{\tilde{Z}\}$ of complex $\tilde{Z}$ is in-phase (180^o out-of phase) with the reference signal if *X* is +*ve* (–*ve*), respectively. The imaginary component $Y = \text{Im}\{\tilde{Z}\}$ of complex $\tilde{Z}$ is +90^o (–90^o) out-of-phase with the reference signal if *Y* is +*ve* (–*ve*), respectively.

The number $i \equiv \sqrt{-1}$. The magnitude of the complex variable $\tilde{Z}$ is denoted as $|\tilde{Z}| \equiv \sqrt{\tilde{Z}\tilde{Z}^*}$ or $|\tilde{Z}|^2 \equiv \tilde{Z}\tilde{Z}^*$ where $\tilde{Z}^*$ is the so-called complex *conjugate* of $\tilde{Z}$, which changes $i \to -i$, such that $i \cdot i^* = \sqrt{-1} \cdot -\sqrt{-1} = +1$ (note that $i^2 = -1 = (i^*)^2$ and $i \cdot i^* = i^* \cdot i = +1$), thus we see that:
$\tilde{Z}^* = (\tilde{Z})^* = (X + iY)^* = X - iY$. Hence we see that:
$|\tilde{Z}| \equiv \sqrt{\tilde{Z}\tilde{Z}^*} = \sqrt{(X+iY)(X-iY)} = \sqrt{X^2 + i\cancel{XY} - i\cancel{XY} + Y^2} = \sqrt{X^2 + Y^2}$. Thus, we realize that the magnitude of $\tilde{Z}$, $|\tilde{Z}|$ is analogous to the hypotenuse, *c* of a right triangle ($c^2 = a^2 + b^2$) and/or *e.g.* the radius of a circle, *r* centered at the origin ($r^2 = x^2 + y^2$).

Because complex variables $\tilde{Z} = X + iY$ consist of two components, $\tilde{Z}$ can be graphically depicted as a 2-component "vector" (*aka* "phasor") $\tilde{Z} = (X, Y)$ lying in the so-called 2-D complex plane, as shown in the figure below.

The ***in-phase***, so-called "***real***" component of $\tilde{Z}$, $X = \text{Re}\{\tilde{Z}\}$ by convention is drawn along the *x*, or horizontal axis (*i.e.* the ***abscissa***), as shown in the figure below.

The 90° ***out-of-phase/quadrature***, so-called "***imaginary***" component of $\tilde{Z}$, $Y = \text{Im}\{\tilde{Z}\}$ by convention is drawn along the *y*, or vertical axis (*i.e.* the ***ordinate***), as shown in the figure below.



It can be readily seen from the above diagram that the endpoint of the complex "vector" (*aka* "phasor"), $\tilde{Z} = X + iY$ lies at a point on the circumference of a circle, centered at (*X*, *Y*) = (0, 0), with "radius" (*i.e.* magnitude) $|\tilde{Z}| = \sqrt{X^2 + Y^2}$ and phase angle, $\varphi = \tan^{-1}(Y/X)$ (*n.b.* defined relative to the *X*-axis), (or equivalently: $\varphi' = \tan^{-1}(X/Y)$, *n.b.* defined relative to the *Y*-axis).

Instead of using Cartesian coordinates, we can alternatively/equivalently express the complex variable, $\tilde{Z}$ in polar coordinate form: $\tilde{Z} = |\tilde{Z}|(\cos\varphi + i\sin\varphi)$, since from the above diagram, we see that $X = |\tilde{Z}|\cos\varphi$ and $Y = |\tilde{Z}|\sin\varphi$. Recall the trigonometric identity: $\cos^2\varphi + \sin^2\varphi = 1$ which is used in obtaining the magnitude of $\tilde{Z}$, $|\tilde{Z}|$ from $\tilde{Z}$ itself:

$$
\begin{aligned}
|\tilde{Z}| &= \sqrt{\tilde{Z}\tilde{Z}^*} = |\tilde{Z}|\sqrt{(\cos\varphi + i\sin\varphi)(\cos\varphi - i\sin\varphi)} \\
&= |\tilde{Z}|\sqrt{\cos^2\varphi + \cancel{i\sin\varphi\cos\varphi} - \cancel{i\sin\varphi\cos\varphi} + \sin^2\varphi} \\
&= |\tilde{Z}|\sqrt{\cos^2\varphi + \sin^2\varphi} = |\tilde{Z}|
\end{aligned}
$$

We can (always) redefine the phase variable φ such that *e.g.* $\varphi \Rightarrow (\omega t + \varphi)$, it can then be seen that:

$$\tilde{Z}(t) = \left|\tilde{Z}(t)\right|\left(\cos(\omega t + \varphi) + i\sin(\omega t + \varphi)\right)$$

with real component: $X(t) = \mathrm{Re}\left\{\tilde{Z}(t)\right\} = \left|\tilde{Z}(t)\right|\cos(\omega t + \varphi)$

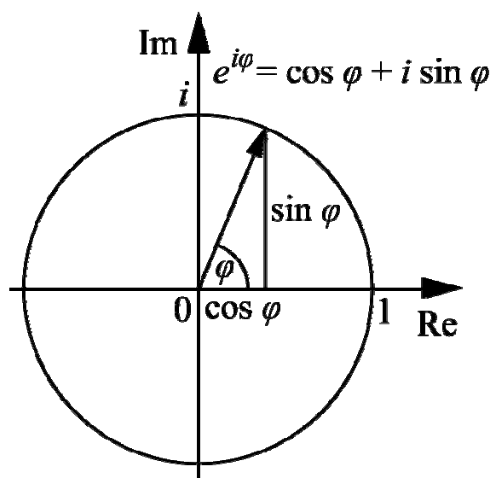
and imaginary component: $Y(t) = \mathrm{Im}\left\{\tilde{Z}(t)\right\} = \left|\tilde{Z}(t)\right|\sin(\omega t + \varphi)$.

Note that at the zero of time $t = 0$, these relations are identical to the above.

If (for simplicity's sake) we take the phase angle $\varphi = 0$, then: $\tilde{Z}(t) = \left|\tilde{Z}(t)\right|\left(\cos(\omega t) + i\sin(\omega t)\right)$.
At time $t = 0$, it can be seen that the complex variable $\tilde{Z}(t=0) = X(t=0) = \left|\tilde{Z}(t=0)\right|$ is a purely real quantity, lying entirely along the x-axis, since $\tilde{Z}(t=0) = \left|\tilde{Z}\right|\cos 0 = \left|\tilde{Z}(t=0)\right|$.
As time t progresses, it can be seen that the complex variable $\tilde{Z}(t) = \left|\tilde{Z}(t)\right|\left(\cos(\omega t) + i\sin(\omega t)\right)$ rotates in a ***counter*-*clockwise*** direction in the complex plane with constant angular frequency $\omega = 2\pi f$ radians/second, where f is the frequency (in cycles/second {cps}, or Hertz {= *Hz*}) completing one revolution in the complex plane every $\tau = 1/f = 2\pi/\omega$ seconds {the variable τ is known as the period of oscillation, or period of vibration}. This rotation of $\tilde{Z}(t)$ in the complex plane can also be seen from the time evolution of the phase:
$\varphi(t) = \tan^{-1}\left(Y(t)/X(t)\right) = \tan^{-1}\left(\cancel{\left|\tilde{Z}(t)\right|}\sin\omega t \big/ \cancel{\left|\tilde{Z}(t)\right|}\cos\omega t\right) = \tan^{-1}(\tan\omega t) = \omega t$.

Complex Exponential Notation:

The famous mathematician-physicist Leonhard Euler showed that for any real number φ, that $e^{i\varphi} = \cos\varphi + i\sin\varphi$. This is known as Euler's formula. Geometrically, the locus of points described by $e^{i\varphi}$ for $0 \le \varphi \le 2\pi$ lie on the unit circle (i.e. radius $\left|e^{i\varphi}\right| = 1$) in the complex plane, centered at (0,0) as shown in the figure on the right. Note that if $e^{i\varphi} = \cos\varphi + i\sin\varphi$, then $\left(e^{i\varphi}\right)^* = e^{-i\varphi} = \cos\varphi - i\sin\varphi$.
We can thus write any "generic" complex quantity $\tilde{Z} = \left|\tilde{Z}\right|(\cos\varphi + i\sin\varphi)$ as $\tilde{Z} = \left|\tilde{Z}\right|e^{i\varphi}$ and write its complex conjugate $\tilde{Z}^* = \left|\tilde{Z}\right|(\cos\varphi - i\sin\varphi)$ as $\tilde{Z}^* = \left|\tilde{Z}\right|e^{-i\varphi}$. Note that:



$\tilde{Z}\tilde{Z}^* = \left(\left|\tilde{Z}\right|e^{i\varphi}\right)\cdot\left(\left|\tilde{Z}\right|e^{-i\varphi}\right) = \left|\tilde{Z}\right|^2 e^{i\varphi}\cdot e^{-i\varphi} = \left|\tilde{Z}\right|^2 e^{i\varphi - i\varphi} = \left|\tilde{Z}\right|^2 e^0 = \left|\tilde{Z}\right|^2\cdot 1 = \left|\tilde{Z}\right|^2$ Note further that since $e^{i\varphi} = \cos\varphi + i\sin\varphi$ and $e^{-i\varphi} = \cos\varphi - i\sin\varphi$, adding and subtracting these two equations from each other, it is easy to show that $\cos\varphi = \frac{1}{2}\left(e^{i\varphi} + e^{-i\varphi}\right)$ and $\sin\varphi = \frac{1}{2i}\left(e^{i\varphi} - e^{-i\varphi}\right)$.

Linear Superposition (Addition) of Two Periodic Signals

It is illustrative to consider the situation associated with the linear superposition of two complex periodic, equal-amplitude, identical-frequency amplitudes at a given observation point $\vec{r}$ in 3-D space {defined from a local origin $O(0,0,0)$}, where one signal differs in relative phase from the other by $\varphi = -90^{\circ} = -\pi/2$. Since the zero of time is arbitrary, we have the freedom to chose one signal to be purely real at time $t = 0$, *e.g.* such that:

$$\tilde{Z}_1(\vec{r},t) = A(\vec{r},t)e^{i\omega t} = A(\vec{r},t)\big(\cos(\omega t) + i\sin(\omega t)\big) \text{ with } \left|\tilde{Z}_1(\vec{r},t)\right| = A(\vec{r},t)$$

and the other signal,

$$\tilde{Z}_2(\vec{r},t) = A(\vec{r},t)e^{i(\omega t-\pi/2)} = A(\vec{r},t)\big(\cos(\omega t-\pi/2) + i\sin(\omega t-\pi/2)\big) \text{ with } \left|\tilde{Z}_2(\vec{r},t)\right| = A(\vec{r},t),$$

i.e. both signals have purely real amplitude, $A(\vec{r},t)$ and angular frequency, ω.

Note also, that at this point in the discussion, the two complex amplitudes $\tilde{Z}_1(\vec{r},t)$ and $\tilde{Z}_2(\vec{r},t)$ are (for the moment) taken to be "generic" acoustic quantities – *i.e.* both could represent *e.g.* complex pressure $\tilde{p}(\vec{r},t)$, complex particle velocity $\tilde{\vec{u}}(\vec{r},t)$, complex displacement $\tilde{\vec{\xi}}(\vec{r},t)$ and/or complex acceleration $\tilde{\vec{a}}(\vec{r},t)$.

