# Euler's Equation for Inviscid Fluid Flow

Euler's equation for *inviscid* (*i.e.* *dissipationless*) fluid flow is a special/limiting case of the more general {non-linear} Navier-Stokes equation – which expresses Newton's 2nd law of motion for {compressible} fluid flow. The N-S eq′n, in the absence of external driving forces is:

$$\tilde{\rho}(\vec{r},t)\frac{D\vec{\tilde{u}}(\vec{r},t)}{Dt}=-\vec{\nabla}\tilde{p}(\vec{r},t)+\left(\tfrac{4}{3}\eta+\eta_B\right)\vec{\nabla}\left(\vec{\nabla}\bullet\vec{\tilde{u}}(\vec{r},t)\right)-\eta\left(\vec{\nabla}\times\left(\vec{\nabla}\times\vec{\tilde{u}}(\vec{r},t)\right)\right)$$

The two ***dissipative*** terms on the right-hand side of the Navier-Stokes equation – a non-zero gradient of the divergence of the particle velocity $\vec{\nabla}\left(\vec{\nabla}\bullet\vec{\tilde{u}}(\vec{r},t)\right)$ and the curl of the ***vorticity*** of the particle velocity $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{\tilde{u}}(\vec{r},t)\right)$ are associated with the ***coefficient of shear viscosity*** of the fluid $\eta$, and the ***coefficient of bulk viscosity*** of the fluid $\eta_B$, both of which have SI units of Pascal-seconds (*Pa-s*).

The time derivative term on the left-hand side of the Navier-Stokes equation, $\frac{D\vec{\tilde{u}}(\vec{r},t)}{Dt}$ is the complex particle ***acceleration*** associated with an infinitesimal volume element $V$ of fluid {*e.g.* air} centered on the space-time point $(\vec{r},t)$. From dimensional analysis, note that $\tilde{\rho}(\vec{r},t)\frac{D\vec{\tilde{u}}(\vec{r},t)}{Dt}\left(\frac{kg\text{-}m/s^2}{m^3}=\frac{N}{m^3}\right)$ is a force ***density***. The term $\frac{D}{Dt}\equiv\frac{\partial}{\partial t}+\vec{\tilde{u}}(\vec{r},t)\bullet\vec{\nabla}$ is known as the ***convective*** (or ***substantive*** , *aka* ***material***) derivative, computed from a ***stationary*** observer's reference frame, *e.g.* fixed in the ***laboratory***:

$$\begin{aligned}\frac{D}{Dt}&\equiv\frac{\partial}{\partial t}+\frac{\partial\tilde{x}(\vec{r},t)}{\partial t}\frac{\partial}{\partial x}+\frac{\partial\tilde{y}(\vec{r},t)}{\partial t}\frac{\partial}{\partial y}+\frac{\partial\tilde{z}(\vec{r},t)}{\partial t}\frac{\partial}{\partial z}\\&=\frac{\partial}{\partial t}+\tilde{u}_x(\vec{r},t)\frac{\partial}{\partial x}+\tilde{u}_y(\vec{r},t)\frac{\partial}{\partial y}+\tilde{u}_z(\vec{r},t)\frac{\partial}{\partial z}=\frac{\partial}{\partial t}+\left(\vec{\tilde{u}}(\vec{r},t)\bullet\vec{\nabla}\right)\end{aligned}$$

Euler's equation for inviscid fluid flow is a first-order, linear, homogeneous differential equation, arising from consideration of momentum conservation in a ***lossless/dissipationless*** compressible fluid (liquid or gas), that in the absence of external driving forces describes the relationship between complex pressure $\tilde{p}(\vec{r},t)$ and complex particle velocity $\vec{\tilde{u}}(\vec{r},t)$ in the compressible fluid, of volume mass density $\tilde{\rho}(\vec{r},t)$ $\left(kg/m^3\right)$. Euler's equation for inviscid fluid flow is thus valid for fluids where the ***viscosity*** of the fluid and/or the ***conduction*** of ***heat*** in the fluid are ***both*** zero {or can both be ***approximated*** as being ***negligible***}:

$$\tilde{\rho}(\vec{r},t)\frac{D\vec{\tilde{u}}(\vec{r},t)}{Dt}=\tilde{\rho}(\vec{r},t)\left(\frac{\partial\vec{\tilde{u}}(\vec{r},t)}{\partial t}+\left(\vec{\tilde{u}}(\vec{r},t)\bullet\vec{\nabla}\right)\vec{\tilde{u}}(\vec{r},t)\right)=-\vec{\nabla}\tilde{p}(\vec{r},t)$$


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Inviscid fluid flow in a compressible liquid or gas occurs whenever the magnitude of ***inertial*** forces $\vec{F}_{inertial}(\vec{r},t)$ acting on an infinitesimal volume element $V$ of the fluid centered on the point $\vec{r}$ in the fluid are ***large*** in comparison to the ***dissipative*** forces $\vec{F}_{viscous}(\vec{r},t)$ acting on that fluid, *e.g.* a fluid with high Reynolds number: $R_e = \left|\vec{F}_{inertial}(\vec{r},t)\right| / \left|\vec{F}_{viscous}(\vec{r},t)\right| \gg 1$. "Free" air, well away from any bounding/confining ***surfaces*** is one such example of an inviscid fluid.

In analogy with electric charge conservation, the ***mass continuity equation*** for fluid flow describes ***conservation*** of ***mass*** at every space-time point $(\vec{r},t)$ within the volume $V$ of the fluid:

$$\boxed{\frac{\partial \tilde{\rho}(\vec{r},t)}{\partial t} + \vec{\nabla}\bullet\left(\tilde{\rho}(\vec{r},t)\tilde{\vec{u}}(\vec{r},t)\right) = 0} \quad \text{or:} \quad \boxed{\frac{\partial \tilde{\rho}(\vec{r},t)}{\partial t} + \vec{\nabla}\bullet\tilde{\vec{J}}_a(\vec{r},t) = 0}$$

where: $\tilde{\vec{J}}_a(\vec{r},t) \equiv \tilde{\rho}(\vec{r},t)\tilde{\vec{u}}(\vec{r},t)\ \left(kg/m^2\text{-}s\right)$ is the 3-D vector acoustic mass current density.

For "everyday" complex sound fields $\tilde{S}(\vec{r},t)$ in air (at NTP) that we are considering in this course (in the audio frequency range: $20\ Hz \le f \le 20\ KHz$), typical sound pressure levels are:

$$SPL(\vec{r},t) = L_p(\vec{r},t) = 20\log_{10}\left(\left|\tilde{p}(\vec{r},t)\right| / p_o\right) \ll 134\ dB.$$

The ***reference*** sound over-pressure amplitude is $p_o \equiv 2\times10^{-5}\ RMS\ Pascals\ \left(= RMS\ N/m^2\right)$ in {bone-dry} air at NTP, and we have shown in a previous P406POM lecture note that a sound over-pressure amplitude of $|\tilde{p}| = 1.0\ RMS\ Pascals$ corresponds to a sound pressure level of $SPL = L_p = 20\log_{10}\left(|\tilde{p}|/p_o\right) = 94\ dB \ll 134\ dB$ in {bone-dry} air at NTP. Note that a sound over-pressure amplitude of $|\tilde{p}| = 1.0\ RMS\ Pascals$ is $\ll$ than the ambient atmospheric pressure $P_{atm} = 1.013\times10^5\ Pascals$ at NTP, or: $|\tilde{p}|/P_{atm} \simeq 10^{-5}$. A sound over-pressure *amplitude* that is as large as the atmospheric pressure itself, $\left|\tilde{p}(\vec{r},t)\right| = P_{atm} = 1.013\times10^5\ RMS\ Pascals$ corresponds to an almost unimaginable sound pressure level of $SPL = L_p = 20\log_{10}\left(p_{atm}/p_o\right) = 194\ dB$ ! {Note that an over-pressure amplitude of $\left|\tilde{p}_{pain}(\vec{r},t)\right| = 20\ RMS\ Pascals$ corresponds to a sound pressure level of $SPL = L_p = 20\log_{10}\left(|\tilde{p}_{pain}|/p_o\right) = 120\ dB$, which is the threshold for pain... }

***Non-linear*** effects in air become increasingly noticeable at over-pressure amplitudes greater than $\left|\tilde{p}_{nl}(\vec{r},t)\right| \simeq 100\ RMS\ Pascals \ll P_{atm} = 1.013\times10^5\ Pascals$, which corresponds to a sound pressure level of $SPL = L_p = 20\log_{10}\left(|\tilde{p}_{nl}|/p_o\right) \simeq 134\ dB$ (See graph below).


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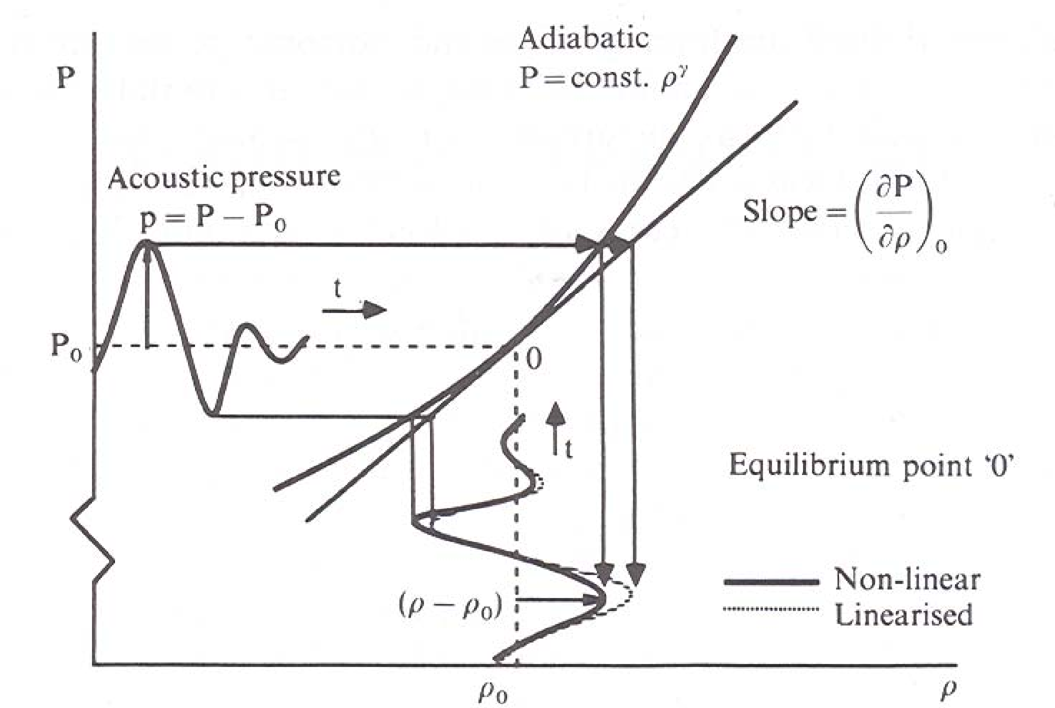
The ***non-linear*** response in air for ***large*** pressure variations $\left(SPL's \gtrsim 134\,dB\right)$ arises from the ***non-linear*** relation between the pressure and the density of air. For ***adiabatic*** changes in air pressure (relevant for sound propagation in air for audio frequency sounds {*i.e.* $f < 20\ KHz$}): $P\left(\vec{r},t\right) = P_{atm} + p\left(\vec{r},t\right) = constant \times \rho^{\gamma}\left(\vec{r},t\right)$ {where for air, $\gamma \equiv C_P/C_V \simeq 7/5 = 1.4$}. The relation between {absolute} pressure $P\left(\vec{r},t\right)$ and volume mass density $\rho^{\gamma}\left(\vec{r},t\right)$ of air is shown in the figure below, where equilibrium (*i.e.* no sound is present) $P_{atm} \equiv P_o$ and $\rho_{atm} \equiv \rho_o$:




We can express the instantaneous absolute pressure $P\left(\vec{r},t\right)$ as a Taylor series expansion about the equilibrium pressure $P_{atm} \equiv P_o$ and mass density $\rho_{atm} \equiv \rho_o$ configuration:

$$\begin{aligned} P\left(\vec{r},t\right) &= P_o + \left.\frac{\partial P\left(\vec{r},t\right)}{\partial \rho\left(\vec{r},t\right)}\right|_{\rho=\rho_o} \left(\rho\left(\vec{r},t\right) - \rho_o\right) + \frac{1}{2}\left.\frac{\partial^2 P\left(\vec{r},t\right)}{\partial \rho^2\left(\vec{r},t\right)}\right|_{\rho=\rho_o} \left(\rho\left(\vec{r},t\right) - \rho_o\right)^2 + \ \ldots \\ &= P_o + \left.\frac{\partial P\left(\vec{r},t\right)}{\partial \rho\left(\vec{r},t\right)}\right|_{\rho=\rho_o} \delta\rho\left(\vec{r},t\right) \quad + \quad \frac{1}{2}\left.\frac{\partial^2 P\left(\vec{r},t\right)}{\partial \rho^2\left(\vec{r},t\right)}\right|_{\rho=\rho_o} \left(\delta\rho\left(\vec{r},t\right)\right)^2 \quad + \ \ldots \end{aligned}$$

For ***small*** pressure variations $\left(\left|\tilde{p}\left(\vec{r},t\right)\right| \ll P_{atm}\right)$ to ***first*** order, a ***linear*** relationship exists between over-pressure $p\left(\vec{r},t\right)$ and the volume mass density $\rho\left(\vec{r},t\right)$ for air:

$$p\left(\vec{r},t\right) = P\left(\vec{r},t\right) - P_o = \delta P\left(\vec{r},t\right) \simeq \left.\frac{\partial P\left(\vec{r},t\right)}{\partial \rho\left(\vec{r},t\right)}\right|_{\rho=\rho_o} \delta\rho\left(\vec{r},t\right)$$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A mathematical statement associated with the conservation of mass within an infinitesimal volume element $V$ of air of equilibrium volume $V_o$ is given by: $\rho V = \rho_o V_o = constant$.
Thus, the ***volumetric strain*** (relevant for sound propagation in air) is: $\delta V/V = -\delta\rho/\rho$
or: $\delta\rho|_{\rho=\rho_o} = -\rho_o(\delta V/V)$, hence to ***first*** order the over-pressure:

$$p = \delta P = P - P_o \simeq \left.\frac{\partial P}{\partial \rho}\right|_{\rho=\rho_o} \delta\rho = -\rho_o \left.\frac{\partial P}{\partial \rho}\right|_{\rho=\rho_o} \frac{\delta V}{V} = -B\frac{\delta V}{V}$$

where $B = \rho_o\, \partial P/\partial\rho|_{\rho=\rho_o}$ is the ***adiabatic bulk modulus*** of air {*e.g.* @ NTP}.

However, for ***adiabatic*** changes, the absolute air pressure $P = constant \times \rho^{\gamma}$ and thus:
$B = \rho_o\, \partial P/\partial\rho|_{\rho=\rho_o} = \gamma P_o$, hence:

$$p = \delta P = \left.\frac{\partial P}{\partial \rho}\right|_{\rho=\rho_o} \delta\rho = -\rho_o \left.\frac{\partial P}{\partial \rho}\right|_{\rho=\rho_o} \frac{\delta V}{V} = -B\frac{\delta V}{V} = +B\frac{\delta\rho}{\rho} = \gamma P_o\left(\frac{\rho-\rho_o}{\rho_o}\right) = \gamma P_o \cdot s$$

The fractional change in volume mass density is known as the ***condensation***: $s \equiv \frac{\delta\rho}{\rho} \simeq \frac{(\rho-\rho_o)}{\rho_o}$

Thus, for "everyday" audio sound over-pressure amplitudes $|\tilde{p}(\vec{r},t)| \ll 100\ RMS\ Pascals$ { $SPL \ll 134\ dB$ }, the response of air as a medium for sound propagation is very nearly ***linear***.

This in turn implies that for "everyday" sound over-pressure amplitudes, the volume mass density of air at NTP is nearly ***constant***, *i.e.* $|\tilde{\rho}(\vec{r},t)| \simeq \rho_o = 1.204\, kg/m^3$ {*i.e.* $|\tilde{s}(\vec{r},t)| \simeq 0$ }.
However, for "everyday" audio sound over-pressure amplitudes, with ***small*** pressure variations $(|\tilde{p}(\vec{r},t)| \ll P_o)$, since: $\tilde{\rho}(\vec{r},t) = \rho_o + \tilde{\rho}_a(\vec{r},t)$, thus: $\tilde{\rho}_a(\vec{r},t) = \delta\tilde{\rho}(\vec{r},t) = \tilde{\rho}(\vec{r},t) - \rho_o$ $(|\rho_a(\vec{r},t)| \ll \rho_o)$ is the {incremental} volume mass density "amplitude" associated with the presence of the acoustic sound field, the time derivatives $\partial\tilde{\rho}(\vec{r},t)/\partial t = \partial\tilde{\rho}_a(\vec{r},t)/\partial t \neq 0$ and $\partial\tilde{s}(\vec{r},t)/\partial t \neq 0$.

However, for $|\tilde{\rho}_a(\vec{r},t)| \ll \rho_o$, the ***non-linear*** $\vec{\nabla}\bullet(\tilde{\rho}(\vec{r},t)\vec{\tilde{u}}(\vec{r},t))$ term in the ***mass continuity equation*** can be ***linearized***:

$$\vec{\nabla}\bullet(\tilde{\rho}(\vec{r},t)\vec{\tilde{u}}(\vec{r},t)) = \vec{\nabla}\bullet(\{\rho_o + \tilde{\rho}_a(\vec{r},t)\}\vec{\tilde{u}}(\vec{r},t))$$
$$= \rho_o\vec{\nabla}\bullet\vec{\tilde{u}}(\vec{r},t) + \underbrace{\vec{\nabla}\bullet(\tilde{\rho}_a(\vec{r},t)\vec{\tilde{u}}(\vec{r},t))}_{neglect} \simeq \rho_o\vec{\nabla}\bullet\vec{\tilde{u}}(\vec{r},t)$$


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Hence, for "everyday" audio sound fields, the ***linearized*** ***mass*** ***continuity*** ***equation*** is:

$$\frac{\partial\tilde{\rho}(\vec{r},t)}{\partial t}+\rho_o\vec{\nabla}\bullet\tilde{\vec{u}}(\vec{r},t)\simeq 0$$

Note also that for "everyday" audio sound fields, the ***linearized*** complex ***acoustic*** ***mass*** ***current*** ***density*** is: $\tilde{\vec{J}}_a(\vec{r},t)\simeq\rho_o\,\tilde{\vec{u}}(\vec{r},t)\;\;(kg/m^2\text{-}s)$.

Likewise, for "everyday" audio sound fields, the ***non***-***linear*** Euler equation can likewise be ***linearized***. For $|\tilde{\rho}_a(\vec{r},t)|\ll\rho_o$, with $\tilde{\rho}(\vec{r},t)=\rho_o+\tilde{\rho}_a(\vec{r},t)$ we first make the approximation:

$$\tilde{\rho}(\vec{r},t)\frac{D\tilde{\vec{u}}(\vec{r},t)}{Dt}\;\Rightarrow\;\rho_o\frac{D\tilde{\vec{u}}(\vec{r},t)}{Dt}=\rho_o\left(\frac{\partial\tilde{\vec{u}}(\vec{r},t)}{\partial t}+\left(\tilde{\vec{u}}(\vec{r},t)\bullet\vec{\nabla}\right)\tilde{\vec{u}}(\vec{r},t)\right)$$

A second approximation that we now make for "everyday" audio sound fields is that it can be shown that the magnitude of the ***non***-***linear*** term $\left(\tilde{\vec{u}}(\vec{r},t)\bullet\vec{\nabla}\right)\tilde{\vec{u}}(\vec{r},t)$ is very small in comparison to the magnitude of the $\partial\tilde{\vec{u}}(\vec{r},t)/\partial t$ term, and hence can be neglected. Thus, the ***linearized*** version of Euler's equation, valid for $SPL\ll 134\,dB$ (over-pressure amplitudes $|\tilde{p}(\vec{r},t)|\ll 100\;RMS\;Pascals$) becomes:

$$\rho_o\frac{\partial\tilde{\vec{u}}(\vec{r},t)}{\partial t}\simeq-\vec{\nabla}\tilde{p}(\vec{r},t)\quad\text{or:}\quad\frac{\partial\tilde{\vec{u}}(\vec{r},t)}{\partial t}\simeq-\frac{1}{\rho_o}\vec{\nabla}\tilde{p}(\vec{r},t)$$

The **Helmholtz** **Theorem** tells us that the vectorial nature of an ***arbitrary*** vector field $\tilde{\vec{F}}(\vec{r})$ is ***fully***-***specified***/***unique*** if $a.)$ $\lim_{r\to\infty}\tilde{\vec{F}}(\vec{r})\to 0$ and $b.)$ the ***divergence*** **.and.** the ***curl*** of $\tilde{\vec{F}}(\vec{r})$ are ***both*** known, *i.e.* $\vec{\nabla}\bullet\tilde{\vec{F}}(\vec{r})=\tilde{C}(\vec{r})$ and $\vec{\nabla}\times\tilde{\vec{F}}(\vec{r})=\tilde{\vec{D}}(\vec{r})$, with the restriction that $\vec{\nabla}\bullet\left(\vec{\nabla}\times\tilde{\vec{F}}(\vec{r})\right)=\vec{\nabla}\bullet\tilde{\vec{D}}(\vec{r})\equiv 0$, since the divergence of the curl of ***any*** vector field is ***always*** zero.

For the 3-D particle velocity $\tilde{\vec{u}}(\vec{r},t)$ associated with sound waves propagating in an inviscid fluid such as air, for "everyday" over-pressure amplitudes of $|\tilde{p}(\vec{r},t)|\ll 100\;RMS\;Pascals$, we showed above that the ***linearized*** mass continuity equation (expressing conservation of mass), tells us that the spatial ***divergence*** of the 3-D particle velocity field is equal to the negative of the normalized (*aka* fractional) time rate of change of the volume mass density:

$$\vec{\nabla}\bullet\tilde{\vec{u}}(\vec{r},t)\simeq-\frac{1}{\rho_o}\frac{\partial\tilde{\rho}(\vec{r},t)}{\partial t}$$


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What is the ***curl*** of the 3-D particle velocity field, $\vec{\nabla}\times\vec{\tilde{u}}(\vec{r},t)=???$ Physically, the ***curl*** of a ***velocity*** field is often associated *e.g.* with a ***rotation*** and/or a velocity ***shear*** – such as the velocity field $\vec{v}(\vec{r},t)$ associated with a whirlpool, or a vortex in water. For this reason, the ***curl*** of a velocity field $\nabla\times\vec{v}(\vec{r},t)$ is sometimes known as/called the ***vorticity***.