At time $t = 0$:

$$\tilde{Z}_1(\vec{r},t=0) = A(\vec{r},t=0)e^{i0} = A(\vec{r},t=0)e^{0} = A(\vec{r},t=0)\big(\cos(0) + \cancel{i\sin(0)}\big) = A(\vec{r},t=0)$$

and:

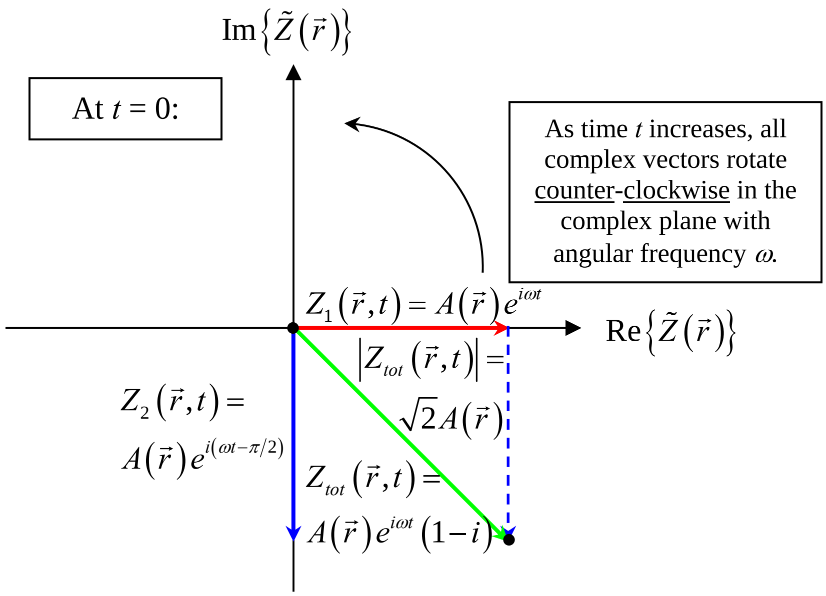
$$\tilde{Z}_2(\vec{r},t=0) = A(\vec{r},t=0)e^{i(0+\pi/2)} = A(\vec{r},t=0)\big(\cancel{\cos(-\pi/2)} + i\sin(-\pi/2)\big) = -iA(\vec{r},t=0)$$

Thus, for this specific example, we see that the 2nd signal $\tilde{Z}_2(\vec{r},t) = A(\vec{r},t)e^{i(\omega t-\pi/2)}$ *lags* (*i.e.* is behind) the 1st signal $\tilde{Z}_1(\vec{r},t) = A(\vec{r},t)e^{i\omega t}$ by 90° in phase, as shown in the figure below, for $t = 0$. The resultant/total complex amplitude $\tilde{Z}_{tot}(\vec{r},t)$ is the {instantaneous} *phasor* sum of the two individual complex amplitudes:

$$\begin{aligned}\tilde{Z}_{tot}(\vec{r},t) &= \tilde{Z}_1(\vec{r},t) + \tilde{Z}_2(\vec{r},t) = A(\vec{r},t)e^{i\omega t} + A(\vec{r},t)e^{i(\omega t-\pi/2)} = A(\vec{r},t)e^{i\omega t}\left(1 + e^{-i\pi/2}\right) \\ &= A(\vec{r},t)e^{i\omega t}\left[1 + \big(\cancel{\cos(-\pi/2)} + i\sin(-\pi/2)\big)\right] = A(\vec{r},t)e^{i\omega t}(1-i)\end{aligned}$$

with magnitude:

$$\left|\tilde{Z}_{tot}(\vec{r},t)\right| = \sqrt{\tilde{Z}_{tot}(\vec{r},t)\tilde{Z}^{*}_{tot}(\vec{r},t)} = A(\vec{r},t)\sqrt{(1-i)(1+i)} = A(\vec{r},t)\sqrt{1-\cancel{i}+\cancel{i}+1} = \sqrt{2}A(\vec{r},t)$$



Note that the reason all complex vectors rotate in a counter-clockwise direction in the complex plane is due to the sign-choice of the $e^{+i\omega t} = \cos(\omega t) + i\sin(\omega t)$ time dependence – it determines the direction complex vectors rotate in the complex plane. Had we instead chosen the $e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t)$ time dependence, then all complex vectors would have instead rotated in a clockwise direction in the complex plane.

Throughout this course, note that we will ***always*** assume/adopt the convention of ***positive*** $e^{+i\omega t} = \cos(\omega t) + i\sin(\omega t)$ time dependence – because it turns out that the {default} way we use the lock-in amplifiers in the various phase-sensitive experiments that we have in the P406POM lab implicitly corresponds mathematically to the $e^{+i\omega t}$ convention – hence it is extremely important to use the correct mathematical descriptions in order to match experimental realities!

Note also that if we had instead chosen the second amplitude to be $\tilde{Z}_2(\vec{r},t) = A(\vec{r},t)e^{i(\omega t+\pi/2)}$, then the 2nd signal $\tilde{Z}_2(\vec{r},t)$ would *lead* (*i.e.* be ahead of) the 1st signal $\tilde{Z}_1(\vec{r},t) = Ae^{i\omega t}$ by 90° degrees in phase. For this situation, the total complex amplitude is:

$$\tilde{Z}_{tot}(\vec{r},t) = \tilde{Z}_1(\vec{r},t) + \tilde{Z}_2(\vec{r},t) = A(\vec{r},t)e^{i\omega t} + A(\vec{r},t)e^{i(\omega t+\pi/2)} = A(\vec{r},t)e^{i\omega t}\left(1+e^{+i\pi/2}\right)$$

$$= A(\vec{r},t)e^{i\omega t}\left[1+\left(\cancel{\cos(\pi/2)} + i\sin(\pi/2)\right)\right] = A(\vec{r},t)e^{i\omega t}(1+i)$$

with the same magnitude as before:

$$\left|\tilde{Z}_{tot}(\vec{r},t)\right| = \sqrt{\tilde{Z}_{tot}(\vec{r},t)\tilde{Z}^*_{tot}(\vec{r},t)} = A(\vec{r},t)\sqrt{(1+i)(1-i)} = A(\vec{r},t)\sqrt{1+\cancel{i}-\cancel{i}+1} = \sqrt{2}A(\vec{r},t)$$

We can now also see that a change in the sign of a complex quantity: $\tilde{Z}(t) \Rightarrow -\tilde{Z}(t)$ physically corresponds to a phase change/shift in phase/phase *retardation* of –180° (*n.b.* which is also mathematically equivalent to a phase *advance* of +180°).
In other words: $\tilde{Z}'(\vec{r},t) = -\tilde{Z}(\vec{r},t) = \tilde{Z}(\vec{r},t)e^{\pm i\pi}$ because $e^{\pm i\pi} = \cos(\pi) \pm \cancel{i\sin(\pi)} = \cos(\pi) = -1$.

The same mathematical formalism can be used for adding together two *arbitrary* complex periodic time-dependent signals $\tilde{Z}_1(\vec{r},t) = A_1(\vec{r},t)e^{i(\omega_1(t)t+\varphi_1(\vec{r},t))}$ and $\tilde{Z}_2(\vec{r},t) = A_2(\vec{r},t)e^{i(\omega_2(t)t+\varphi_2(\vec{r},t))}$. Note that *here*, the individual amplitudes, frequencies and phases may all be time-dependent. The resultant overall complex amplitude in this case is:

$$\tilde{Z}_{tot}(\vec{r},t) = \tilde{Z}_1(\vec{r},t) + \tilde{Z}_2(\vec{r},t) = A_1(\vec{r},t)e^{i(\omega_1(t)t+\varphi_1(\vec{r},t))} + A_2(\vec{r},t)e^{i(\omega_2(t)t+\varphi_2(\vec{r},t))}$$

Because the zero of time is (always) arbitrary, we are again free to choose/redefine $t = 0$ in such a way as to rotate away one of the two phases – absorbing it as an overall/absolute phase (which is physically unobservable). Since $e^{(x+y)} = e^x \cdot e^y$, the above formula can be rewritten as:

$$\tilde{Z}_{tot}(\vec{r},t) = \tilde{Z}_1(\vec{r},t) + \tilde{Z}_2(\vec{r},t) = A_1(\vec{r},t)e^{i\omega_1(t)t}e^{i\varphi_1(\vec{r},t)} + A_2(\vec{r},t)e^{i\omega_2(t)t}e^{i\varphi_2(\vec{r},t)}$$

Multiplying both sides of this equation by $e^{-i\varphi_1(t)}$:

$$\begin{aligned}\tilde{Z}_{tot}(\vec{r},t)e^{-i\varphi_1(\vec{r},t)} &= A_1(\vec{r},t)e^{i\omega_1(t)t}\cancel{e^{i\varphi_1(\vec{r},t)}e^{-i\varphi_1(\vec{r},t)}} + A_2(\vec{r},t)e^{i\omega_2(t)t}e^{i\varphi_2(\vec{r},t)}e^{-i\varphi_1(\vec{r},t)} \\ &= A_1(\vec{r},t)e^{i\omega_1(t)t} + A_2(\vec{r},t)e^{i\omega_2(t)t}e^{i(\varphi_2(\vec{r},t)-\varphi_1(\vec{r},t))}\end{aligned}$$

This shift in overall phase, by an amount $e^{-i\varphi_1(\vec{r},t)}$ is formally equivalent to a redefinition to the zero of time, and also physically corresponds to a (simultaneous) rotation of both of the (mutually-perpendicular) real and imaginary axes in the complex plane by an angle, $\varphi_1(\vec{r},t)$.

The physical meaning of the remaining phase after this redefinition of time/shift in overall phase is a phase *difference* between the second complex amplitude, $\tilde{Z}_2(\vec{r},t)$ relative to the first, $\tilde{Z}_1(\vec{r},t)$. The relative phase difference is $\Delta\varphi_{12}(\vec{r},t) \equiv (\varphi_1(\vec{r},t) - \varphi_2(\vec{r},t))$. Thus, at the (newly) redefined time $t^* = t - \varphi_1(r,t)/\omega_1(t) = 0$ (and then substituting $t^* \Rightarrow t$) the resulting overall, time-redefined amplitude is:

$$\tilde{Z}_{tot}(\vec{r},t) = A_1(\vec{r},t)e^{i\omega_1(t)t} + A_2(\vec{r},t)e^{i\omega_2(t)t}e^{i(\varphi_2(\vec{r},t)-\varphi_1(\vec{r},t))}$$

or:

$$\tilde{Z}_{tot}(\vec{r},t) = A_1(\vec{r},t)e^{i\omega_1(t)t} + A_2(\vec{r},t)e^{i\omega_2(t)t}e^{i\Delta\varphi_{21}(\vec{r},t)}$$

The magnitude of the resulting overall amplitude, $|\tilde{Z}_{tot}(\vec{r},t)|$ can be obtained from (temporarily suppressing the $(\vec{r},t)$-dependence, for clarity's sake):

$$\begin{aligned}|\tilde{Z}_{tot}|^2 &= \tilde{Z}_{tot} \cdot \tilde{Z}_{tot}^* = (\tilde{Z}_1 + \tilde{Z}_2)\cdot(\tilde{Z}_1 + \tilde{Z}_2)^* = (\tilde{Z}_1 + \tilde{Z}_2)\cdot(\tilde{Z}_1^* + \tilde{Z}_2^*) \\ &= \tilde{Z}_1 \cdot \tilde{Z}_1^* + \tilde{Z}_1 \cdot \tilde{Z}_2^* + \tilde{Z}_2 \cdot \tilde{Z}_1^* + \tilde{Z}_2 \cdot \tilde{Z}_2^* \\ &= |\tilde{Z}_1|^2 + \tilde{Z}_1 \cdot \tilde{Z}_2^* + \tilde{Z}_2 \cdot \tilde{Z}_1^* + |\tilde{Z}_2|^2\end{aligned}$$

Let us now work on simplifying the sum of the two cross terms in the above expression. Since $\tilde{Z}_1(\vec{r},t)$ and $\tilde{Z}_2(\vec{r},t)$ are complex quantities, they can always be written as:

$$\tilde{Z}_1(\vec{r},t) = X_1(\vec{r},t) + iY_1(\vec{r},t) \text{ and } \tilde{Z}_2(\vec{r},t) = X_2(\vec{r},t) + iY_2(\vec{r},t).$$

Then (again, for clarity's sake, temporarily suppressing the $(\vec{r},t)$-dependence of these quantities):

$$
\begin{aligned}
\tilde{Z}_1\cdot\tilde{Z}_2^* + \tilde{Z}_2\cdot\tilde{Z}_1^* &= (X_1+iY_1)\cdot(X_2+iY_2)^* + (X_2+iY_2)\cdot(X_1+iY_1)^* \\
&= (X_1+iY_1)\cdot(X_2-iY_2) \; + (X_2+iY_2)\cdot(X_1-iY_1) \\
&= (X_1\cdot X_2 + iY_1\cdot X_2 - iY_2\cdot X_1 + Y_1\cdot Y_2) \; + \; (X_2\cdot X_1 + iY_2\cdot X_1 - iY_1\cdot X_2 + Y_2\cdot Y_1) \\
&= \left(X_1\cdot X_2 + i\,\cancel{X_2\cdot Y_1} - i\,\cancel{X_1\cdot Y_2} + Y\cdot_1 Y_2\right) + \left(X_1\cdot X_2 + i\,\cancel{X_1\cdot Y_2} - i\,\cancel{X_2\cdot Y_1} + Y_1\cdot Y_2\right) \\
&= 2\left(X_1\cdot X_2 + Y_1\cdot Y_2\right) = 2\,\mathrm{Re}\left\{Z_1\cdot Z_2^*\right\}
\end{aligned}
$$

i.e. $\mathrm{Re}\left\{\tilde{Z}_1\cdot\tilde{Z}_2^*\right\} = \mathrm{Re}\left\{\tilde{Z}_2\cdot\tilde{Z}_1^*\right\}$. We will explicitly prove this statement – the 1st term is:

$$
\tilde{Z}_1\cdot\tilde{Z}_2^* = X_1\cdot X_2 + iX_2\cdot Y_1 - iX_1\cdot Y_2 + Y\cdot_1 Y_2 = \underbrace{\left(X_1\cdot X_2 + Y_1\cdot Y_2\right)}_{=\mathrm{Re}\left\{\tilde{Z}_1\cdot\tilde{Z}_2^*\right\}} + i\underbrace{\left(X_2\cdot Y_1 - X_1\cdot Y_2\right)}_{=\mathrm{Im}\left\{\tilde{Z}_1\cdot\tilde{Z}_2^*\right\}}
$$

whereas the 2nd term (= changing in indices $1 \rightleftarrows 2$ in the above expression) is:

$$
\begin{aligned}
\tilde{Z}_2\cdot\tilde{Z}_1^* &= X_2\cdot X_1 + iX_1\cdot Y_2 - iX_2\cdot Y_1 + Y_2\cdot Y_1 = X_1\cdot X_2 + iX_1\cdot Y_2 - iX_2\cdot Y_1 + Y_1\cdot Y_2 \\
&= \underbrace{\left(X_1\cdot X_2 + Y_1\cdot Y_2\right)}_{=\mathrm{Re}\left\{\tilde{Z}_2\cdot\tilde{Z}_1^*\right\}} + i\underbrace{\left(X_1\cdot Y_2 - X_2\cdot Y_1\right)}_{=\mathrm{Im}\left\{\tilde{Z}_2\cdot\tilde{Z}_1^*\right\}}
\end{aligned}
$$

Separately comparing the real and imaginary parts of each of these two terms, we see that indeed

$$
\mathrm{Re}\left\{\tilde{Z}_1\cdot\tilde{Z}_2^*\right\} = \mathrm{Re}\left\{\tilde{Z}_2\cdot\tilde{Z}_1^*\right\} = \left(X_1\cdot X_2 + Y_1\cdot Y_2\right)
$$

Whereas: $\quad \mathrm{Im}\left\{\tilde{Z}_1\cdot\tilde{Z}_2^*\right\} = -\mathrm{Im}\left\{\tilde{Z}_2\cdot\tilde{Z}_1^*\right\} = \left(X_2\cdot Y_1 - X_1\cdot Y_2\right).$

Alternatively, we can equivalently see this another way, simply by working with the explicit expressions for complex $\tilde{Z}_1(\vec{r},t) = A_1(\vec{r},t)e^{i(\omega_1(t)t+\varphi_1(\vec{r},t))}$ and $\tilde{Z}_2(\vec{r},t) = A_2(\vec{r},t)e^{i(\omega_2(t)t+\varphi_2(\vec{r},t))}$:

$$
\begin{aligned}
\tilde{Z}_1\cdot\tilde{Z}_2^* + \tilde{Z}_2\cdot\tilde{Z}_1^* &= A_1e^{i(\omega_1 t+\varphi_1)}\cdot A_2e^{-i(\omega_2 t+\varphi_2)} + A_2e^{i(\omega_2 t+\varphi_2)}\cdot A_1e^{-i(\omega_1 t+\varphi_1)} \\
&= A_1\cdot A_2e^{i(\omega_1 t+\varphi_1)}e^{-i(\omega_2 t+\varphi_2)} + A_1\cdot A_2e^{i(\omega_2 t+\varphi_2)}e^{-i(\omega_1 t+\varphi_1)}
\end{aligned}
$$

Let us define $x \equiv \left(\omega_1(t)t + \varphi_1(t)\right)$ and $y \equiv \left(\omega_2(t)t + \varphi_2(t)\right)$. Rewriting the above expression:

$$
\begin{aligned}
\tilde{Z}_1\cdot\tilde{Z}_2^* + \tilde{Z}_2\cdot\tilde{Z}_1^* &= A_1\cdot A_2e^{ix}\cdot e^{-iy} + A_1\cdot A_2e^{iy}\cdot e^{-ix} \\
&= A_1\cdot A_2e^{i(x-y)} \;\; + A_1\cdot A_2e^{i(y-x)} = A_1\cdot A_2\left(e^{i(x-y)} + e^{i(y-x)}\right) \\
&= A_1\cdot A_2\underbrace{\left(e^{i(x-y)} + e^{-i(x-y)}\right)}_{=2\cos(x-y)} = 2\underbrace{A_1\cdot A_2\cos(x-y)}_{=\mathrm{Re}\left\{\tilde{Z}_1(t)\cdot\tilde{Z}_2^*(t)\right\}\,!!!} \\
&= 2\,\mathrm{Re}\left\{\tilde{Z}_1\cdot\tilde{Z}_2^*\right\}
\end{aligned}
$$

Thus, finally we see that:

$$\begin{aligned}|\tilde{Z}_{tot}|^2 &= |\tilde{Z}_1|^2 + \tilde{Z}_1 \cdot \tilde{Z}_2^* + \tilde{Z}_2 \cdot \tilde{Z}_1^* + |\tilde{Z}_2|^2 \\ &= |\tilde{Z}_1|^2 + |\tilde{Z}_2|^2 + (\tilde{Z}_1 \cdot \tilde{Z}_2^* + \tilde{Z}_2 \cdot \tilde{Z}_1^*) \\ &= |\tilde{Z}_1|^2 + |\tilde{Z}_2|^2 + 2\,\mathrm{Re}\{\tilde{Z}_1 \cdot \tilde{Z}_2^*\}\end{aligned}$$

If we now insert the explicit expressions for complex $\tilde{Z}_1(\vec{r},t) = A_1(\vec{r},t)e^{i(\omega_1(t)t+\varphi_1(t))}$ and $\tilde{Z}_2(\vec{r},t) = A_2(\vec{r},t)e^{i(\omega_2(t)t+\varphi_2(t))}$ in the above formula:

$$\begin{aligned}|\tilde{Z}_{tot}|^2 &= A_1^2 + A_2^2 + 2A_1 \cdot A_2 \cos[(\omega_1 t + \varphi_1) - (\omega_2 t + \varphi_2)] \\ &= A_1^2 + A_2^2 + 2A_1 \cdot A_2 \cos[(\omega_1 t - \omega_2 t) + (\varphi_1 - \varphi_2)] \\ &= A_1^2 + A_2^2 + 2A_1 \cdot A_2 \cos[(\omega_1 - \omega_2)t + (\varphi_1 - \varphi_2)]\end{aligned}$$

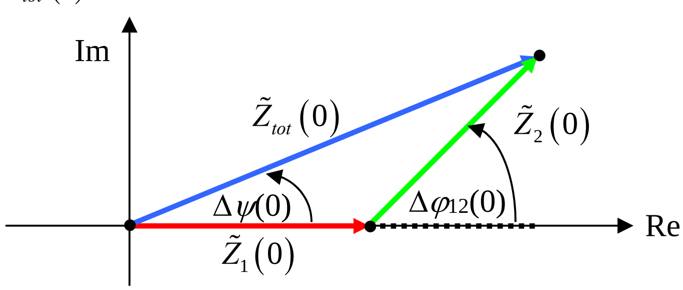
Let us now define $\Delta\omega_{12}(t) \equiv (\omega_1(t) - \omega_2(t))$ and $\Delta\varphi_{12}(\vec{r},t) \equiv (\varphi_1(\vec{r},t) - \varphi_2(\vec{r},t))$.
Then we see that:

$$|\tilde{Z}_{tot}|^2 = A_1^2 + A_2^2 + 2A_1 \cdot A_2 \cos[\Delta\omega_{12} t + \Delta\varphi_{12}]$$

If the frequencies of the two complex amplitudes are equal, then $\Delta\omega_{12}(t) \equiv (\omega_1(t) - \omega_2(t)) = 0$ and thus:

$$|\tilde{Z}_{tot}|^2 = A_1^2 + A_2^2 + 2A_1 \cdot A_2 \cos\Delta\varphi_{12}$$

Note that this expression is simply the formula for the **law of cosines** associated with a triangle lying in the complex plane! The *phasor diagram* associated with the two complex amplitudes $\tilde{Z}_1(\vec{r},t) = A_1(\vec{r},t)e^{i(\omega_1(t)t+\varphi_1(\vec{r},t))}$ and $\tilde{Z}_2(t) = A_2(t)e^{i(\omega_2(t)t+\varphi_2(\vec{r},t))}$ and their resulting overall amplitude $\tilde{Z}_{tot}(t)$ in the complex plane is shown in the figure below, for $t = 0$.



Note that as time t increases, the phasor triangle diagram rotates counter-clockwise in the complex plane – and also potentially in a quite complicated manner, *e.g.* if $\omega_1(t) \neq \omega_2(t)$.

For any complex quantity $\tilde{Z}(\vec{r},t) = X(\vec{r},t) + iY(\vec{r},t)$, the phase angle $\varphi(\vec{r},t)$ relative to the real axis (*i.e.* the x-axis) in the complex plane is given by the simple trigonometric formula: $\tan\varphi = Y/X$ or: $\varphi(t) = \tan^{-1}(Y/X)$.

Thus, in the above figure, the phase angle $\Delta\psi(\vec{r},t)$ associated with the overall/resultant amplitude $\tilde{Z}_{tot}(\vec{r},t)$ is $\tan\Delta\psi = \mathrm{Im}\{\tilde{Z}_{tot}\}/\mathrm{Re}\{\tilde{Z}_{tot}\}$ or: $\Delta\psi = \tan^{-1}\left(\mathrm{Im}\{\tilde{Z}_{tot}\}/\mathrm{Re}\{\tilde{Z}_{tot}\}\right)$.

Writing the "zero-of-time redefined" complex amplitude "vectors" $\tilde{Z}_1(\vec{r},t) = A_1(\vec{r},t)e^{i(\omega_1(t)t)}$, $\tilde{Z}_2(\vec{r},t) = A_2(\vec{r},t)e^{i(\omega_2(t)t+\Delta\varphi_{21}(\vec{r},t))}$ and $\tilde{Z}_{tot}(\vec{r},t) = A_1(\vec{r},t)e^{i\omega_1(t)t} + A_2(\vec{r},t)e^{i\omega_2(t)t}e^{i\Delta\varphi_{21}(\vec{r},t)}$ in terms of their respective real (*x*-) and imaginary (*y*-) components, it is straightforward to show that:

$$\Delta\psi(\vec{r},t) = \tan^{-1}\left(\frac{\mathrm{Im}\{\tilde{Z}_{tot}(\vec{r},t)\}}{\mathrm{Re}\{\tilde{Z}_{tot}(\vec{r},t)\}}\right) = \tan^{-1}\left(\frac{A_1(\vec{r},t)\sin(\omega_1 t) + A_2(\vec{r},t)\sin(\omega_2 t + \Delta\varphi_{12})}{A_1(\vec{r},t)\cos(\omega_1 t) + A_2(\vec{r},t)\cos(\omega_2 t + \Delta\varphi_{12})}\right)$$

Note that at $t = 0$:

$$\Delta\psi(\vec{r},t=0) = \tan^{-1}\left(\frac{\mathrm{Im}\{\tilde{Z}_{tot}(\vec{r},t=0)\}}{\mathrm{Re}\{\tilde{Z}_{tot}(\vec{r},t=0)\}}\right) = \tan^{-1}\left(\frac{A_2(\vec{r},t=0)\sin\Delta\varphi_{12}}{A_1(\vec{r},t=0) + A_2(\vec{r},t=0)\cos\Delta\varphi_{12}}\right)$$

If the two frequencies are equal to each other, *i.e.* $\omega_1(t) = \omega_2(t) = \omega$, then $\Delta\omega_{12}(t) \equiv (\omega_1(t) - \omega_2(t)) = 0$ and this expression simplifies to:

$$\Delta\psi(\vec{r},t) = \tan^{-1}\left(\frac{\mathrm{Im}\{\tilde{Z}_{tot}(\vec{r},t)\}}{\mathrm{Re}\{\tilde{Z}_{tot}(\vec{r},t)\}}\right) = \tan^{-1}\left(\frac{A_1(\vec{r},t)\sin\omega t + A_2(\vec{r},t)\sin(\omega t + \Delta\varphi_{12})}{A_1(\vec{r},t)\cos\omega t + A_2(\vec{r},t)\cos(\omega t + \Delta\varphi_{12})}\right)$$

At $t = 0$: $\Delta\psi(\vec{r},t=0) = \tan^{-1}\left(\frac{\mathrm{Im}\{\tilde{Z}_{tot}(\vec{r},t=0)\}}{\mathrm{Re}\{\tilde{Z}_{tot}(\vec{r},t=0)\}}\right) = \tan^{-1}\left(\frac{A_2(\vec{r},t=0)\sin\Delta\varphi_{12}}{A_1(\vec{r},t=0) + A_2(\vec{r},t=0)\cos\Delta\varphi_{12}}\right)$