However, in an ***inviscid*** fluid (*i.e.* one which is ***dissipationless***/has ***zero*** viscosity) such as air, the ***vorticity*** $\nabla\times\vec{v}(\vec{r},t)\equiv 0$, because an ***inviscid*** fluid ***cannot*** support velocity ***shears*** and/or ***vortices*** in the ***inviscid*** fluid. We can explicitly show/prove that $\vec{\nabla}\times\vec{\tilde{u}}(\vec{r},t)=0$ for "everyday" audio sound over-pressure amplitudes in air at NTP of $\left|\tilde{p}(\vec{r},t)\right|\ll 100\ RMS\ Pascals$. First, we take the partial derivative of $\vec{\nabla}\times\vec{\tilde{u}}(\vec{r},t)$ with respect to time:

$$\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{\tilde{u}}(\vec{r},t)\right)=\vec{\nabla}\times\frac{\partial\vec{\tilde{u}}(\vec{r},t)}{\partial t}$$

However, the Euler equation for inviscid fluid flow is: $\dfrac{\partial\vec{\tilde{u}}(\vec{r},t)}{\partial t}=-\dfrac{1}{\rho_o}\vec{\nabla}\tilde{p}(\vec{r},t)$, thus:

$$\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{\tilde{u}}(\vec{r},t)\right)=\vec{\nabla}\times\frac{\partial\vec{\tilde{u}}(\vec{r},t)}{\partial t}=-\frac{1}{\rho_o}\left(\vec{\nabla}\times\vec{\nabla}\tilde{p}(\vec{r},t)\right)$$

However, the ***curl*** of the ***gradient*** of any ***arbitrary*** scalar field $f(\vec{r},t)$ is also ***always*** zero, *i.e.* $\vec{\nabla}\times\vec{\nabla}f(\vec{r},t)\equiv 0$, thus:

$$\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{\tilde{u}}(\vec{r},t)\right)=\vec{\nabla}\times\frac{\partial\vec{\tilde{u}}(\vec{r},t)}{\partial t}=-\frac{1}{\rho_o}\left(\vec{\nabla}\times\vec{\nabla}\tilde{p}(\vec{r},t)\right)\equiv 0$$

This tells us that: $\vec{\nabla}\times\vec{\tilde{u}}(\vec{r},t)=constant\neq fcn(t)$. Thus, if for any time $-\infty\leq t\leq+\infty$, there is ***no*** vorticity in the inviscid fluid ($\vec{\nabla}\times\vec{\tilde{u}}(\vec{r},t)=0$), then it must ***remain*** = 0 for ***all*** time. *Q.E.D.*

If we take the time derivative of both sides of the {linearized} mass continuity equation, and the divergence of both sides of the {linearized} Euler equation:

$$\vec{\nabla}\bullet\frac{\partial\vec{\tilde{u}}(\vec{r},t)}{\partial t}\simeq-\frac{1}{\rho_o}\frac{\partial^2\tilde{\rho}(\vec{r},t)}{\partial t^2}=-\frac{1}{\rho_o c^2}\frac{\partial^2\tilde{p}(\vec{r},t)}{\partial t^2}\ \text{ and: }\ \vec{\nabla}\bullet\frac{\partial\vec{\tilde{u}}(\vec{r},t)}{\partial t}\simeq-\frac{1}{\rho_o}\vec{\nabla}\bullet\vec{\nabla}\tilde{p}(\vec{r},t)=-\frac{1}{\rho_o}\nabla^2\tilde{p}(\vec{r},t)$$

and then using the {linearized} adiabatic relationship between complex overpressure, $\tilde{p}$ and mass density, $\tilde{\rho}(\vec{r},t)=\frac{1}{c^2}\tilde{p}(\vec{r},t)$, we also have the relation: $\partial\tilde{\rho}(\vec{r},t)/\partial t\simeq\frac{1}{c^2}\partial\tilde{p}(\vec{r},t)/\partial t$. Hence, we obtain the {linearized} wave equation for complex overpressure:

$$\boxed{\nabla^2\tilde{p}(\vec{r},t)-\frac{1}{c^2}\frac{\partial^2\tilde{p}(\vec{r},t)}{\partial t^2}=0}$$


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If we now take the spatial gradient of both sides of the linearized mass continuity equation, and the time derivative of both sides of the linearized Euler equation, and again use the {linearized} adiabatic relationship between complex overpressure, $\tilde{p}$ and mass density, $\tilde{\rho}(\vec{r},t)=\frac{1}{c^2}\tilde{p}(\vec{r},t)$, we also have the relation: $\nabla\tilde{\rho}(\vec{r},t)=\frac{1}{c^2}\nabla\tilde{p}(\vec{r},t)$, then:

$$\nabla\left(\vec{\nabla}\bullet\vec{\tilde{u}}(\vec{r},t)\right)\simeq-\frac{1}{\rho_o}\frac{\partial\nabla\tilde{\rho}(\vec{r},t)}{\partial t}=-\frac{1}{\rho_o c^2}\frac{\partial\nabla\tilde{p}(\vec{r},t)}{\partial t}\quad\text{and:}\quad\frac{\partial^2\vec{\tilde{u}}(\vec{r},t)}{\partial t^2}\simeq-\frac{1}{\rho_o}\frac{\partial\vec{\nabla}\tilde{p}(\vec{r},t)}{\partial t}$$

Combining these two equations, we obtain:

$$\vec{\nabla}\left(\vec{\nabla}\bullet\vec{\tilde{u}}(\vec{r},t)\right)=-\frac{1}{\rho_o c^2}\frac{\partial\vec{\nabla}\tilde{p}(\vec{r},t)}{\partial t}=\frac{1}{c^2}\frac{\partial^2\vec{\tilde{u}}(\vec{r},t)}{\partial t^2}$$

If the complex vector acoustic particle velocity field is ***irrotational*** (*i.e.* $\vec{\nabla}\times\vec{\tilde{u}}(\vec{r},t)=0$), then using the vector relation $\nabla^2\vec{\tilde{u}}=\vec{\nabla}\left(\vec{\nabla}\bullet\vec{\tilde{u}}\right)-\vec{\nabla}\times\left(\cancel{\vec{\nabla}\times\vec{\tilde{u}}}\right)=\vec{\nabla}\left(\vec{\nabla}\bullet\vec{\tilde{u}}\right)$, we also obtain the {linearized} wave equation for complex vector particle velocity:

$$\boxed{\nabla^2\vec{\tilde{u}}(\vec{r},t)-\frac{1}{c^2}\frac{\partial^2\vec{\tilde{u}}(\vec{r},t)}{\partial t^2}=0}$$

## The Complex Particle Velocity Potential, $\tilde{\Phi}_u(\vec{r},t)$

Since an inviscid (*i.e.* dissipationless) fluid does not support vorticity, *i.e.* $\vec{\nabla}\times\vec{\tilde{u}}(\vec{r},t)=0$ then since the ***curl*** of the ***gradient*** of any ***arbitrary*** scalar field $f(\vec{r},t)$ is also ***always*** zero, *i.e.* $\vec{\nabla}\times\vec{\nabla}f(\vec{r},t)\equiv 0$, we can write $\vec{\tilde{u}}(\vec{r},t)=\vec{\nabla}\tilde{\Phi}_u(\vec{r},t)$, where $\tilde{\Phi}_u(\vec{r},t)$ is the ***complex particle velocity potential*** associated with $\vec{\tilde{u}}(\vec{r},t)$. Then $\vec{\nabla}\times\vec{\nabla}\tilde{\Phi}_u(\vec{r},t)=0$.

Note that since $\vec{\tilde{u}}(\vec{r},t)$ and the gradient operator $\vec{\nabla}\equiv\partial/\partial x\,\hat{x}+\partial/\partial y\,\hat{y}+\partial/\partial z\,\hat{z}$ {in Cartesian coordinates} have SI units of $m/s$ and $m^{-1}$ respectively, the complex velocity potential $\tilde{\Phi}_u(\vec{r},t)$ has SI units of $m^2/s$. Physically, note also that lines/contours {and/or 3-D surfaces} of constant $\tilde{\Phi}_u(\vec{r},t)=\tilde{K}=k+i\kappa=constant$ are thus {complex!} "***equipotentials***", which are {everywhere} ***perpendicular*** to the complex particle velocity $\vec{\tilde{u}}(\vec{r},t)$.

Note additionally that $\tilde{\Phi}_u(\vec{r},t)$ with $\vec{\nabla}\times\vec{\tilde{u}}(\vec{r},t)=\vec{\nabla}\times\vec{\nabla}\tilde{\Phi}_u(\vec{r},t)=0$ is the acoustic analog of the electrostatic potential $\Phi_e(\vec{r})$ associated with the electrostatic field $\vec{E}(\vec{r})\equiv-\vec{\nabla}\tilde{\Phi}_e(\vec{r})$, since in ***electrostatics*** $\vec{\nabla}\times\vec{E}(\vec{r})\equiv-\vec{\nabla}\times\vec{\nabla}\tilde{\Phi}_e(\vec{r})=0$ {whereas in ***electrodynamics***, $\vec{\nabla}\times\vec{E}(\vec{r},t)\equiv-\partial\vec{B}(\vec{r},t)/\partial t\neq 0$ }.


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Exploiting the analog of the concept of electrical "voltage" – *i.e*. a difference in electrical potential $\Delta\Phi_e^{b\text{-}a} \equiv \Phi_e^b - \Phi_e^a = \int_a^b \vec{\nabla}\Phi_e(\vec{r})\cdot d\vec{\ell} = -\int_a^b \vec{E}(\vec{r})\cdot d\vec{\ell}$ we can also define a complex particle velocity potential ***difference*** (*aka* particle velocity "voltage") as:

$$\Delta\tilde{\Phi}_u^{b\text{-}a}(t) \equiv \tilde{\Phi}_u^b(t) - \Phi_u^a(t) = \int_a^b \vec{\nabla}\tilde{\Phi}_u(\vec{r},t)\cdot d\vec{\ell} = -\int_a^b \vec{\tilde{u}}(\vec{r},t)\cdot d\vec{\ell}$$

From the mass continuity equation: $\vec{\nabla}\bullet\vec{\tilde{u}}(\vec{r},t) = -\tfrac{1}{\rho_o}\left(\partial\tilde{\rho}(\vec{r},t)/\partial t\right)$ and: $\vec{\tilde{u}}(\vec{r},t) = \vec{\nabla}\tilde{\Phi}_u(\vec{r},t)$, then for "everyday" audio sound over-pressure amplitudes in {bone-dry} air at NTP of $\left|\tilde{p}(\vec{r},t)\right| \ll 100\ RMS\ Pascals$ $\{SPL \ll 134\,dB\}$, then: $\vec{\nabla}\bullet\vec{\nabla}\tilde{\Phi}_u(\vec{r},t) = -\tfrac{1}{\rho_o}\left(\partial\tilde{\rho}(\vec{r},t)/\partial t\right)$, which can be written as $\nabla^2\tilde{\Phi}_u(\vec{r},t) = -\tfrac{1}{\rho_o}\left(\partial\tilde{\rho}(\vec{r},t)/\partial t\right)$; this is Poisson's equation for the complex particle velocity potential!

Thus, we can thus solve {certain classes of} acoustical physics problems simply by solving Poisson's equation $\nabla^2\tilde{\Phi}_u(\vec{r},t) = -\tfrac{1}{\rho_o}\left(\partial\tilde{\rho}(\vec{r},t)/\partial t\right)$ for the complex particle velocity potential $\tilde{\Phi}_u(\vec{r},t)$, subject to the boundary condition(s) {and/or initial conditions at $t = 0$} associated with the specific problem using techniques/methodology similar to that used for solving Poisson's equation $\nabla^2\tilde{\Phi}_e(\vec{r}) \neq 0$ in E&M problems!

Note that {again} using the {linearized} adiabatic relationship between complex overpressure and mass density, $\tilde{\rho}(\vec{r},t) = \tfrac{1}{c^2}\tilde{p}(\vec{r},t)$ we also have: $\partial\tilde{\rho}(\vec{r},t)/\partial t \simeq \tfrac{1}{c^2}\partial\tilde{p}(\vec{r},t)/\partial t$. Hence for "everyday" audio sound fields, the above differential equation for the complex velocity potential can equivalently be written as: $\nabla^2\tilde{\Phi}_u(\vec{r},t) = -\tfrac{1}{\rho_o c^2}\left(\partial\tilde{p}(\vec{r},t)/\partial t\right)$.

If $\vec{\tilde{u}}(\vec{r},t) = \vec{\nabla}\tilde{\Phi}_u(\vec{r},t)$, the {linearized} Euler equation can be written as: $\dfrac{\partial\vec{\nabla}\tilde{\Phi}_u(\vec{r},t)}{\partial t} = \vec{\nabla}\dfrac{\partial\tilde{\Phi}_u(\vec{r},t)}{\partial t} \simeq -\dfrac{1}{\rho_o}\vec{\nabla}\tilde{p}(\vec{r},t)$, which implies that: $\dfrac{\partial\tilde{\Phi}_u(\vec{r},t)}{\partial t} \simeq -\dfrac{1}{\rho_o}\tilde{p}(\vec{r},t)$, and hence that: $\dfrac{\partial^2\tilde{\Phi}_u(\vec{r},t)}{\partial t^2} \simeq -\dfrac{1}{\rho_o}\dfrac{\partial\tilde{p}(\vec{r},t)}{\partial t}$. From above, we also have: $\dfrac{\partial\tilde{p}(\vec{r},t)}{\partial t} \simeq c^2\dfrac{\partial\tilde{\rho}(\vec{r},t)}{\partial t}$, thus: $\dfrac{\partial^2\tilde{\Phi}_u(\vec{r},t)}{\partial t^2} \simeq -\dfrac{c^2}{\rho_o}\dfrac{\partial\tilde{\rho}(\vec{r},t)}{\partial t}$, but from the above Poisson equation: $\nabla^2\tilde{\Phi}_u(\vec{r},t) = -\dfrac{1}{\rho_o}\dfrac{\partial\tilde{\rho}(\vec{r},t)}{\partial t}$, thus, we obtain the wave equation for the complex velocity potential:

$$\boxed{\nabla^2\tilde{\Phi}_u(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2\tilde{\Phi}_u(\vec{r},t)}{\partial t^2} = 0}$$


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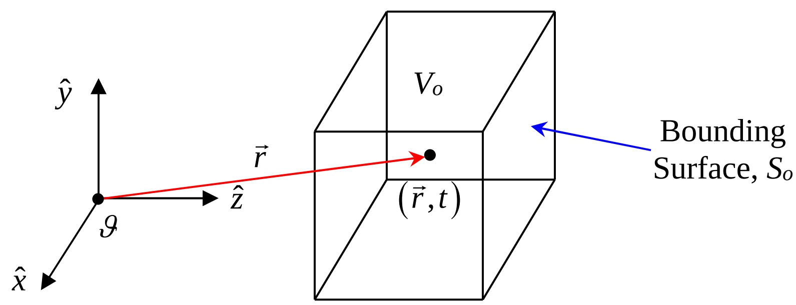
**Derivation of Euler's Equation for Inviscid Fluid Flow from Newton's Second Law of Motion:**

We can derive Euler's equation for inviscid fluid flow using Newton's 2nd law of motion $\left(\vec{F}_{net} = m\vec{a}\right)$ and at the same time gain some useful insight into the physical meaning of particle velocity, $\vec{u}\left(\vec{r},t\right)$.

Consider an infinitesimal volume element $V_o = 1\left(\mu m\right)^3$ bounded by the surface $S_o$ centered on the space-time point $\left(\vec{r},t\right)$ {= center of mass of the infinitesimal volume element $V_o$} containing {bone-dry} air at NTP, in thermal equilibrium with the air surrounding it, and with equilibrium volume mass density $\rho_o = 1.204\,kg/m^3$, as shown in the figure below:




Rather than work in the fixed laboratory reference frame, we deliberately choose to work in a reference frame that is co-moving with the infinitesimal volume element $V_o$ of air. Note that the pressure $p\left(\vec{r},t\right)$ associated with the infinitesimal volume element $V_o$ as measured in the co-moving reference frame of the infinitesimal volume element $V_o$ is the same pressure as measured in the fixed laboratory frame, this is because pressure $p\left(\vec{r},t\right)$ is intrinsically a scalar quantity.

The air {at NPT} contained within the infinitesimal volume element $V_o$ is at a static / equilibrium absolute pressure of one atmosphere, *i.e.* $p_{atm} = 1.013\times10^5\,Pascals$ and a finite temperature $T = 20^o\ C$ (= 293.15 $K$). At the microscopic level, the air molecules within the infinitesimal volume element $V_o$ each have mean thermal energy $\left\langle E_{mol}^{th}\right\rangle = \frac{3}{2}k_B T$ where $k_B = 1.381\times10^{-23}\ Joules/Kelvin$ and collide randomly with each other, undergoing Brownian random-walk type motions.

Suppose that a sound wave with over-pressure amplitude $\left|p\left(\vec{r},t\right)\right| \ll 100\ RMS\ Pascals$ { $SPL \ll 134\,dB$ } is incident on the {initially static} air contained within the infinitesimal volume element $V_o$. When the over-pressure amplitude $p\left(\vec{r},t\right)$ is instantaneously greater (less) than the ambient pressure $p_{atm}$, the air contained within $V_o$ momentarily compresses (expands), respectively. Note that conceptually, the surface $S_o$ that bounds the infinitesimal volume element $V_o$ is endowed with "magical" properties, in that it is a fictitious, Gaussian-type surface (*e.g.* as commonly used in *E&M* problems), the nature of the bounding surface $S_o$ also is one which expands and/or contracts as the air contained within the infinitesimal volume element $V_o$ expands or contracts. Operationally this means we need only keep track of linear/leading-order terms in various expansions…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Furthermore, if the nature of incident sound wave is such as to cause the air molecules within the infinitesimal volume element $V_o$ to collectively move in a given direction, *i.e*. to be displaced by a collective 3-D distance $\vec{\xi}(\vec{r},t)$ from its equilibrium position, with collective velocity $\vec{u}(\vec{r},t)$ and collective acceleration $\vec{a}(\vec{r},t)$, the "magical" Gaussian surface $S_o$ co-moves with the air contained within $V_o$.

An infinitesimal volume element of size *e.g*. a cubic micron $V_o = 1(\mu m)^3$ is statistically large enough for our purposes. The air contained within this infinitesimal volume element $V_o$ is in thermal equilibrium with itself and with the air surrounding it. Avogadro's number $N_A = 6.022\times 10^{23}\, molecules/mole$ and recall that one mole of {bone-dry} air @ NTP has mean/average molar mass of $m_{mol}^{air} = 28.97\, gms/mole$. Thus, for a volume mass density of air $\rho_o = 1.204\, kg/m^3$ at NTP there are $24.06\ cm^3/mole$, or ~ 25 *billion* molecules of air per cubic micron at NTP. The average/mean velocity vector associated with the mean/average thermal energy $\langle U_{th}(\vec{r},t)\rangle$ of this number of air molecules contained within the infinitesimal volume element $V_o$ is $\langle \vec{u}_{mol}(\vec{r},t)\rangle = 0$, however the thermal energies $\langle E_{mol}^{th}\rangle = \frac{3}{2}k_B T = \frac{1}{2}m|\vec{u}_{mol}|^2$ associated with individual air molecules contained within $V_o$ may be such that individual molecules within $V_o$ leave through the bound surface $S_o$ via exiting through one of the top, bottom or side surfaces associated with $S_o$. However, one of the other "magical" properties endowed with the co-moving surface $S_o$ associated with the air contained within the infinitesimal volume element $V_o$ is that if an air molecule leaves (enters) the bounding surface $S_o$ at a given point $\vec{r}_{mol}$ on one side of the volume element with velocity vector $\vec{u}_{mol}(\vec{r}_{mol},t)$, it instantaneously enters (leaves) the surface $S_o$ again with velocity vector $\vec{u}_{mol}(\vec{r}_{mol}^{\,conj},t)$, but on the other side of the volume element, at its conjugate point $\vec{r}_{mol}^{\,conj}$ relative to the center point $(\vec{r},t)$ of the infinitesimal volume element, $V_o$. Thus the total air mass $m_{air}$, the average/mean linear momentum $\langle \vec{P}_{air}(\vec{r},t)\rangle$ and the average/mean thermal energy $\langle U_{th}(\vec{r},t)\rangle$ are all conserved by this "magical" property of the fictitious Gaussian surface $S$ bounding the infinitesimal volume element $V_o$.

From Newton's 2nd law of motion, $\vec{F}_{net} = m\vec{a}$, we can calculate the force(s) acting on the air within the infinitesimal volume element $V$ due to an over-pressure amplitude $p(\vec{r},t)$. The mass of air contained within the infinitesimal volume element $V_o$ is $m = \rho_o V_o\ (kg)$. Newton's 2nd law tells us that $\vec{F}_{net}(\vec{r},t) = m\vec{a}(\vec{r},t)$ or that: $\vec{a}(\vec{r},t) = \vec{F}_{net}(\vec{r},t)/m = \vec{F}_{net}(\vec{r},t)/\rho_o V_o$. We define the {net} force per unit volume acting on the infinitesimal volume element as: $\vec{f}_{net}(\vec{r},t) \equiv \vec{F}_{net}(\vec{r},t)/V_o$. Thus the acceleration $\vec{a}(\vec{r},t) = \vec{f}_{net}(\vec{r},t)/\rho_o$.