Finally, if additionally the two individual amplitudes are also equal to each other, *i.e.* $A_1(\vec{r},t) = A_2(\vec{r},t) = A(\vec{r},t)$ then:

$$\left|\tilde{Z}_{tot}(\vec{r},t)\right|^2 = 2A^2(1 + \cos\Delta\varphi_{12}) \text{ and:}$$

$$\Delta\psi(\vec{r},t) = \tan^{-1}\left(\frac{\cancel{A}\sin\omega t + \cancel{A}\sin(\omega t + \Delta\varphi_{12})}{\cancel{A}\cos\omega t + \cancel{A}\cos(\omega t + \Delta\varphi_{12})}\right) = \tan^{-1}\left(\frac{\sin\omega t + \sin(\omega t + \Delta\varphi_{12})}{\cos\omega t + \cos(\omega t + \Delta\varphi_{12})}\right)$$

At $t = 0$: $\Delta\psi(\vec{r},t=0) = \tan^{-1}\left(\frac{\sin\Delta\varphi_{12}}{1 + \cos\Delta\varphi_{12}}\right)$

Beats Phenomenon

The phenomenon of beats is actually one of the most general cases of wave interference. Suppose at the observation point $\vec{r}$ in 3-D space we linearly superpose (*i.e.* add) together two signals with "zero-of-time re-defined" complex amplitudes $\tilde{Z}_1(\vec{r},t) = A_1(\vec{r},t)e^{i(\omega_1(t)t)}$ and $\tilde{Z}_2(\vec{r},t) = A_2(\vec{r},t)e^{i(\omega_2(t)t+\Delta\varphi_{21}(\vec{r},t))}$, which have similar/comparable frequencies, $\omega_1(t) \sim \omega_2(t)$ with $\Delta\omega_{12}(t) \equiv (\omega_1(t) - \omega_2(t))$ and instantaneous phase of the second signal relative to the first of $\Delta\varphi_{12}(\vec{r},t) \equiv (\varphi_1(\vec{r},t) - \varphi_2(\vec{r},t))$, the total/overall complex amplitude at the observation point $\vec{r}$ in 3-D space is $\tilde{Z}_{tot}(\vec{r},t) = \tilde{Z}_1(\vec{r},t) + \tilde{Z}_2(\vec{r},t) = A_1(\vec{r},t)e^{i\omega_1(t)t} + A_2(\vec{r},t)e^{i\omega_2(t)t}e^{i\Delta\varphi_{21}(\vec{r},t)}$.

Note that at the amplitude level, there is nothing explicitly overt and/or obvious in the above mathematical expression for the overall/total/resultant complex amplitude $\tilde{Z}_{tot}(\vec{r},t)$ that *easily* at-a-glance explains the phenomenon of beats associated with adding together two complex signals that have comparable amplitudes and frequencies.

However, let's consider the (instantaneous) phasor relationship between the two complex amplitudes $\tilde{Z}_1(\vec{r},t) = A_1(\vec{r},t)e^{i(\omega_1(t)t)}$ and $\tilde{Z}_2(\vec{r},t) = A_2(\vec{r},t)e^{i(\omega_2(t)t+\Delta\varphi_{21}(\vec{r},t))}$ respectively. Their relative phase difference at time $t = 0$ is $\Delta\varphi_{12}(\vec{r},t=0) \equiv (\varphi_1(\vec{r},t=0) - \varphi_2(\vec{r},t=0))$; the resultant/total complex amplitude $\tilde{Z}_{tot}(\vec{r},t)$ is shown in the above phasor diagram at time $t = 0$.

From the law of cosines, we showed above that magnitude^2 of the resultant/total complex amplitude at the observation point $\vec{r}$ in 3-D space was:

$$\left|\tilde{Z}_{tot}(\vec{r},t)\right|^2 = A_1^2(\vec{r},t) + A_2^2(\vec{r},t) + 2A_1(\vec{r},t)\cdot A_2(\vec{r},t)\cos\left[\Delta\omega_{12}t + \Delta\varphi_{12}(\vec{r},t)\right]$$

Then:

$$\left|\tilde{Z}_{tot}(\vec{r},t)\right| = \sqrt{\left|\tilde{Z}_{tot}(\vec{r},t)\right|^2} = \sqrt{A_1^2(\vec{r},t) + A_2^2(\vec{r},t) + 2A_1(\vec{r},t)\cdot A_2(\vec{r},t)\cos\left[\Delta\omega_{12}t + \Delta\varphi_{12}(\vec{r},t)\right]}$$

For equal amplitudes: $A_1(\vec{r},t) = A_2(\vec{r},t) = A(\vec{r},t) = A = constant$ and zero relative phase: $\Delta\varphi_{12}(\vec{r},t) \equiv (\varphi_1(\vec{r},t) - \varphi_2(\vec{r},t)) = 0$ {*i.e.* $\varphi_1(\vec{r},t) = \varphi_2(\vec{r},t)$} and constant (*i.e.* time-independent) frequencies ω_2 and ω_1, this expression simplifies to:

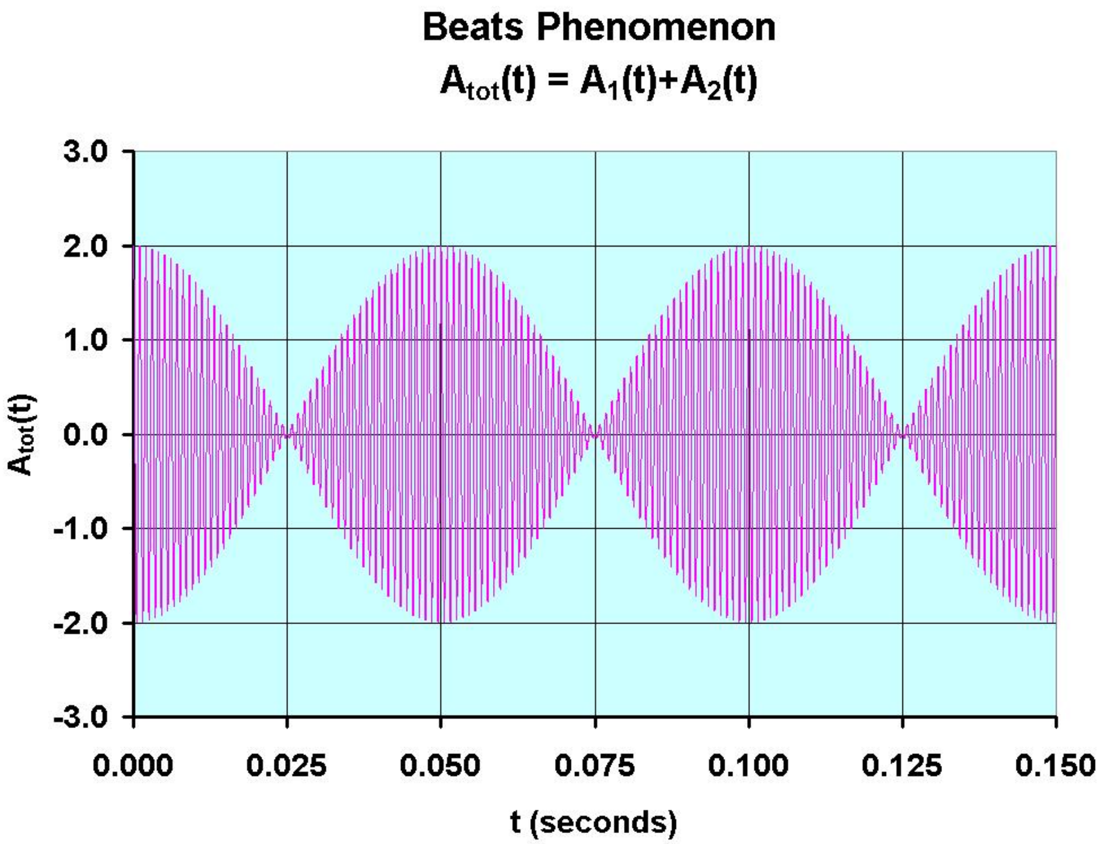
$$\left|\tilde{Z}_{tot}(\vec{r},t)\right| = \sqrt{A^2 + A^2 + 2A^2\cos\Delta\omega_{12}t} = \sqrt{2}A\cdot\sqrt{1+\cos\Delta\omega_{12}t}$$

Note that as time t increases, that $0 \le \left|\tilde{Z}_{tot}(\vec{r},t)\right| \le 2$.

The phase $\Delta\psi(\vec{r},t)$ associated with the total amplitude $\tilde{Z}_{tot}(\vec{r},t)$ for this specialized case is:

$$\Delta\psi(\vec{r},t) = \tan^{-1}\left(\frac{\mathrm{Im}\left\{\tilde{Z}_{tot}(\vec{r},t)\right\}}{\mathrm{Re}\left\{\tilde{Z}_{tot}(\vec{r},t)\right\}}\right) = \tan^{-1}\left(\frac{\sin(\omega_1 t)+\sin(\omega_2 t)}{\cos(\omega_1 t)+\cos(\omega_2 t)}\right)$$

The magnitude of the total complex amplitude, $\left|\tilde{Z}_{tot}(\vec{r},t)\right| = \left|\tilde{Z}_1(\vec{r},t) + \tilde{Z}_2(\vec{r},t)\right|$ *vs*. time, *t* is shown in the figure below for time-independent/constant frequencies of f_1 = 1000 *Hz* and f_2 = 980 *Hz*, equal amplitudes of unit strength, *i.e.* $A_1(\vec{r},t) = A_2(\vec{r},t) = A(\vec{r},t) = A = 1.0$ and zero relative phase, $\Delta\varphi_{12}(\vec{r},t) = 0$:



The beats phenomenon can clearly be seen in the above waveform of the magnitude of the total amplitude $\left|\tilde{Z}_{tot}(\vec{r},t)\right| = \left|\tilde{Z}_1(\vec{r},t) + \tilde{Z}_2(\vec{r},t)\right|$ *vs*. time, *t*. From the above graph, it is obvious that the beat period, $\tau_{beat} = 1/f_{beat}$ = 0.050 sec = 1/20th sec, corresponding to a beat frequency, $f_{beat} = 1/\tau_{beat}$ = 20 *Hz*, which is simply the frequency difference, $f_{beat} \equiv |f_1 - f_2|$ between f_1 = 1000 *Hz* and f_2 = 980 *Hz*. Thus, the beat period, $\tau_{beat} = 1/f_{beat} = 1/|f_1 - f_2|$. When $f_1 = f_2$, the beat period becomes infinitely long, and thus no beats are heard!

In terms of the phasor diagram, as time progresses the individual amplitudes $\tilde{Z}_1(\vec{r},t)$ and $\tilde{Z}_2(\vec{r},t)$ precess (*i.e*. rotate) counter-clockwise in the complex plane at (angular) rates of $\omega_1 = 2\pi f_1$ and $\omega_2 = 2\pi f_2$ radians per second respectively, completing one revolution in the phasor diagram for each cycle/each period of $\tau_1 = 2\pi/\omega_1 = 1/f_1$ and $\tau_2 = 2\pi/\omega_2 = 1/f_2$, respectively.

If at time $t = 0$ the two phasors are precisely in phase with each other (*i.e.* with initial relative phase $\Delta\varphi_{21} = 0.0$), then the resultant/total amplitude, $\tilde{Z}_{tot}(\vec{r}, t=0) = \tilde{Z}_1(\vec{r}, t=0) + \tilde{Z}_2(\vec{r}, t=0)$ at time $t = 0$ will be as shown in the figure below.

$Z_{tot}(t = 0) = Z_1(t = 0) + Z_2(t = 0)$

$Z_1(t = 0)$ $Z_2(t = 0)$

As time progresses, if $\omega_1 \neq \omega_2$, (noting that phasor 1 has angular frequency $\omega_1 = 2\pi f_1 = 2{*}1000\pi = 2000\pi$ radians/sec and phasor 2 has angular frequency $\omega_2 = 2\pi f_2 = 2{*}980\pi = 1960\pi$ radians/sec in our example above) phasor 1, with higher angular frequency will precess more rapidly than phasor 2 (by the difference in angular frequencies, $\Delta\omega = (\omega_1 - \omega_2) = (2000\pi - 1960\pi) = 40\pi$ radians/second). Thus as time increases, if $\omega_1 > \omega_2$, phasor 1 will *lead* phasor 2.