Next, let us (initially) consider only the *x*-component of the net force due to an over-pressure $p(\vec{r},t)$ acting on the infinitesimal volume element $V_o$ of air, as shown in a side view in the figure below:


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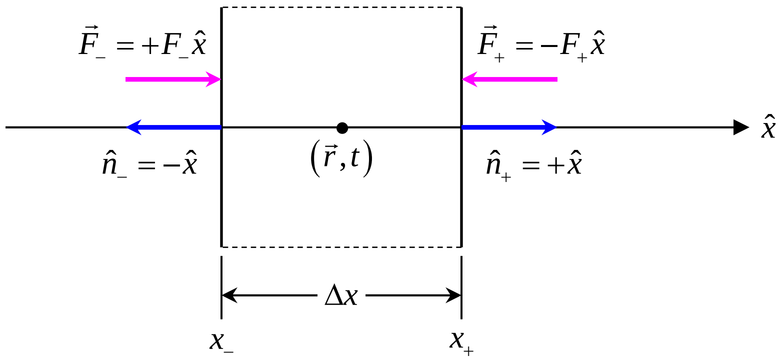


Note that here we must be mindful of the nature of the compressive force(s) due to the {small} over-pressure $p(\vec{r},t)$ acting on the infinitesimal volume element $V_o$ – namely, that thermal equilibrium of the air contained within the volume $V_o$, as well as *all* *other* adjacent / neighboring infinitesimal volume elements of air must be maintained at all times during this process. The restriction that $|p(\vec{r},t)| \ll 100\ RMS\ Pascals$ { $SPL \ll 134\ dB$ } for harmonic/periodic over-pressure amplitudes with frequencies in the audio range of human hearing (20 $Hz < f <$ 20 $KHz$) guarantees that thermal equilibrium holds during this process. From a thermodynamic perspective, this is clearly a reversible, adiabatic, and hence isentropic process.

The infinitesimal vector area elements associated with the $x_-$ (LHS) and $x_+$ (RHS) of the infinitesimal volume element $V_o$ are: $\vec{A}_- = A\hat{n}_- = -A_o\hat{x}\ (m^2)$ and $\vec{A}_+ = A\hat{n}_+ = +A_o\hat{x}\ (m^2)$. Note that the unit normal vectors $\hat{n}_- = -\hat{x}$ and $\hat{n}_+ = +\hat{x}$ associated with these two surfaces, by convention, point outward from/perpendicular to the surface $S_o$.

The $x$-force acting on the LHS surface located at $x_-$ is: $\vec{F}_- = +F_-\hat{x} = -p_-\vec{A}_- = +p_-A_o\hat{x}$.

The $x$-force acting on the RHS surface located at $x_+$ is: $\vec{F}_+ = -F_+\hat{x} = -p_+\vec{A}_+ = -p_+A_o\hat{x}$.

The ***net*** $x$-force acting on the infinitesimal volume element $V$ is: $\vec{F}_{net_x} = \vec{F}_+ + \vec{F}_- = -(p_+ - p_-)A_o\hat{x}$.

The ***net*** $x$-force *per unit volume* acting on the infinitesimal volume element $V_o = A_o \cdot \Delta x$ is:

$$\vec{f}_{net_x} = \frac{\vec{F}_{net_x}}{V_o} = \frac{-\overbrace{(p_+ - p_-)}^{\equiv \Delta p} \cancel{A_o}\hat{x}}{\cancel{A_o} \cdot \Delta x} = -\frac{\Delta p}{\Delta x}\hat{x}$$

In the limit that the volume $V_o$ of the infinitesimal volume element $\rightarrow 0$:

$$\vec{f}_{net_x}(\vec{r},t) = -\frac{\partial p(\vec{r},t)}{\partial x}\hat{x}$$

We can repeat this analysis for the $y$- and $z$-components of the ***net*** force per unit volume due to the overpressure amplitude acting on the infinitesimal volume element $V_o$ of air, the results are similar:

$$\vec{f}_{net_y}(\vec{r},t) = -\frac{\partial p(\vec{r},t)}{\partial y}\hat{y} \quad \text{and:} \quad \vec{f}_{net_z}(\vec{r},t) = -\frac{\partial p(\vec{r},t)}{\partial z}\hat{z}$$


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The total ***net*** 3-D vector force per unit volume is therefore:

$$\vec{f}_{net}(\vec{r},t) = f_{net_x}(\vec{r},t)\hat{x} + f_{net_y}(\vec{r},t)\hat{y} + f_{net_z}(\vec{r},t)\hat{z}$$

$$= -\frac{\partial p(\vec{r},t)}{\partial x}\hat{x} - \frac{\partial p(\vec{r},t)}{\partial y}\hat{y} - \frac{\partial p(\vec{r},t)}{\partial y}\hat{z} = -\underbrace{\left(\frac{\partial}{\partial x}\hat{x} - \frac{\partial}{\partial y}\hat{y} - \frac{\partial}{\partial y}\hat{z}\right)}_{\equiv\vec{\nabla}} p(\vec{r},t) = -\vec{\nabla}p(\vec{r},t)$$

Thus we have: $\vec{a}(\vec{r},t) = \vec{f}(\vec{r},t)/\rho_o$ and: $\vec{f}_{net}(\vec{r},t) = \vec{f}(\vec{r},t) = -\vec{\nabla}p(\vec{r},t)$, hence: $\vec{a}(\vec{r},t) = -\vec{\nabla}p(\vec{r},t)/\rho_o$. Recall that (for $\left|p(\vec{r},t)\right| \ll 100\ RMS\ Pascals$ { $SPL \ll 134\ dB$ }, the particle acceleration $\vec{a}(\vec{r},t)$ is the time rate of change of the particle velocity $\vec{u}(\vec{r},t)$, *i.e.* $\vec{a}(\vec{r},t) = \partial\vec{u}(\vec{r},t)/\partial t$, hence we obtain Euler's equation for inviscid fluid flow, valid for air with $\left|p(\vec{r},t)\right| \ll 100\ RMS\ Pascals$ { $SPL \ll 134\ dB$ }:

$$\vec{a}(\vec{r},t) = \frac{\partial\vec{u}(\vec{r},t)}{\partial t} = -\frac{1}{\rho_o}\vec{\nabla}p(\vec{r},t) \quad Q.E.D.$$

"Complexifying" this equation, we have:

$$\boxed{\vec{\tilde{a}}(\vec{r},t) = \frac{\partial\vec{\tilde{u}}(\vec{r},t)}{\partial t} = -\frac{1}{\rho_o}\vec{\nabla}\tilde{p}(\vec{r},t)}$$

Although this relationship between the complex particle acceleration $\vec{\tilde{a}}(\vec{r},t)$, particle velocity $\vec{\tilde{u}}(\vec{r},t)$ and complex pressure $\tilde{p}(\vec{r},t)$ was derived in the co-moving/center-of-mass reference frame associated with the infinitesimal volume element $V_o$ centered on the space-time point $(\vec{r},t)$, superimposed on top of a <u>static</u> pressure field $p_{atm} = 1.013\times10^5\ Pascals$, it can be seen that for <u>small</u>, <u>harmonic</u>/<u>periodic</u> over-pressure amplitude variations, *e.g.* $\tilde{p}(\vec{r},t) = \tilde{p}_o(\vec{r})e^{i\omega t}$ with $\left|\tilde{p}(\vec{r},t)\right| \ll p_{atm}$ that each of these quantities are the same in the laboratory reference frame.

We can now also see that the complex particle <u>displacement</u> $\vec{\tilde{\xi}}(\vec{r},t)$ $(m)$ {from equilibrium position}, complex particle <u>velocity</u> $\vec{\tilde{u}}(\vec{r},t) = \partial\vec{\tilde{\xi}}(\vec{r},t)/\partial t$ $(m/s)$ and complex particle <u>acceleration</u> $\vec{\tilde{a}}(\vec{r},t) = \partial\vec{\tilde{u}}(\vec{r},t)/\partial t$ $(m/s^2)$ are associated with the <u>***collective***</u>, random-thermal energy-averaged-out motion of the air molecules contained within the infinitesimal volume element $V_o$ bounded by the {co-moving} surface $S_o$ centered on the space-time point $(\vec{r},t)$.


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**<u>Complex Sound Fields</u>** $\tilde{S}(\vec{r},t)$**:**

The acoustical physics properties associated with an <u>arbitrary</u> "everyday" audio complex sound field $\tilde{S}(\vec{r},t)$ can be ***<u>completely</u>*/*<u>uniquely</u>* *<u>determined</u>*** at the space-time point $(\vec{r},t)$ by measuring ***<u>two</u>*** physical quantities associated with the complex sound field:

(*a*.) the complex over-pressure $\tilde{p}(\vec{r},t)$ at the space-time point $(\vec{r},t)$ - a ***<u>scalar</u>*** quantity,

**.and.**

(*b*.) the complex particle velocity $\vec{\tilde{u}}(\vec{r},t)$ at the space-time point $(\vec{r},t)$ - a 3-D ***<u>vector</u>*** quantity with: $\lim_{r\to\infty}\vec{\tilde{u}}(\vec{r})\to 0$, $\vec{\nabla}\bullet\vec{\tilde{u}}(\vec{r},t)\simeq -\frac{1}{\rho_o}\left(\partial\tilde{\rho}(\vec{r},t)/\partial t\right)$ and: $\vec{\nabla}\times\vec{\tilde{u}}(\vec{r},t)=0$ {or = constant}.

**<u>Complex Sound Field Quantities: Working in the *Time-Domain vs*. the *Frequency-Domain*</u>**

It is extremely important whenever working with any/all complex sound field quantities to understand/distinguish as to whether one is working with such quantities in the ***time-domain*** *vs*. working with such quantities in the ***frequency-domain*** – they are ***<u>not</u>*** the same/indentical…

Complex quantities in the ***time-domain*** *vs*. their ***frequency-domain*** counterparts are related by ***Fourier transforms*** of each other – because time *t* (units = seconds) and frequency $f=\omega/2\pi$ (units = 1/sec = *Hz*) are so-called ***Fourier conjugate variables*** of each other. We thus use the notation $\tilde{S}(\vec{r},t)$ *vs*. $\tilde{S}(\vec{r},\omega)$ to indicate a ***time-domain*** complex sound field *vs*. ***frequency-domain*** complex sound field at the space-point $\vec{r}$, respectively.

How do we know whether we are working in the ***time-domain*** *vs*. the ***frequency domain***?

A time-dependent ***instantaneous*** voltage signal $V_{p\text{-}mic}(\vec{r},t)=V_o^{p\text{-}mic}(\omega_o)\cos(\omega_o t+\varphi_p(\vec{r},\omega_o))$, *e.g*. output from a pressure sensitive microphone placed at the point $\vec{r}=(x\hat{x},y\hat{y},z\hat{z})$ in the sound field of a loudspeaker {located at the origin $(0,0,0)$} and driven by a sine-wave function generator (of angular frequency $\omega_o=2\pi f_o$) + power amplifier is a typical example of a ***time-domain*** signal – it is observable *e.g*. on an oscilloscope, which displays the ***instantaneous*** voltage signal $V_{p\text{-}mic}(\vec{r},t)=V_o^{p\text{-}mic}(\vec{r},\omega_o)\cos(\omega_o t+\varphi_p(\vec{r},\omega_o))$ output from the microphone as a function of time, *t*.

We specify, for clarity/definiteness′ sake that the oscilloscope trace of the display of the *p*-mic signal $V_{p\text{-}mic}(\vec{r},t)=V_o^{p\text{-}mic}(\vec{r},\omega_o)\cos(\omega_o t+\varphi_p(\vec{r},\omega_o))$ is triggered ***externally*** by the ***sync signal*** output from the sine-wave generator – which serves as the ***reference*** signal and thus gives physical meaning to the (overall) phase $\varphi_p(\vec{r},\omega_o)$ of the *p*-mic signal, which is defined ***<u>relative</u>*** to the ***time-domain*** sine-wave voltage signal $V_{FG}(t)=V_o^{FG}\cos\omega_o t$ output from the sine-wave generator, since (by industry standard, the positive-going edge of ) the TTL-level ***sync signal*** output from the sine-wave generator is used to ***in-phase*** trigger the start of the oscilloscope trace displaying the microphone signal $V_{p\text{-}mic}(\vec{r},t)=V_o^{p\text{-}mic}(\vec{r},\omega_o)\cos(\omega_o t+\varphi_p(\vec{r},\omega_o))$.