Eventually (at time $t = ½\,\tau_{beat} = 0.025 = 1/40^{th}$ sec in our above example) phasor 2 will be lagging precisely $\Delta\varphi = \pi$ radians, or 180^o behind in phase relative to phasor 1. At time $t = ½\,\tau_{beat} = 0.025$ sec $= 1/40^{th}$ sec phasor 1 will be oriented exactly as it was at time $t = 0.0$ (having precessed exactly $N_1 = \omega_1 t / 2\pi = 2\pi f_1 t / 2\pi = f_1 t = 25.0$ revolutions in this time period), whereas phasor 2 will be pointing in the opposite direction at this instant in time (having precessed only $N_2 = \omega_2 t / 2\pi = 2\pi f_2 t / 2\pi = f_2 t = 24.5$ revolutions in this same time period), and thus the total amplitude $\tilde{Z}_{tot}(\vec{r}, t = \tfrac{1}{2}\tau_{beat}) = \tilde{Z}_1(\vec{r}, t = \tfrac{1}{2}\tau_{beat}) + \tilde{Z}_2(\vec{r}, t = \tfrac{1}{2}\tau_{beat})$ will be zero at this instant in time (if the magnitudes of the two individual amplitudes are precisely equal to each other), or minimal (if the magnitudes of the two individual amplitudes are not precisely equal to each other), as shown in the figure below.

$Z_{tot}(t = ½\,\tau_{beat}) = Z_1(t = ½\,\tau_{beat}) + Z_2(t = ½\,\tau_{beat}) = 0$

$Z_2(t = ½\,\tau_{beat}) = -Z_1(t = ½\,\tau_{beat})$

As time progresses further, phasor 2 will continue to lag further and further behind phasor 1, and eventually (at time $t = \tau_{beat} = 0.050$ sec $= 1/20^{th}$ sec in our above example) phasor 2, having precessed through $N_2 = 49.0$ revolutions will now be exactly $\Delta\varphi = 2\pi$ radians, or 360^o (or one full revolution) behind in phase relative to phasor 1 (which has precessed through $N_1 = 50.0$ full revolutions), thus, the net/overall result here is the same as being exactly in phase with phasor 1! At this instant in time, $Z_{tot}(t = \tau_{beat}) = Z_1(t = \tau_{beat}) + Z_2(t = \tau_{beat}) = 2Z_1(t = \tau_{beat})$, and the phasor diagram at time $t = \tau_{beat}$ looks precisely like that as shown above for time $t = 0$.

Thus, it should (hopefully) now be clear to the reader that the phenomenon of beats is manifestly that of time-dependent alternating constructive/destructive interference between two periodic signals of comparable frequency, at the amplitude level. This is by no means a trivial point, as often the beats phenomenon is discussed in many physics textbooks in the context of intensity, $\left|\tilde{I}_{tot}(\vec{r}, t)\right| \propto \left|\tilde{Z}_{tot}(\vec{r}, t)\right|^2$. However, from the above discussion, it should be clear that the physics *origin* of the beats phenomenon has absolutely *nothing* to do with the intensity of the overall/ resultant signal, it arises from wave interference at the amplitude level.

A Special/Limiting Case – Amplitude Modulation:

Suppose at the observation point $\vec{r}$ in 3-D space that $A_1(\vec{r},t) \gg A_2(\vec{r},t)$ and $f_1 \gg f_2$, then the *exact* expression for the complex total/resultant amplitude:

$$\left|\tilde{Z}_{tot}(\vec{r},t)\right| = \sqrt{\left|\tilde{Z}_{tot}(\vec{r},t)\right|^2} = \sqrt{A_1^2(\vec{r},t) + A_2^2(\vec{r},t) + 2A_1(\vec{r},t)\cdot A_2(\vec{r},t)\cos\left[\Delta\omega_{12}t + \Delta\varphi_{12}(\vec{r},t)\right]}$$

can be *approximated*, neglecting terms of order $m^2 \equiv \left(A_2(\vec{r},t)/A_1(\vec{r},t)\right)^2 \ll 1$ under the radical sign, and noting that for $f_1 \gg f_2$, then $\omega_1 \gg \omega_2$ and hence $\Delta\omega_{12} \equiv (\omega_1 - \omega_2) \cong \omega_1$. For simplicity in this discussion, we set the phase difference $\Delta\varphi_{12}(\vec{r},t) \equiv \varphi_1(\vec{r},t) - \varphi_2(\vec{r},t) = 0$ (its effect is merely to shift the overall beats pattern to the left or right along the time axis). Then:

$$\left|\tilde{Z}_{tot}(\vec{r},t)\right| = A_1(\vec{r},t)\sqrt{1 + \left(A_2(\vec{r},t)/A_1(\vec{r},t)\right)^2 + 2\left(A_2(\vec{r},t)/A_1(\vec{r},t)\right)\cos\left[\Delta\omega_{12}t + \cancel{\Delta\varphi_{12}(\vec{r},t)}\right]}$$

$$= A_1(\vec{r},t)\sqrt{1 + \cancel{m^2} + 2m\cos\left[\Delta\omega_{12}t\right]} \simeq A_1(\vec{r},t)\sqrt{1 + 2m\cos\omega_1 t}$$

Using the Taylor series expansion $\sqrt{1+\varepsilon} \simeq 1 + \tfrac{1}{2}\varepsilon$ for the case $\varepsilon \equiv 2m\cos\omega_1 t \ll 1$, the magnitude of the total complex amplitude is $\left|\tilde{Z}_{tot}(\vec{r},t)\right| \simeq A_1(\vec{r},t)\left(1 + m\cos\omega_1 t\right)$.

The ratio $m \equiv \left(A_2(\vec{r},t)/A_1(\vec{r},t)\right) \ll 1$ is known as the (amplitude) *modulation depth* associated with the high-frequency *carrier* wave $\tilde{Z}_1(\vec{r},t)$, with amplitude $A_1(\vec{r},t) \gg A_2(\vec{r},t)$ and frequency $f_1 \gg f_2$, *modulated* by the low frequency wave $\tilde{Z}_2(\vec{r},t)$ with amplitude $A_2(\vec{r},t)$ and frequency f_2. This is the underlying principle of how *AM radio* works – note that *AM* stands for ***A**mplitude **M**odulation*. In *AM* radio broadcasting, $540\ KHz \lesssim f_1 = f_{carrier} \lesssim 1600\ KHz$ whereas $20Hz \lesssim f_2 = f_{audio} \lesssim 20\ KHz$.

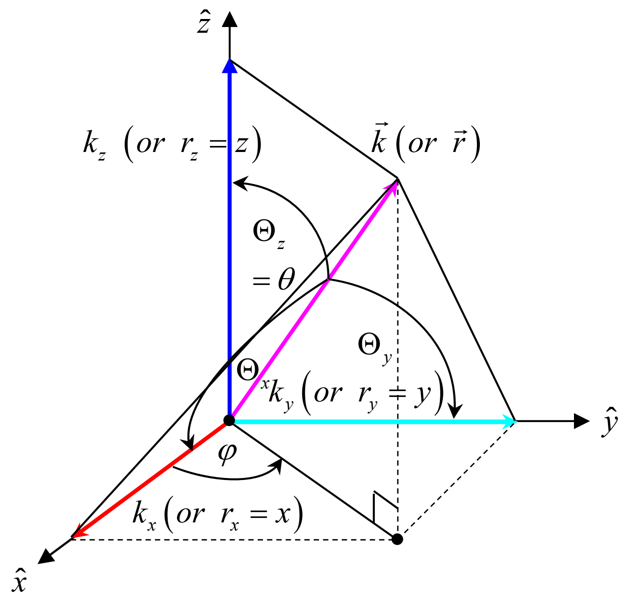
Propagation of Complex Sound Waves in Three Dimensions:

In previous lectures, we have discussed the propagation of purely real sound waves in one dimension, *e.g.* a monochromatic traveling plane wave propagating in the $\pm x$-direction: $\psi(x,t) = A\cos(\omega t \mp kx)$ where A is the amplitude of the wave, the wavenumber $k = 2\pi/\lambda\ (m^{-1})$, the wavelength $\lambda = v/f\ (m)$ and the ***phase*** speed of propagation of the monochromatic traveling wave in the medium is $v_\varphi = f\lambda = \omega/k\ (m/s)$, which in "free air" {*i.e.* "The Great Wide Open"} is also equal to the speed of propagation of energy v_E in that medium.

We can "complexify" the purely real 1-D monochromatic traveling plane wave description(s) $\psi(x,t) = A\cos(\omega t \mp kx)$ to become complex 1-D monochromatic traveling plane waves simply by adding on a purely imaginary term: $iA\sin(\omega t \mp kx)$, *i.e.* complex 1-D monochromatic traveling plane waves in the $\pm x$-direction are mathematically described by:

$$\tilde{\psi}(x,t) = A\left\{\cos(\omega t \mp kx) + i\sin(\omega t \mp kx)\right\} = Ae^{i(\omega t \mp kx)}.$$

In order to describe monochromatic traveling plane waves propagating in an *arbitrary* direction in 3-D space, in analogy to the 3-D position vector $\vec{r} = r_x\hat{x} + r_y\hat{y} + r_z\hat{z} = x\hat{x} + y\hat{y} + z\hat{z}$, we introduce the concept of a ***wavevector*** $\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$. The wavevector $\vec{k}$ is an important physical quantity because it tells us the ***propagation direction*** of the wave – it is in the $\hat{k} = \vec{k}/|\vec{k}| = \vec{k}/k$ direction. The k_x, k_y, k_z are the components of the wavevector $\vec{k}$ along (*i.e.* ***projections*** onto) the $\hat{x}, \hat{y}, \hat{z}$ axes, respectively as shown in the figure below:



The ***magnitude*** of the observer's position vector $\vec{r}$ is: $r = |\vec{r}| = \sqrt{\vec{r}\bullet\vec{r}} = \sqrt{r_x^2 + r_y^2 + r_z^2} = \sqrt{x^2 + y^2 + z^2}$.

Likewise, the ***magnitude*** of the wavevector $\vec{k}$ is: $k = |\vec{k}| = \sqrt{\vec{k}\bullet\vec{k}} = \sqrt{k_x^2 + k_y^2 + k_z^2}$.

The three *x,y,z*-***direction cosines*** associated with the position vector $\vec{r}$ are obtained from dot products (*aka* ***inner products***) of the ***unit position vector*** $\hat{r}$ with the $\hat{x}, \hat{y}, \hat{z}$ axes, respectively:

$$\cos\Theta_x \equiv \hat{r}\bullet\hat{x}\,, \quad \cos\Theta_y \equiv \hat{r}\bullet\hat{y} \quad \text{and} \quad \cos\Theta_z \equiv \hat{r}\bullet\hat{z}$$

Since $\vec{r} = r\hat{r}$, then: $\hat{r} = \vec{r}/r = \vec{r}/|\vec{r}| = \vec{r}\Big/\sqrt{x^2 + y^2 + z^2} = x\hat{x} + y\hat{y} + z\hat{z}\Big/\sqrt{x^2 + y^2 + z^2}$.

Thus we see that:

$$\cos\Theta_x \equiv \hat{r}\bullet\hat{x} = x\Big/\sqrt{x^2 + y^2 + z^2} = x/|r| = x/r\,,$$

$$\cos\Theta_y \equiv \hat{r}\bullet\hat{y} = y\Big/\sqrt{x^2 + y^2 + z^2} = y/|r| = y/r \quad \text{and}$$

$$\cos\Theta_z \equiv \hat{r}\bullet\hat{z} = z\Big/\sqrt{x^2 + y^2 + z^2} = z/|r| = z/r\,.$$

Note also that: $|\hat{r}\bullet\hat{r}|^2 = \cos^2\Theta_x + \cos^2\Theta_y + \cos^2\Theta_z = 1$.

In terms of the usual 3-D spherical-polar coordinate system's ***polar*** and ***azimuthal*** angles θ and φ, respectively it is straightforward to show that:

$$\cos\Theta_x = \sin\theta\cos\varphi\,,$$
$$\cos\Theta_y = \sin\theta\sin\varphi \quad \text{and}$$
$$\cos\Theta_z = \cos\theta\,,\ \textit{i.e.}\ \text{that}\ \ \Theta_z = \theta\,.$$

Likewise, the wavevector $\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$ has its own direction cosines:

$$\cos\Theta_{k_x} \equiv \hat{k}\bullet\hat{x} = k_x\Big/\sqrt{k_x^2 + k_y^2 + k_z^2} = k_x\Big/\left|\vec{k}\right| = k_x/k\,,$$
$$\cos\Theta_{k_y} \equiv \hat{k}\bullet\hat{y} = k_y\Big/\sqrt{k_x^2 + k_y^2 + k_z^2} = k_y\Big/\left|\vec{k}\right| = k_y/k \quad \text{and}$$
$$\cos\Theta_{k_z} \equiv \hat{k}\bullet\hat{z} = k_z\Big/\sqrt{k_x^2 + k_y^2 + k_z^2} = k_z\Big/\left|\vec{k}\right| = k_z/k\,.$$

Note again that: $\left|\hat{k}\bullet\hat{k}\right|^2 = \cos^2\Theta_{k_x} + \cos^2\Theta_{k_y} + \cos^2\Theta_{k_z} = 1$.