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Note that the ***instantaneous time-domain*** voltage signals $V_{FG}(t) = V_o^{FG}\cos\omega_o t$ and $V_{p\text{-}mic}(\vec{r},t) = V_o^{p\text{-}mic}(\vec{r},\omega_o)\cos(\omega_o t + \varphi_p(\vec{r},\omega_o))$ are ***purely real*** quantities. We can "***complexify***" these ***instantaneous time-domain*** quantities by adding ***quadrature/imaginary*** terms to them:

$$\tilde{V}_{FG}(t) = V_o^{FG}\cos\omega_o t + iV_o^{FG}\sin\omega_o t = V_o^{FG}(\cos\omega_o t + i\sin\omega_o t) = V_o^{FG}e^{i\omega_o t} \quad \text{and:}$$

$$\begin{aligned}\tilde{V}_{p\text{-}mic}(\vec{r},t) &= V_o^{p\text{-}mic}(\vec{r},\omega_o)\cos(\omega_o t + \varphi_p(\vec{r},\omega_o)) + iV_o^{p\text{-}mic}(\vec{r},\omega_o)\sin(\omega_o t + \varphi_p(\vec{r},\omega_o)) \\ &= V_o^{p\text{-}mic}(\vec{r},\omega_o)\left\{\cos(\omega_o t + \varphi_p(\vec{r},\omega_o)) + i\sin(\omega_o t + \varphi_p(\vec{r},\omega_o))\right\} = V_o^{p\text{-}mic}(\vec{r},\omega_o)e^{i(\omega_o t + \varphi_p(\vec{r},\omega_o))}\end{aligned}$$

A {dual-channel} ***lock-in amplifier*** is ***<u>manifestly</u>*** a ***frequency-domain*** device that is routinely used in many types of physics experiments to simultaneously measure the real (*i.e.* in-phase) and imaginary/quadrature (*i.e.* 90° out-of-phase) components of a complex harmonic (*i.e.* single-frequency) signal, ***relative*** to a ***reference*** sine-wave signal of the same angular frequency $\omega_o = 2\pi f_o$.

In the above example, we could *e.g.* additionally simultaneously connect the microphone's ***time-domain*** output signal $V_{p\text{-}mic}(\vec{r},t) = V_o^{p\text{-}mic}(\vec{r},\omega_o)\cos(\omega_o t + \varphi_p(\vec{r},\omega_o))$ to the input of the lock-in amplifier and then ***<u>also</u>*** connect the TTL-level ***sync output*** of the sine-wave generator to the ***reference input*** of the lock-in amplifier, which is ***phase-locked*** to the actual instantaneous {***time-domain***} sine-wave voltage signal $V_{FG}(t) = V_o^{FG}\cos\omega_o t$ output from the sine-wave generator.

The lock-in amplifier then outputs ***frequency-domain*** real ("$X(\omega_o)$") and imaginary ("$Y(\omega_o)$") components of the complex *p*-mic signal that are respectively in-phase (90° out-of-phase) ***relative*** to the lock-in amplifier's ***reference input signal*** – in this case, the TTL-level ***sync signal*** output from the sine-wave generator:

$$\begin{aligned}X(\omega_o) &\equiv \mathrm{Re}\{\tilde{V}_{p\text{-}mic}(\vec{r},\omega_o)\} = \mathrm{Re}\{V_o^{p\text{-}mic}(\vec{r},\omega_o)e^{i\varphi_p(\vec{r},\omega_o)}\} = V_o^{p\text{-}mic}(\vec{r},\omega_o)\,\mathrm{Re}\{e^{i\varphi_p(\vec{r},\omega_o)}\} \\ &= V_o^{p\text{-}mic}(\vec{r},\omega_o)\,\mathrm{Re}\{\cos\varphi_p(\vec{r},\omega_o) + i\cancel{\sin\varphi_p(\vec{r},\omega_o)}\} = V_o^{p\text{-}mic}(\vec{r},\omega_o)\cos\varphi_p(\vec{r},\omega_o) \\ Y(\omega_o) &\equiv \mathrm{Im}\{\tilde{V}_{p\text{-}mic}(\vec{r},\omega_o)\} = \mathrm{Im}\{V_o^{p\text{-}mic}(\vec{r},\omega_o)e^{i\varphi_p(\vec{r},\omega_o)}\} = V_o^{p\text{-}mic}(\vec{r},\omega_o)\,\mathrm{Im}\{e^{i\varphi_p(\vec{r},\omega_o)}\} \\ &= V_o^{p\text{-}mic}(\vec{r},\omega_o)\,\mathrm{Im}\{\cancel{\cos\varphi_p(\vec{r},\omega_o)} + i\sin\varphi_p(\vec{r},\omega_o)\} = V_o^{p\text{-}mic}(\vec{r},\omega_o)\sin\varphi_p(\vec{r},\omega_o)\end{aligned}$$

Thus, we see that the lock-in amplifier outputs the real (*i.e.* in-phase) and imaginary/quadrature {*i.e.* 90° out-of-phase) components of the ***frequency-domain*** complex voltage ***<u>amplitude</u>*** associated with the pressure microphone's output signal, obtained at the point $\vec{r}$ in the (complex) sound field of the loudspeaker:

$$\begin{aligned}\tilde{V}_{p\text{-}mic}(\vec{r},\omega_o) &= \mathrm{Re}\{\tilde{V}_{p\text{-}mic}(\vec{r},\omega_o)\} + i\,\mathrm{Im}\{\tilde{V}_{p\text{-}mic}(\vec{r},\omega_o)\} \\ &= V_o^{p\text{-}mic}(\vec{r},\omega_o)\cos\varphi_p(\vec{r},\omega_o) + iV_o^{p\text{-}mic}(\vec{r},\omega_o)\sin\varphi_p(\vec{r},\omega_o) \\ &= V_o^{p\text{-}mic}(\vec{r},\omega_o)\{\cos\varphi_p(\vec{r},\omega_o) + i\sin\varphi_p(\vec{r},\omega_o)\} = V_o^{p\text{-}mic}(\vec{r},\omega_o)e^{i\varphi_p(\vec{r},\omega_o)}\end{aligned}$$


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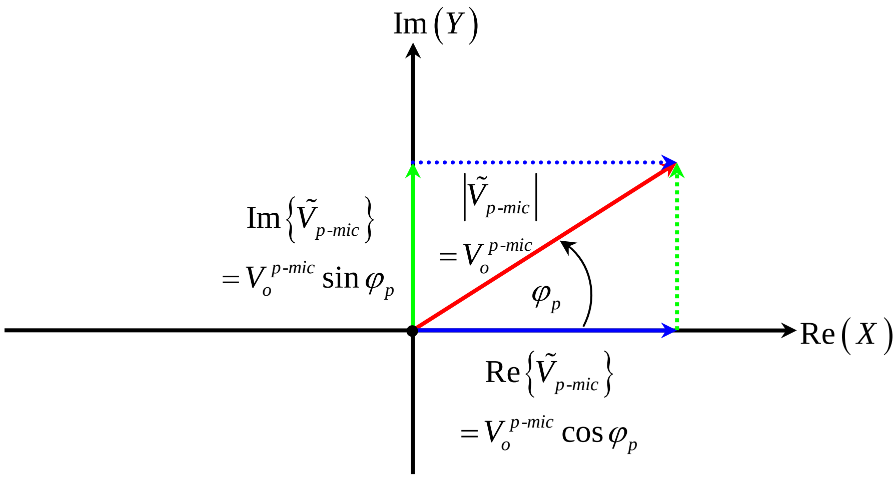
In the 2-D Re-Im complex plane, the complex ***frequency-domain*** phasor diagram for complex $\tilde{V}_{p\text{-}mic}(\vec{r},\omega_o)$ is ***static*** (*i.e.* does not rotate) and appears as shown below:

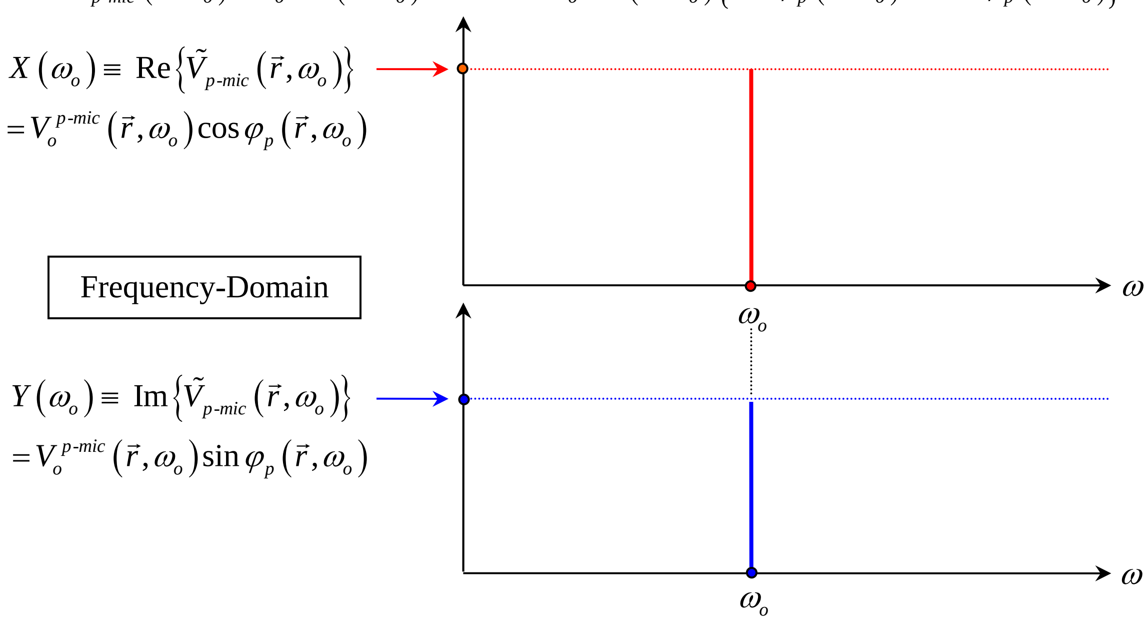



In the complex ***time-domain***, the entire phasor diagram for complex $\tilde{V}_{p\text{-}mic}(\vec{r},t)$ rotates CCW in the complex plane at angular frequency $\omega_o$.

Please see/read Physics 406 Lect. Notes 13 Part 2 for additional details on how lock-in amplifiers work, and their use(s) in the laboratory…

Graphically, the real and imaginary ***frequency-domain*** components of the complex voltage amplitude signal output from the *p*-mic might look something like that shown in the figures below, for a ***pure*** (*i.e.* single-frequency) sine-wave signal output from the sine-wave generator + power amplifier driving a loudspeaker:

$$\tilde{V}_{p\text{-}mic}(\vec{r},\omega_o) = V_o^{p\text{-}mic}(\vec{r},\omega_o)e^{i\varphi_p(\vec{r},\omega_o)} = V_o^{p\text{-}mic}(\vec{r},\omega_o)\{\cos\varphi_p(\vec{r},\omega_o) + i\sin\varphi_p(\vec{r},\omega_o)\}$$





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Note that the angular frequency "spikes" in the above figure at $\omega' = \omega$ associated with the real and imaginary components of the complex ***frequency-domain*** amplitude $\tilde{V}_{p\text{-}mic}(\vec{r},\omega_o)$ are in fact 1-D ***delta-functions*** {in angular-frequency space}, which can be mathematically represented as $V_o^{p\text{-}mic}(\vec{r},\omega_o)\cos\varphi_p(\vec{r},\omega_o)\cdot\delta(\omega_o-\omega)$ and $V_o^{p\text{-}mic}(\vec{r},\omega_o)\sin\varphi_p(\vec{r},\omega_o)\cdot\delta(\omega_o-\omega)$, respectively. Note one of the many interesting/intriguing properties of the 1-D delta function: Since $\omega = 2\pi f$, hence $d\omega = 2\pi df$, and thus:

$$\int_{-\infty}^{+\infty}\delta(\omega_o-\omega)\,d\omega = \int_{-\infty}^{+\infty}\delta(2\pi f_o - 2\pi f)\cdot 2\pi df = \int_{-\infty}^{+\infty}\delta\left[2\pi(f_o - f)\right]\cdot 2\pi df$$
$$= \int_{-\infty}^{+\infty}\tfrac{1}{|2\pi|}\delta(f_o - f)\cdot 2\pi df = \int_{-\infty}^{+\infty}\tfrac{1}{\cancel{2\pi}}\delta(f_o - f)\cdot \cancel{2\pi}\, df = \int_{-\infty}^{+\infty}\delta(f_o - f)\,df = 1$$

Note further that the 1-D delta function $\delta(\omega_o-\omega)$ has physical units of ***inverse*** angular frequency, $\omega^{-1} = 1/\omega$ (*i.e.* sec/radian) and that the 1-D delta function $\delta(f_o - f)$ has physical units of ***inverse*** frequency, $f^{-1} = 1/f$ (*i.e.* seconds), since the 1-D integrals $\int_{-\infty}^{+\infty}\delta(\omega_o-\omega)\,d\omega = 1$ and $\int_{-\infty}^{+\infty}\delta(f_o - f)\,df$ are both dimensionless…

The above complex ***frequency-domain*** result(s) should be compared with their complex ***time-domain*** counterparts:

$$\tilde{V}_{p\text{-}mic}(\vec{r},t) = V_o^{p\text{-}mic}(\vec{r},\omega_o)\,e^{i(\omega_o t+\varphi_p(\vec{r},\omega_o))} = V_o^{p\text{-}mic}(\vec{r},\omega_o)\,e^{i\varphi_p(\vec{r},\omega_o)}\cdot e^{i\omega_o t}$$
$$= V_o^{p\text{-}mic}(\vec{r},\omega_o)\left\{\cos\left(\omega_o t+\varphi_p(\vec{r},\omega_o)\right) + i\sin\left(\omega_o t+\varphi_p(\vec{r},\omega_o)\right)\right\}$$

$$X(t) \equiv \operatorname{Re}\left\{\tilde{V}_{p\text{-}mic}(\vec{r},t)\right\} = \operatorname{Re}\left\{V_o^{p\text{-}mic}(\vec{r},\omega_o)\,e^{i(\omega_o t+\varphi_p(\vec{r},\omega_o))}\right\} = V_o^{p\text{-}mic}(\vec{r},\omega_o)\operatorname{Re}\left\{e^{i(\omega_o t+\varphi_p(\vec{r},\omega_o))}\right\}$$
$$= V_o^{p\text{-}mic}(\vec{r},\omega_o)\operatorname{Re}\left\{\cos\left(\omega_o t+\varphi_p(\vec{r},\omega_o)\right) + \cancel{i\sin\left(\omega_o t+\varphi_p(\vec{r},\omega_o)\right)}\right\}$$
$$= V_o^{p\text{-}mic}(\vec{r},\omega_o)\cos\left(\omega_o t+\varphi_p(\vec{r},\omega_o)\right)$$

$$Y(t) \equiv \operatorname{Im}\left\{\tilde{V}_{p\text{-}mic}(\vec{r},t)\right\} = \operatorname{Im}\left\{V_o^{p\text{-}mic}(\vec{r},\omega_o)\,e^{i(\omega_o t+\varphi_p(\vec{r},\omega_o))}\right\} = V_o^{p\text{-}mic}(\vec{r},\omega_o)\operatorname{Im}\left\{e^{i(\omega_o t+\varphi_p(\vec{r},\omega_o))}\right\}$$
$$= V_o^{p\text{-}mic}(\vec{r},\omega_o)\operatorname{Im}\left\{\cancel{\cos\left(\omega_o t+\varphi_p(\vec{r},\omega_o)\right)} + i\sin\left(\omega_o t+\varphi_p(\vec{r},\omega_o)\right)\right\}$$
$$= V_o^{p\text{-}mic}(\vec{r},\omega_o)\sin\left(\omega_o t+\varphi_p(\vec{r},\omega_o)\right)$$

As mentioned above, the ***frequency-domain*** counterparts of complex ***time-domain*** quantities such as $\tilde{V}_{FG}(t) = V_o^{FG}e^{i\omega_o t}$ and $\tilde{V}_{p\text{-}mic}(\vec{r},t) = V_o^{p\text{-}mic}(\vec{r},\omega_o)\,e^{i(\omega_o t+\varphi_p(\vec{r},\omega_o))}$ are obtained by taking the ***Fourier transform*** of the ***time-domain*** quantities, and vice-versa.


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What is a ***Fourier transform***?

For ***continuous*** complex ***time-domain*** functions $\tilde{f}(t)$, the ***Fourier transform*** of the complex ***time-domain*** function $\tilde{f}(t)$ to the complex ***frequency-domain*** is: $\boxed{\tilde{f}(\omega) \equiv \int_{-\infty}^{+\infty} \tilde{f}(t) e^{-i\omega t} dt}$ where $t$ is treated as a {dummy} variable in the integration over {all} time, from $-\infty \le t \le +\infty$.

The ***inverse Fourier transform*** of a ***continuous*** complex ***frequency-domain*** function $\tilde{f}(\omega)$ to the ***time-domain*** is: $\boxed{\tilde{f}(t) \equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{f}(\omega) e^{+i\omega t} d\omega}$ where $\omega = 2\pi f$ is treated as a {dummy} variable in the integration over {all negative **.and.** positive} angular frequencies: $-\infty \le \omega \le +\infty$.

Note also that the factor of $1/2\pi$ appears here pre-multiplying the latter integral over the angular frequency variable $\omega$ because we are using the angular frequency $\omega \equiv 2\pi f$ in the integral rather than the frequency $f$ itself as a {dummy} variable of integration – technically speaking, frequency $f$ (*sec*$^{-1}$) and time $t$ (*seconds*) are ***true Fourier conjugate variables*** of each other, and ***not*** angular frequency $\omega = 2\pi f$ (***radians***/*sec*$^{-1}$) and time $t$ (*seconds*).

For monochromatic/single-frequency (*aka* harmonic) sound fields the relationship between "generic" complex ***time-domain*** *vs*. complex ***frequency-domain*** quantities is simply given by $\tilde{f}(t) = \tilde{f}(\omega) \cdot e^{i\omega t}$. Thus, *e.g*. the relations between complex ***time-domain*** *vs*. complex ***frequency-domain*** scalar over-pressure and/or 3-D complex vector particle velocity are:

$$\tilde{p}(t) = \tilde{p}(\omega) \cdot e^{i\omega t} = \left|\tilde{p}(\omega)\right| \cdot e^{i\varphi_p(\omega)} \cdot e^{i\omega t}$$

and:

$$\begin{aligned}\vec{\tilde{u}}(t) = \vec{\tilde{u}}(\omega) \cdot e^{i\omega t} &= \left(\tilde{u}_x(\omega)\hat{x} + \tilde{u}_y(\omega)\hat{y} + \tilde{u}_z(\omega)\hat{z}\right) \cdot e^{i\omega t} \\ &= \left(\left|\tilde{u}_x(\omega)\right| \cdot e^{i\varphi_{u_x}(\omega)}\hat{x} + \left|\tilde{u}_y(\omega)\right| \cdot e^{i\varphi_{u_y}(\omega)}\hat{y} + \left|\tilde{u}_z(\omega)\right| \cdot e^{i\varphi_{u_z}(\omega)}\hat{z}\right) \cdot e^{i\omega t}\end{aligned}$$

There are several useful relations associated with Fourier transforms which we list here:

| | Time-Domain: | | Frequency Domain: |
|---|---|---|---|
| Linearity: | $\tilde{h}(t) = a\tilde{f}(t) + b\tilde{g}(t)$ | $\Rightarrow$ | $\tilde{h}(\omega) = a\tilde{f}(\omega) + b\tilde{g}(\omega)$ |
| Translation: | $\tilde{h}(t) = \tilde{f}(t - t_o)$ | $\Rightarrow$ | $\tilde{h}(\omega) = \tilde{f}(\omega)e^{i\omega t_o}$ |
| Modulation: | $\tilde{h}(t) = \tilde{f}(t)e^{i\omega_o t}$ | $\Rightarrow$ | $\tilde{h}(\omega) = \tilde{f}(\omega - \omega_o)$ |
| Scaling: | $\tilde{h}(t) = \tilde{f}(at)$ | $\Rightarrow$ | $\tilde{h}(\omega) = \frac{1}{\lvert a \rvert}\tilde{f}\left(\frac{\omega}{a}\right)$ |
| Conjugation: | $\tilde{h}(t) = \tilde{f}^*(t)$ | $\Rightarrow$ | $\tilde{h}(\omega) = \tilde{f}^*(-\omega)$ |


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**Complex *Specific* Acoustic Immittances - Admittance and Impedance of a Medium:**

The ***medium*** (solid, liquid or gas) in which sound waves propagate has associated with it the property of how easy (or how difficult) it is for sound waves to propagate through that medium – the so-called complex ***specific*** acoustic ***immittances*** – complex ***specific*** acoustic ***admittance*** and/or complex ***specific*** acoustic ***impedance*** (the reciprocal of complex ***specific*** acoustic ***admittance***) give us such information.

For propagation of 1-D sound waves in a medium, the complex ***specific*** acoustic immittances – *i.e*. collectively the complex ***specific*** acoustic admittance and/or complex ***specific*** acoustic impedance are both well-defined quantities. They are defined in analogy to the complex form of Ohm's Law ($\tilde{V} = \tilde{I}\tilde{Z}$ , $\tilde{I} = \tilde{V}\tilde{Y}$) as used *e.g*. in electrical circuit theory, since complex over-pressure $\tilde{p}$ is the analog of complex *AC* voltage $\tilde{V}$ , and particle velocity $\tilde{\vec{u}}$ is ~ the analog of complex *AC* electric current $\tilde{I}_e$ {Note that $\tilde{\vec{J}}_a(\vec{r},t) \equiv \rho_o \tilde{\vec{u}}(\vec{r},t)$ $(kg/s\text{-}m^2)$ is the complex acoustic ***mass*** current density}, whereas $\tilde{\vec{J}}_e \equiv \tilde{I}/\vec{A}_\perp = n_e q_e \tilde{\vec{v}}_e = \rho_e \tilde{\vec{v}}$ $(Amp/m^2 = Coul/s\text{-}m^2)$ is the complex ***electrical*** current density}. Note also that both $\tilde{\vec{J}}_e$ and $\tilde{\vec{J}}_a$ are 3-D ***vector*** quantities.