If we were to imagine 1-D complex monochromatic traveling plane waves propagating in the $\hat{k}_x = \pm\hat{x}$, $\hat{k}_y = \pm\hat{y}$ and $\hat{k}_z = \pm\hat{z}$ -directions we would describe each of these mathematically as:

Prop. in $\hat{k}_x = \pm\hat{x}$-direction: $\tilde{\psi}_x(\vec{r},t) = A\left\{\cos\left(\omega t \mp k_x x\right) + i\sin\left(\omega t \mp k_x x\right)\right\} = Ae^{i(\omega t \mp k_x x)}$

Prop. in $\hat{k}_y = \pm\hat{y}$-direction: $\tilde{\psi}_y(\vec{r},t) = A\left\{\cos\left(\omega t \mp k_y y\right) + i\sin\left(\omega t \mp k_y y\right)\right\} = Ae^{i(\omega t \mp k_y y)}$

Prop. in $\hat{k}_z = \pm\hat{z}$-direction: $\tilde{\psi}_z(\vec{r},t) = A\left\{\cos\left(\omega t \mp k_z z\right) + i\sin\left(\omega t \mp k_z z\right)\right\} = Ae^{i(\omega t \mp k_z z)}$

From these relations, noting that $\vec{k}\bullet\vec{r} = \left(k_x\hat{x} + k_y\hat{y} + k_z\hat{z}\right)\bullet\left(x\hat{x} + y\hat{y} + z\hat{z}\right) = k_x x + k_y y + k_z z$, we can generalize to the case for a complex monochromatic traveling plane wave propagating in an *arbitrary* direction $\hat{k} = \left(k_x\hat{x} + k_y\hat{y} + k_z\hat{z}\right)\Big/\left|\vec{k}\right| = \left(k_x\hat{x} + k_y\hat{y} + k_z\hat{z}\right)\Big/k$ in 3-D space:

Prop. in $\hat{k}$-direction: $\tilde{\psi}(\vec{r},t) = A\left\{\cos\left(\omega t \mp \vec{k}\bullet\vec{r}\right) + i\sin\left(\omega t \mp \vec{k}\bullet\vec{r}\right)\right\} = Ae^{i\left(\omega t \mp \vec{k}\bullet\vec{r}\right)}$

The above expression for a complex monochromatic traveling plane wave propagating in an arbitrary direction $\hat{k}$ in 3-D space is an appropriate description for a complex ***scalar*** field – *e.g.* complex pressure $\tilde{p}(\vec{r},t)$ – because ***scalar*** fields $\tilde{\psi}(\vec{r},t)$ at each/every space-time point $(\vec{r},t)$ have ***no*** explicit direction associated with them, other than their propagation direction $\hat{k}$.

We can also easily generalize the above complex scalar traveling wave description $\tilde{\psi}(\vec{r},t)$ to describe 3-D complex ***vector*** monochromatic traveling plane waves propagating in an arbitrary direction $\hat{k}$ in 3-D space – *e.g.* complex particle displacement $\tilde{\vec{\xi}}(\vec{r},t)$, particle velocity $\tilde{\vec{u}}(\vec{r},t)$ and/or particle acceleration $\tilde{\vec{a}}(\vec{r},t)$. We can mathematically describe "generic" 3-D complex vector fields *e.g.* in Cartesian / rectangular coordinates in the following form:

$$\tilde{\vec{\psi}}(\vec{r},t) = \tilde{\psi}_x(\vec{r},t)\,\hat{x} + \tilde{\psi}_y(\vec{r},t)\,\hat{y} + \tilde{\psi}_z(\vec{r},t)\,\hat{z}\,.$$

Vector fields $\tilde{\vec{\psi}}(\vec{r},t)$ at each/every space-time point $(\vec{r},t)$ have ***do*** have an explicit direction associated with them – namely the $\hat{\tilde{\psi}}(\vec{r},t)=\tilde{\vec{\psi}}(\vec{r},t)\big/\left|\tilde{\vec{\psi}}(\vec{r},t)\right|$ direction. Thus, for a 3-D complex ***vector*** monochromatic traveling plane wave propagating in an arbitrary direction $\hat{k}$ in 3-D space:

$$\text{Prop. in } \hat{k}\text{-direction:}\quad \tilde{\vec{\psi}}(\vec{r},t)=\vec{A}\left\{\cos\left(\omega t\mp\vec{k}\bullet\vec{r}\right)+i\sin\left(\omega t\mp\vec{k}\bullet\vec{r}\right)\right\}=\vec{A}e^{i\left(\omega t\mp\vec{k}\bullet\vec{r}\right)}$$

where the complex vector ***amplitude*** associated with the 3-D complex vector monochromatic traveling plane wave $\tilde{\vec{\psi}}(\vec{r},t)$ propagating in an arbitrary direction $\hat{k}$ in 3-D space is given by:

$$\vec{A}=A_x\hat{x}+A_y\hat{y}+A_z\hat{z}$$

The above expressions for 3-D complex ***scalar*** and ***vector*** monochromatic traveling plane waves propagating in an arbitrary direction $\hat{k}$ in 3-D space are formal mathematical solutions to their corresponding 3-D wave equations:

$$\nabla^2\tilde{\psi}(\vec{r},t)-\frac{1}{v_\varphi^2}\frac{\partial^2\tilde{\psi}(\vec{r},t)}{\partial t^2}=0 \quad\text{and}\quad \nabla^2\tilde{\vec{\psi}}(\vec{r},t)-\frac{1}{v_\varphi^2}\frac{\partial^2\tilde{\vec{\psi}}(\vec{r},t)}{\partial t^2}=0 \ \text{ respectively.}$$

The 3-D complex monochromatic traveling plane wave solution(s) to these two linear, homogeneous, 2nd-order differential equations physically correspond, respectively to ***scalar*** and ***vector*** waves propagating in the $\hat{k}$ direction in a 3-D medium which has the following physical properties:

(a) The medium is *lossless*, *i.e.* no friction/no damping and/or dissipative processes exist.

(b) The medium is also *dispersionless*, *i.e.* there is no frequency dependence of the phase speed of propagation *v* in the medium, *i.e.* $v_\varphi = constant \neq fcn(f,\vec{r},...)$, such that the ***dispersion relationship*** $v_\varphi = f\lambda=\omega/k = constant \neq fcn(f,\vec{r},...)$ is valid/holds in the medium.

If the medium ***is*** dissipative and/or dispersive, the above wave equation(s) and their solutions will necessarily be modified/change as a consequence of such phenomena.

Note also that the 3-D vector wave equation $\nabla^2\tilde{\vec{\psi}}(\vec{r},t)-\frac{1}{v_\varphi^2}\frac{\partial^2\tilde{\vec{\psi}}(\vec{r},t)}{\partial t^2}=0$ is actually three *separate/independent* wave equations, since $\tilde{\vec{\psi}}(\vec{r},t)=\tilde{\psi}_x(\vec{r},t)\hat{x}+\tilde{\psi}_y(\vec{r},t)\hat{y}+\tilde{\psi}_z(\vec{r},t)\hat{z}$:

$$\nabla^2\tilde{\psi}_x(\vec{r},t)-\frac{1}{v_\varphi^2}\frac{\partial^2\tilde{\psi}_x(\vec{r},t)}{\partial t^2}=0,\ \nabla^2\tilde{\psi}_y(\vec{r},t)-\frac{1}{v_\varphi^2}\frac{\partial^2\tilde{\psi}_y(\vec{r},t)}{\partial t^2}=0 \ \text{and}\ \nabla^2\tilde{\psi}_z(\vec{r},t)-\frac{1}{v_\varphi^2}\frac{\partial^2\tilde{\psi}_z(\vec{r},t)}{\partial t^2}=0$$

i.e. $\nabla^2\tilde{\vec{\psi}}(\vec{r},t)-\frac{1}{v_\varphi^2}\frac{\partial^2\tilde{\vec{\psi}}(\vec{r},t)}{\partial t^2}=0$

$$=\left[\nabla^2\tilde{\psi}_x(\vec{r},t)-\frac{1}{v_\varphi^2}\frac{\partial^2\tilde{\psi}_x(\vec{r},t)}{\partial t^2}\right]\hat{x}+\left[\nabla^2\tilde{\psi}_y(\vec{r},t)-\frac{1}{v_\varphi^2}\frac{\partial^2\tilde{\psi}_y(\vec{r},t)}{\partial t^2}\right]\hat{y}+\left[\nabla^2\tilde{\psi}_z(\vec{r},t)-\frac{1}{v_\varphi^2}\frac{\partial^2\tilde{\psi}_z(\vec{r},t)}{\partial t^2}\right]\hat{z}=0$$

The complex monochromatic scalar and vector traveling plane waves $\tilde{\psi}(\vec{r},t)$, $\tilde{\vec{\psi}}(\vec{r},t)$ {obviously} must respectively satisfy the 3-D wave equations:

$$\nabla^2\tilde{\psi}(\vec{r},t)-\frac{1}{v_\varphi^2}\frac{\partial^2}{\partial t^2}\tilde{\psi}(\vec{r},t)=0 \text{ and: } \nabla^2\tilde{\vec{\psi}}(\vec{r},t)-\frac{1}{v_\varphi^2}\frac{\partial^2\tilde{\vec{\psi}}(\vec{r},t)}{\partial t^2}=0$$

In 3-D Cartesian/rectangular coordinates the ***Laplacian*** operator $\nabla^2 \equiv \vec{\nabla}\bullet\vec{\nabla}$ {where $\vec{\nabla}$ is the ***gradient*** operator} has the form:

$$\nabla^2 \equiv \vec{\nabla}\bullet\vec{\nabla}=\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\bullet\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

Explicitly carrying out the differentiation(s), we obtain the ***dispersion relation*** associated with propagation of a complex monochromatic traveling plane wave {in "***free air***"}: $k_x^2+k_y^2+k_z^2=\omega^2/v_\varphi^2$, which since $k^2\equiv|\vec{k}|^2=\vec{k}\bullet\vec{k}=k_x^2+k_y^2+k_z^2$ can be equivalently written as: $k^2=\omega^2/v_\varphi^2$ or: $k=\omega/v_\varphi$, hence the phase velocity: $v_\varphi=\omega/k=f\lambda=constant\neq fcn(f,\vec{r},...)$.

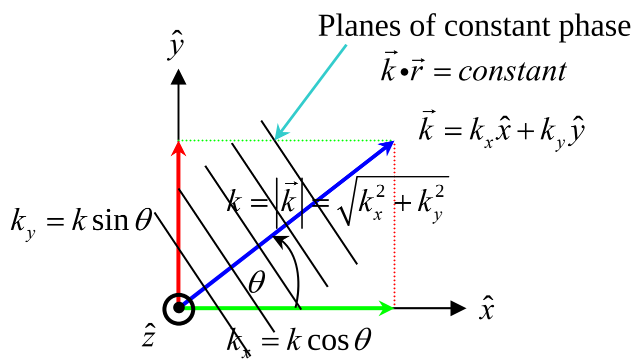
The surface{s} of ***constant phase*** associated with a traveling plane wave occur for $\vec{k}\bullet\vec{r}=constant$ in the argument of the $e^{i(\omega t\mp\vec{k}\bullet\vec{r})}$ factor in $\tilde{\psi}(\vec{r},t)$ and/or $\tilde{\vec{\psi}}(\vec{r},t)$.

From the fundamental/mathematical definition of a {spatial} gradient, the vector wavenumber:

$$\vec{k}\equiv\vec{\nabla}(\vec{k}\bullet\vec{r})=\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)(k_x x+k_y y+k_z z)=k_x\hat{x}+k_y\hat{y}+k_z\hat{z}$$

is a vector quantity that points in a direction *perpendicular* to the surface(s) of constant phase, $\vec{k}\bullet\vec{r}=constant$. Physically, it points in the direction of propagation of the traveling plane wave.

If *e.g.* the vector wavenumber $\vec{k}$ lies only in the *x*-*y* plane {thus making an angle θ with respect to the $\hat{x}$-axis}, then $\tilde{\psi}(\vec{r},t)=Ae^{i(\omega t-\vec{k}\bullet\vec{r})}=Ae^{i(\omega t-k_x x-k_y y)}$ and 2-D planar surfaces of constant phase are oriented parallel to the $\hat{z}$-axis as shown in the figure below {for a "snapshot-in-time", *e.g.* at $t = 0$}:



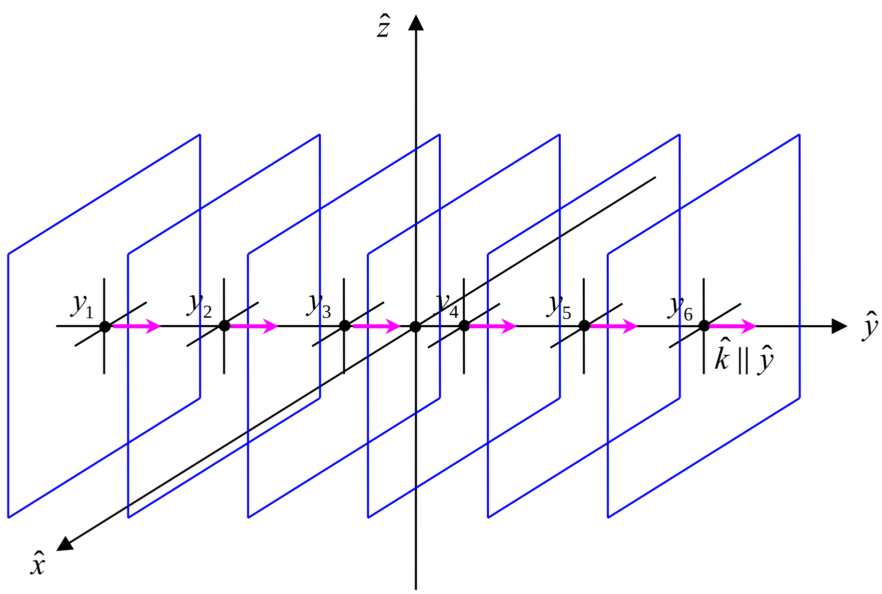
From the above figure, we see that: $\vec{k}=k_x\hat{x}+k_y\hat{y}+\cancel{k_z\hat{z}}=k_x\hat{x}+k_y\hat{y}=k\cos\theta\hat{x}+k\sin\theta\hat{y}$.

Surfaces of constant phase are $\vec{k}\cdot\vec{r} = k_x x + k_y y = constant$ or: $y = -(k_x/k_y)x + constant$, which is the equation of a straight line $y(x) = mx + b$ with slope: $m = -(k_x/k_y) = -(k\cos\theta/k\sin\theta) = -\cot\theta$ and y-intercept: $b = constant$.

At *e.g.* fixed $y = 0$, this traveling wave is: $\tilde{\psi}(\vec{r},t) = \tilde{\psi}(x,0,z,t) = Ae^{i(\omega t - k_x x)} = Ae^{i(\omega t - kx\cos\theta)}$.

At *e.g.* fixed $x = 0$, this traveling wave is: $\tilde{\psi}(\vec{r},t) = \tilde{\psi}(0,y,z,t) = Ae^{i(\omega t - k_y y)} = Ae^{i(\omega t - ky\sin\theta)}$.

The 3-D complex monochromatic traveling plane wave solution(s) to the above linear, homogeneous, 2nd-order differential equations also physically means that ***propagating*** 2-D planes (*aka* *wavefronts*) of constant phase $\varphi(\vec{r},t) = \omega t \mp \vec{k}\cdot\vec{r}$ also exist, as shown in the figure below, *e.g.* for a ***scalar*** 3-D complex monochromatic traveling plane wave propagating in the $\hat{k} = +\hat{y}$ direction with $\vec{k} = k_y\hat{y}$ and observer position $\vec{r} = y\hat{y}$, thus, here: $\tilde{\psi}(\vec{r},t) = Ae^{i(\omega t - \vec{k}\cdot\vec{r})} = Ae^{i(\omega t - k_y y)}$:



For each of the $i = 1:6$ planes located at $y = y_i$ in the above figure, at a specific instant in time, t the phase $\varphi_i(\vec{r},t) = \varphi(x, y = y_i, z, t) = \omega t - ky_i$ associated with the complex traveling plane wave propagating in the $\hat{k} = +\hat{y}$ direction is the same (*i.e.* constant) for ***every*** (x,z) point on that $y = y_i$ plane. Note also that the phase difference $\Delta\varphi_{i,i-1}(\vec{r},t)$ between successive planes i and i–1 is *also* constant, as well as time-independent:

$$\Delta\varphi_{i,i-1}(\vec{r},t) \equiv \varphi(x, y = y_i, z, t) - \varphi(x, y = y_{i-1}, z, t) = (\cancel{\omega t} - ky_i) - (\cancel{\omega t} - ky_{i-1}) = -k\underbrace{(y_i - y_{i-1})}_{\equiv\Delta y} = -k\Delta y$$

<u>**Complex Standing Waves**</u>:

Suppose that we linearly superpose (*i.e.* add together) *e.g.* two counter-propagating <u>*scalar*</u> 1-D complex monochromatic traveling plane waves of the same frequency *f* and amplitude *A*, propagating in the $\hat{k}_1 = +\hat{y}$ and $\hat{k}_2 = -\hat{y}$ directions, respectively in a lossless/dispersionless medium. Then $\vec{k}_1 = +k\hat{y} = +(\omega/v)\,\hat{y} = +(2\pi f/v)\,\hat{y}$ and $\vec{k}_2 = -k\hat{y} = -(\omega/v)\,\hat{y} = -(2\pi f/v)\,\hat{y}$. At the observer's space-time position $(\vec{r},t) = (x,y,z,t)$, the total/resultant wave is:

$$
\begin{aligned}
\tilde{\psi}_{tot}(\vec{r},t) = \tilde{\psi}_1(\vec{r},t) + \tilde{\psi}_2(\vec{r},t) &= Ae^{i\left(\omega t - \vec{k}_1\bullet\vec{r} + \varphi_1(\vec{r},t)\right)} + Ae^{i\left(\omega t - \vec{k}_2\bullet\vec{r} + \varphi_2(\vec{r},t)\right)} \\
&= Ae^{i\left(\omega t - ky + \varphi_1(\vec{r},t)\right)} + Ae^{i\left(\omega t + ky + \varphi_2(\vec{r},t)\right)} \\
&= A\left\{e^{i\left(\omega t - ky + \varphi_1(\vec{r},t)\right)} + e^{i\left(\omega t + ky + \varphi_2(\vec{r},t)\right)}\right\} \\
&= Ae^{i\omega t}\left\{e^{-i\left(ky - \varphi_1(\vec{r},t)\right)} + e^{+i\left(ky + \varphi_2(\vec{r},t)\right)}\right\}
\end{aligned}
$$

The magnitude (*i.e.* length) of the total/resultant wave is:

$$
\begin{aligned}
\left|\tilde{\psi}_{tot}(\vec{r},t)\right| &\equiv \sqrt{\tilde{\psi}_{tot}(\vec{r},t)\cdot\tilde{\psi}^*_{tot}(\vec{r},t)} \\
&= \sqrt{A\,\cancel{e^{i\omega t}}\left\{e^{-i\left(ky-\varphi_1(\vec{r},t)\right)} + e^{+i\left(ky+\varphi_2(\vec{r},t)\right)}\right\}\cdot A\,\cancel{e^{-i\omega t}}\left\{e^{+i\left(ky-\varphi_1(\vec{r},t)\right)} + e^{-i\left(ky+\varphi_2(\vec{r},t)\right)}\right\}} \\
&= A\sqrt{\left\{e^{-i\left(ky-\varphi_1(\vec{r},t)\right)} + e^{+i\left(ky+\varphi_2(\vec{r},t)\right)}\right\}\cdot\left\{e^{+i\left(ky-\varphi_1(\vec{r},t)\right)} + e^{-i\left(ky+\varphi_2(\vec{r},t)\right)}\right\}} \\
&= A\sqrt{1 + e^{+i\left(2ky + \left[\varphi_2(\vec{r},t) - \varphi_1(\vec{r},t)\right]\right)} + e^{-i\left(2ky + \left[\varphi_2(\vec{r},t) - \varphi_1(\vec{r},t)\right]\right)} + 1} \\
&= A\sqrt{2 + e^{+i\left(2ky + \left[\varphi_2(\vec{r},t) - \varphi_1(\vec{r},t)\right]\right)} + e^{-i\left(2ky + \left[\varphi_2(\vec{r},t) - \varphi_1(\vec{r},t)\right]\right)}}
\end{aligned}
$$

We define: $\Delta\varphi_{21}(\vec{r},t) \equiv \varphi_2(\vec{r},t) - \varphi_1(\vec{r},t)$, thus: $\left|\tilde{\psi}_{tot}(\vec{r},t)\right| = A\sqrt{2 + e^{+i\left(2ky + \Delta\varphi_{21}(\vec{r},t)\right)} + e^{-i\left(2ky + \Delta\varphi_{21}(\vec{r},t)\right)}}$

We then define: $\Phi(\vec{r},t) \equiv 2ky + \Delta\varphi_{21}(\vec{r},t)$, thus: $\left|\tilde{\psi}_{tot}(\vec{r},t)\right| = A\sqrt{2 + \underbrace{e^{+i\Phi(\vec{r},t)} + e^{-i\Phi(\vec{r},t)}}_{=2\cos\Phi(\vec{r},t)}}$.

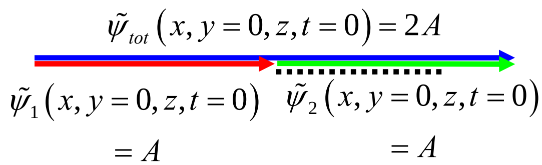
But: $e^{+i\Phi(\vec{r},t)} + e^{-i\Phi(\vec{r},t)} = 2\cos\Phi(\vec{r},t)$, thus we see: $\left|\tilde{\psi}_{tot}(\vec{r},t)\right| = \sqrt{2}A\sqrt{1 + \cos\Phi(\vec{r},t)}$.

Thus, we see that when: $\cos\Phi(\vec{r},t) = +1$, *i.e.* when $\Phi(\vec{r},t) = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi ... = \pm n_{even}\pi$, n_{even} = 0, 2, 4, 6, … the total/resultant wave will be maximal (*i.e.* constructive interference of the two counter-propagating traveling waves): $\left|\tilde{\psi}_{tot}(\vec{r},t)\right| = 2A$, but when $\cos\Phi(\vec{r},t) = -1$, *i.e.* when $\Phi(\vec{r},t) = \pm 1\pi, \pm 3\pi, \pm 5\pi ... = \pm n_{odd}\pi$, n_{odd} = 1, 3, 5, 7, … the total/resultant wave will be minimal, (*i.e.* destructive interference of the two counter-propagating traveling waves): $\left|\tilde{\psi}_{tot}(\vec{r},t)\right| = 0$.

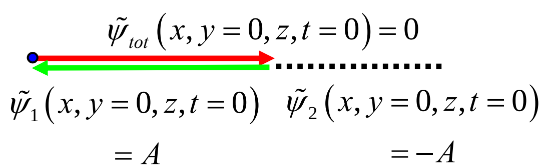
If we choose the observer's position *e.g.* to be at $\vec{r} = (x, y=0, z)$ {*i.e.* anywhere in the *x*-*z* plane, at *y* = 0}, then: $\Phi(x, y=0, z, t) \equiv \Delta\varphi_{21}(x, y=0, z, t) = \varphi_2(x, y=0, z, t) - \varphi_1(x, y=0, z, t)$ and we see that: $\cos\Phi(x, y=0, z, t) = \cos\Delta\varphi_{21}(x, y=0, z, t)$, hence the total/resultant plane wave is: $\left|\tilde{\psi}_{tot}(x,y=0,z,t)\right| = \sqrt{2}A\sqrt{1+\cos\Delta\varphi_{21}(x,y=0,z,t)}$, and thus when: $\cos\Delta\varphi_{21}(x,y=0,z,t)=+1$, *i.e.* when $\Delta\varphi_{21}(x,y=0,z,t)=0,\pm 2\pi,\pm 4\pi,\pm 6\pi ... = \pm n_{even}\pi$, n_{even} = 0, 2, 4, 6, … the total/resultant plane wave will be maximal (i.e. constructive interference of the two counter-propagating traveling waves): $\left|\tilde{\psi}_{tot}(x,y=0,z,t)\right| = 2A$, but when $\cos\Phi(\vec{r},t)=-1$, *i.e.* when $\Delta\varphi_{21}(x,y=0,z,t)=\pm 1\pi,\pm 3\pi,\pm 5\pi ... = \pm n_{even}\pi$, n_{odd} = 1, 3, 5, 7, … the total/resultant plane wave will be minimal, (i.e. destructive interference of the two counter-propagating traveling waves): $\left|\tilde{\psi}_{tot}(\vec{r},t)\right| = 0$.

<u>In terms of phasor diagrams</u>:

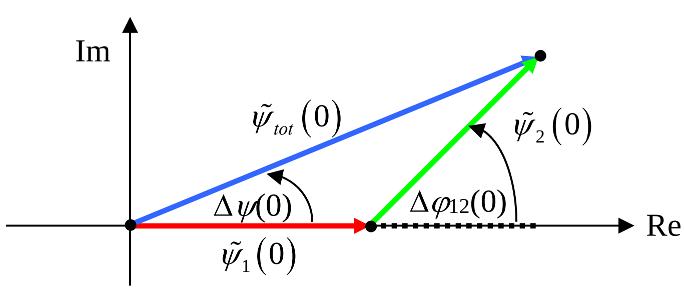
Maximal/constructive interference occurs when the relative phase $\Delta\varphi_{21}$ of the two counter-propagating traveling waves is an ***<u>even</u>*** integer multiple of 2π, *i.e.* 360^o, such that the two individual amplitudes add linearly together, because they are precisely in-phase with each other, as shown in the figure below, at time t = 0:



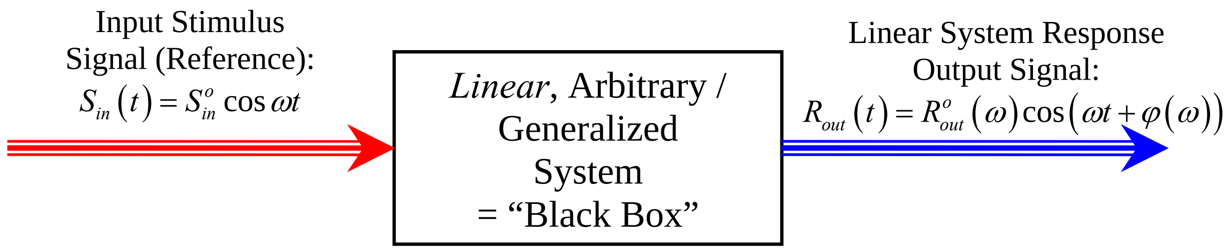
Minimal/destructive interference occurs when the relative phase $\Delta\varphi_{21}$ of the two counter-propagating traveling waves is an ***<u>odd</u>*** integer multiple of π, *i.e.* 180^o, such that the two individual amplitudes completely cancel, because they are in precisely out-of-phase with each other, as shown in the figure below, at time t = 0:



When the relative phase difference $\Delta\varphi_{21}(x,y=0,z,t)=\varphi_2(x,y=0,z,t)-\varphi_1(x,y=0,z,t)$ is anywhere in between these special points, *i.e.* $n_{odd}\pi < \Delta\varphi_{21} < n_{even}\pi$, then only partial/incomplete interference occurs, and the phasor diagram in the complex plane at time t = 0 will in general be something like that shown in the figure below:



Now let us return to our input stimulus/"black-box" system output response problem that we mentioned at the beginning of these lecture notes, and discuss *this* situation in greater detail:



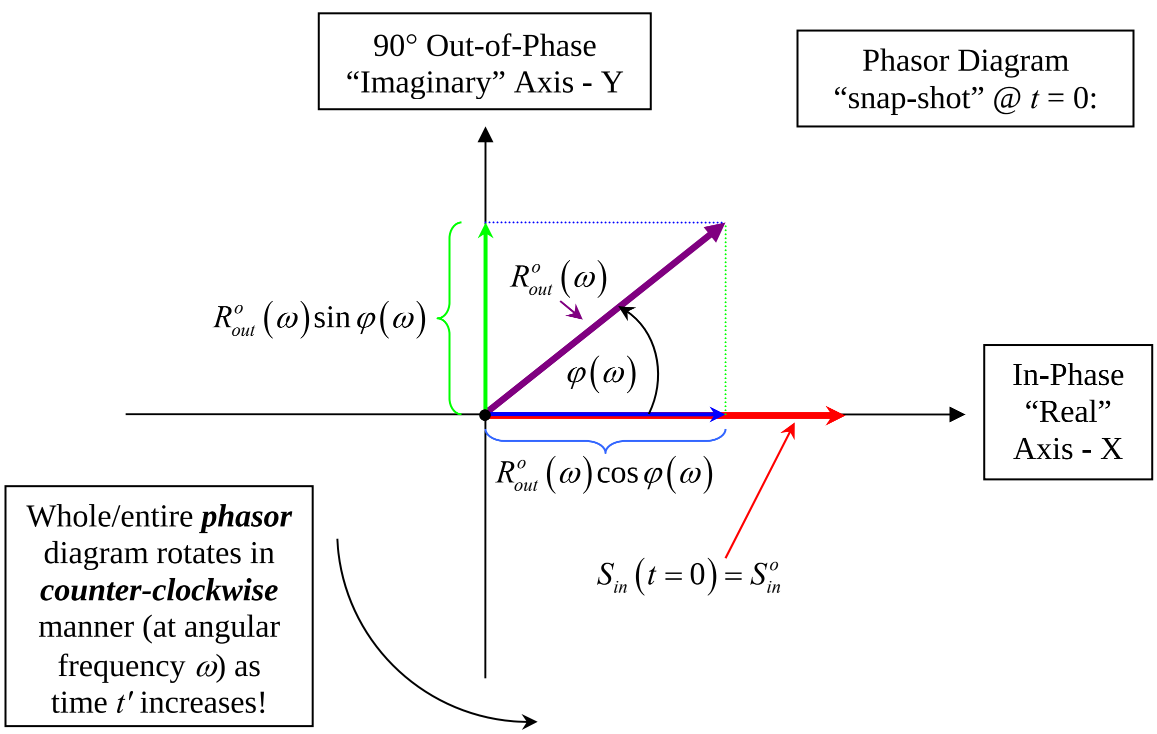
The ***instantaneous*** input ***stimulus*** signal $S_{in}(t) = S_{in}^{o}\cos\omega t$ and the ***instantaneous*** output ***response*** signal $R_{out}(t) = R_{out}^{o}(\omega)\cos(\omega t + \varphi(\omega))$ are ***purely real time-domain*** quantities.

We can "***complexify***" the ***instantaneous*** input/output ***time-domain*** signals just as we have done above by adding suitable / appropriate "***imaginary***" (*aka* ***quadrature***) terms to each, which are {±} 90° ***out-of-phase*** with the above ***purely real*** time-domain quantities:

$$\tilde{S}_{in}(t) = S_{in}^{o}\cos\omega t + iS_{in}^{o}\sin\omega t = S_{in}^{o}e^{i\omega t}$$

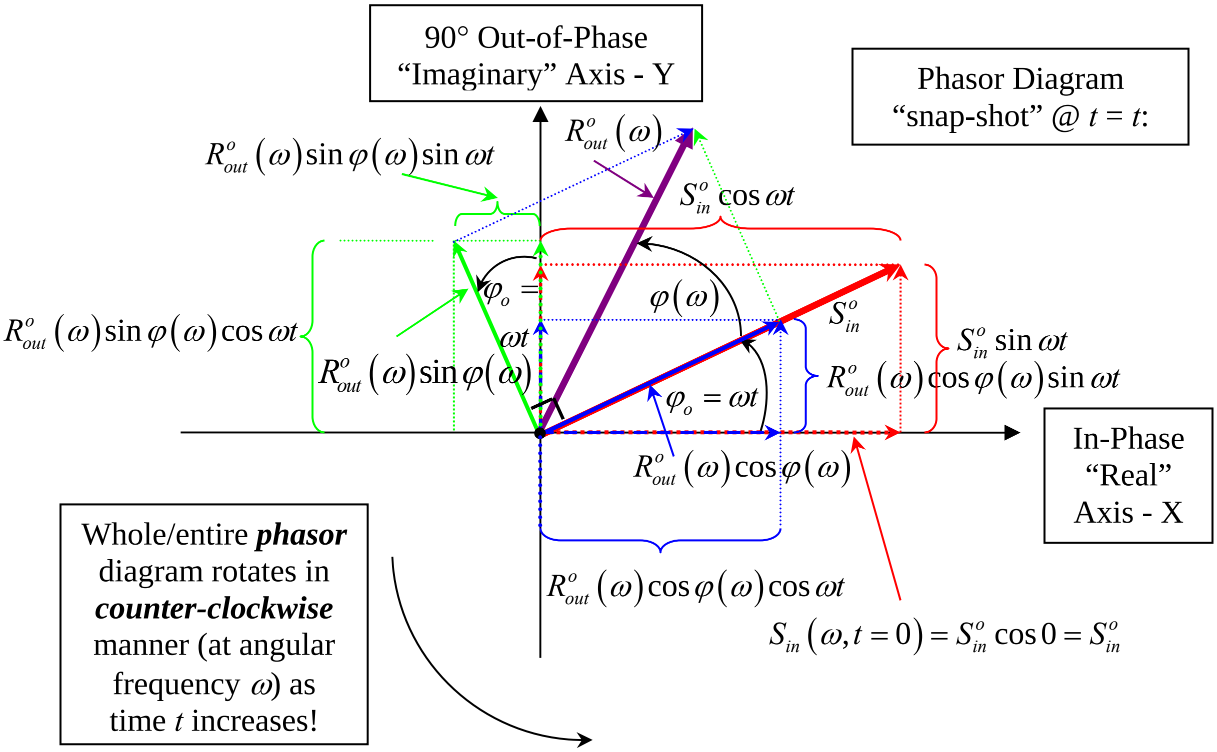
$$\tilde{R}_{out}(t) = R_{out}^{o}(\omega)\cos(\omega t + \varphi(\omega)) + iR_{out}^{o}(\omega)\sin(\omega t + \varphi(\omega)) = R_{out}^{o}(\omega)e^{i(\omega t + \varphi(\omega))}$$

The $t = 0$ phasor diagram associated with these two complex phasors is shown in the figure below:



As time t increases, the entire phasor diagram rotates counter-clockwise in the complex plane, with angular frequency ω as shown in the figure below, for a "snapshot-in-time" at time $t = t$.

The entire $t = t$ phasor diagram below is rotated CCW relative to the above $t = 0$ phasor diagram by an angle $\varphi_o = \omega t$:



If we write out/expand:

$$\tilde{R}_{out}(t) = R_{out}^o(\omega)\cos(\omega t + \varphi(\omega)) + iR_{out}^o(\omega)\sin(\omega t + \varphi(\omega)) = R_{out}^o(\omega)e^{i(\omega t + \varphi(\omega))}$$

$$= \underbrace{R_{out}^o(\omega)\{\cos\omega t\cos\varphi(\omega) - \sin\omega t\sin\varphi(\omega)\}}_{\equiv \mathrm{Re}\{\tilde{R}_{out}(t)\}} + i\,\underbrace{R_{out}^o(\omega)\{\sin\omega t\cos\varphi(\omega) + \cos\omega t\sin\varphi(\omega)\}}_{\equiv \mathrm{Im}\{\tilde{R}_{out}(t)\}}$$

$$= \mathrm{Re}\{\tilde{R}_{out}(t)\} + i\,\mathrm{Im}\{\tilde{R}_{out}(t)\}$$

We can equivalently write this expression in ***matrix notation*** as follows:

$$\begin{pmatrix} \mathrm{Re}\{\tilde{R}_{out}(t)\} \\ \mathrm{Im}\{\tilde{R}_{out}(t)\} \end{pmatrix} = \begin{pmatrix} R_{out}^o(\omega)\{\cos\omega t\cos\varphi(\omega) - \sin\omega t\sin\varphi(\omega)\} \\ R_{out}^o(\omega)\{\sin\omega t\cos\varphi(\omega) + \cos\omega t\sin\varphi(\omega)\} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} \begin{pmatrix} R_{out}^o(\omega)\cos\varphi(\omega) \\ R_{out}^o(\omega)\sin\varphi(\omega) \end{pmatrix}$$

The above $t = t$ phasor diagram has been rotated CCW by an angle $\varphi_o = \omega t$ relative to the $t = 0$ phasor diagram. We can thus rewrite the above matrix equation as:

$$\begin{pmatrix} \mathrm{Re}\{\tilde{R}_{out}(t)\} \\ \mathrm{Im}\{\tilde{R}_{out}(t)\} \end{pmatrix} = \begin{pmatrix} \cos\varphi_o & -\sin\varphi_o \\ \sin\varphi_o & \cos\varphi_o \end{pmatrix} \begin{pmatrix} R_{out}^o(\omega)\cos\varphi(\omega) \\ R_{out}^o(\omega)\sin\varphi(\omega) \end{pmatrix}$$

The 2×2 matrix $\begin{pmatrix} \cos\varphi_o & -\sin\varphi_o \\ \sin\varphi_o & \cos\varphi_o \end{pmatrix}$ is in fact none other than the 2-D ***rotation matrix***, which takes a 2-D vector $\begin{pmatrix} X \\ Y \end{pmatrix}$ and rotates it {in a CCW direction} by an angle $\varphi_o = \omega t$ in the X-Y plane to $\begin{pmatrix} X' \\ Y' \end{pmatrix}$:

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} \cos\varphi_o & -\sin\varphi_o \\ \sin\varphi_o & \cos\varphi_o \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

precisely as shown in the above $t = t$ phasor diagram!

⇒ See/hear complex sound demo using loudspeaker, *p*/*u* mics + 4 ‘scopes and 2 lock-in amplifiers…

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