**Complex Scalar Electrical Immittances:**

Complex Electrical Impedance: $$\boxed{\tilde{Z}_e(t;\omega) \equiv \frac{\tilde{V}(t;\omega)}{\tilde{I}_e(t;\omega)} \quad (Ohms = Volts/Amps)}$$

Complex Electrical Admittance: $$\boxed{\tilde{Y}_e(t;\omega) \equiv \frac{\tilde{I}_e(t;\omega)}{\tilde{V}(t;\omega)} \quad (Siemens = Ohms^{-1} = Amps/Volts)}$$

If we write out these relations using complex polar notation: $\tilde{V}(t;\omega) = |\tilde{V}(\omega)| e^{i\varphi_V(\omega)} \cdot e^{i\omega t}$ , $\tilde{I}(t;\omega) = |\tilde{I}(\omega)| e^{i\varphi_I(\omega)} \cdot e^{i\omega t}$ , then, noting the cancellation of $e^{i\omega t}$ time dependence factors:

$$\tilde{Z}_e(t;\omega) \equiv \frac{\tilde{V}(t;\omega)}{\tilde{I}_e(t;\omega)} = \frac{|\tilde{V}(\omega)| e^{i\varphi_V(\omega)} \cdot \cancel{e^{i\omega t}}}{|\tilde{I}_e(\omega)| e^{i\varphi_I(\omega)} \cdot \cancel{e^{i\omega t}}} = \frac{|\tilde{V}(\omega)| e^{i\varphi_V(\omega)}}{|\tilde{I}_e(\omega)| e^{i\varphi_I(\omega)}} = \frac{|\tilde{V}(\omega)|}{|\tilde{I}_e(\omega)|} e^{i[\varphi_V(\omega)-\varphi_I(\omega)]} = |\tilde{Z}_e(\omega)| e^{i\varphi_Z(\omega)} = \tilde{Z}_e(\omega)$$

$$\tilde{Y}_e(t;\omega) \equiv \frac{\tilde{I}_e(t;\omega)}{\tilde{V}(t;\omega)} = \frac{|\tilde{I}_e(\omega)| e^{i\varphi_I(\omega)} \cdot \cancel{e^{i\omega t}}}{|\tilde{V}(\omega)| e^{i\varphi_V(\omega)} \cdot \cancel{e^{i\omega t}}} = \frac{|\tilde{I}_e(\omega)| e^{i\varphi_I(\omega)}}{|\tilde{V}(\omega)| e^{i\varphi_V(\omega)}} = \frac{|\tilde{I}_e(\omega)|}{|\tilde{V}(\omega)|} e^{i[\varphi_I(\omega)-\varphi_V(\omega)]} = |\tilde{Y}_e(\omega)| e^{i\varphi_Y(\omega)} = \tilde{Y}_e(\omega)$$

Now: $|\tilde{Z}_e(\omega)| = 1/|\tilde{Y}_e(\omega)|$ or: $|\tilde{Y}_e(\omega)| = 1/|\tilde{Z}_e(\omega)|$, and we see that: $\varphi_Z(\omega) = \varphi_V(\omega) - \varphi_I(\omega) = -\varphi_Y(\omega)$, hence: $\tilde{Y}_e(\omega) = |\tilde{Y}_e(\omega)| e^{i\varphi_Y(\omega)} = \{1/|\tilde{Z}_e(\omega)|\} e^{-i\varphi_Z(\omega)} = 1/\{|\tilde{Z}_e(\omega)| e^{i\varphi_Z(\omega)}\} = 1/\tilde{Z}_e(\omega)$. Thus:

$$\boxed{\tilde{Z}_e(t;\omega) \equiv \frac{\tilde{V}(t;\omega)}{\tilde{I}_e(t;\omega)} = \frac{\tilde{V}(\omega)}{\tilde{I}_e(\omega)} = \tilde{Z}_e(\omega) = \frac{1}{\tilde{Y}_e(\omega)}} \text{ and: } \boxed{\tilde{Y}_e(t;\omega) \equiv \frac{\tilde{I}_e(t;\omega)}{\tilde{V}(t;\omega)} = \frac{\tilde{I}_e(\omega)}{\tilde{V}(\omega)} = \tilde{Y}_e(\omega) = \frac{1}{\tilde{Z}_e(\omega)}}$$


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**<u>Complex 3-D Vector *Specific* Acoustic Immittances</u>:**

<u>Cmplx *Spec.* Acoust. Impedance</u>: $\vec{\tilde{z}}_a(\vec{r},t) \equiv \dfrac{\tilde{p}(\vec{r},t)}{\vec{\tilde{u}}(\vec{r},t)} = \dfrac{1}{\vec{\tilde{y}}_a(\vec{r},t)}\ \left(\begin{matrix} Acoustic \\ Ohms \end{matrix} \equiv \begin{matrix} Pa\text{-}s/m \\ = N\text{-}s/m^3 \end{matrix} = Rayl\right)$

<u>Cmplx *Spec.* Acoust. Admittance</u>: $\vec{\tilde{y}}_a(\vec{r},t) \equiv \dfrac{\vec{\tilde{u}}(\vec{r},t)}{\tilde{p}(\vec{r},t)} = \dfrac{1}{\vec{\tilde{z}}_a(\vec{r},t)}\ \left(\begin{matrix} Acoustic \\ Siemens \end{matrix} \equiv \begin{matrix} m/Pa\text{-}s \\ = m^3/N\text{-}s \end{matrix} = Rayl^{-1}\right)$

Note that the complex ***<u>specific</u>*** acoustic immittances $\vec{\tilde{z}}_a(\vec{r},t)$ and $\vec{\tilde{y}}_a(\vec{r},t) = 1/\vec{\tilde{z}}_a(\vec{r},t)$ are 3-D ***<u>vector</u>*** quantities.

The complex 3-D vector ***<u>specific</u>*** acoustic ***<u>admittance</u>*** $\vec{\tilde{y}}_a(\vec{r},t) \equiv \vec{\tilde{u}}(\vec{r},t)/\tilde{p}(\vec{r},t)$ is clearly a mathematically well-defined vector quantity:

$$\vec{\tilde{y}}_a(\vec{r},t) \equiv \frac{\vec{\tilde{u}}(\vec{r},t)}{\tilde{p}(\vec{r},t)} = \tilde{y}_{a_x}(\vec{r},t)\hat{x} + \tilde{y}_{a_y}(\vec{r},t)\hat{y} + \tilde{y}_{a_z}(\vec{r},t)\hat{z} = \frac{\tilde{u}_x(\vec{r},t)}{\tilde{p}(\vec{r},t)}\hat{x} + \frac{\tilde{u}_y(\vec{r},t)}{\tilde{p}(\vec{r},t)}\hat{y} + \frac{\tilde{u}_z(\vec{r},t)}{\tilde{p}(\vec{r},t)}\hat{z}$$

$$= \frac{\left[\tilde{u}_x(\vec{r},t)\hat{x} + \tilde{u}_y(\vec{r},t)\hat{y} + \tilde{u}_z(\vec{r},t)\hat{z}\right]}{\tilde{p}(\vec{r},t)} = \frac{\vec{\tilde{u}}(\vec{r},t)}{\tilde{p}(\vec{r},t)}$$

where: $\tilde{y}_{a_x}(\vec{r},t) = \dfrac{\tilde{u}_x(\vec{r},t)}{\tilde{p}(\vec{r},t)},\quad \tilde{y}_{a_y}(\vec{r},t) = \dfrac{\tilde{u}_y(\vec{r},t)}{\tilde{p}(\vec{r},t)},\quad \tilde{y}_{a_z}(\vec{r},t) = \dfrac{\tilde{u}_z(\vec{r},t)}{\tilde{p}(\vec{r},t)}$

The complex 3-D vector ***<u>specific</u>*** acoustic impedance $\vec{\tilde{z}}_a(\vec{r},t) \equiv \tilde{p}(\vec{r},t)/\vec{\tilde{u}}(\vec{r},t)$ may initially seem like a mathematically less well-defined vector quantity. However, on physical/common sense grounds, we know that *e.g.* the ***<u>magnitudes</u>*** of the complex 3-D vector ***<u>specific</u>*** acoustic immittances, $\left|\vec{\tilde{y}}_a(\vec{r},t)\right|$ and $\left|\vec{\tilde{z}}_a(\vec{r},t)\right|$ must both be ***<u>invariant</u>*** (*i.e.* unchanged) under simple coordinate transformations – *e.g.* rotations and/or translations of the local coordinate system, as well as ***<u>invariant</u>*** under *e.g.* simple rotations of the sound source under investigation.

Consider a simple, 1-D monochromatic/single-frequency sound field – such as an acoustic traveling plane wave propagating *e.g.* in the local $+\hat{x}$ direction. Then $\tilde{u}_x(\vec{r},t) = u_o e^{i(\omega t - k_x x)} \neq 0$, with $\tilde{p}(\vec{r},t) = p_o e^{i(\omega t - k_x x)} \neq 0$, whereas $\tilde{u}_y(\vec{r},t) = \tilde{u}_z(\vec{r},t) = 0$. The components of the complex 3-D vector ***<u>specific</u>*** acoustic admittance are $\tilde{y}_{a_x}(\vec{r},t) = \tilde{u}_x(\vec{r},t)/\tilde{p}(\vec{r},t) = u_o \cancel{e^{i(\omega t - k_x x)}} / p_o \cancel{e^{i(\omega t - k_x x)}} = u_o/p_o \neq 0$, whereas $\tilde{y}_{a_y}(\vec{r},t) = \tilde{y}_{a_z}(\vec{r},t) = 0$.


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Obviously, if we carry out *e.g*. a simple rotation of our local 3-D coordinate system, the individual *x*, *y*, *z* components of $\vec{\tilde{y}}_a(\vec{r},t)$ will change accordingly, however the magnitude $\left|\vec{\tilde{y}}_a(\vec{r},t)\right| = \sqrt{\vec{\tilde{y}}_a(\vec{r},t)\bullet\vec{\tilde{y}}_a^*(\vec{r},t)} = \sqrt{\left|\tilde{y}_{a_x}(\vec{r},t)\right|^2 + \left|\tilde{y}_{a_y}(\vec{r},t)\right|^2 + \left|\tilde{y}_{a_z}(\vec{r},t)\right|^2}$ will ***not*** change.

Likewise, the individual *x*, *y*, *z* components of $\vec{\tilde{z}}_a(\vec{r},t)$ will change accordingly under a simple rotation of our local 3-D coordinate system, however the magnitude $\left|\vec{\tilde{z}}_a(\vec{r},t)\right| = \sqrt{\vec{\tilde{z}}_a(\vec{r},t)\bullet\vec{\tilde{z}}_a^*(\vec{r},t)} = \sqrt{\left|\tilde{z}_{a_x}(\vec{r},t)\right|^2 + \left|\tilde{z}_{a_y}(\vec{r},t)\right|^2 + \left|\tilde{z}_{a_z}(\vec{r},t)\right|^2}$ will ***not*** change.

We thus write the complex 3-D vector ***specific*** acoustic impedance $\vec{\tilde{z}}_a(\vec{r},t)$, *e.g*. in Cartesian coordinates as follows:

$$\begin{aligned}
\vec{\tilde{z}}_a(\vec{r},t) &\equiv \frac{\tilde{p}(\vec{r},t)}{\vec{\tilde{u}}(\vec{r},t)} = \tilde{z}_{a_x}(\vec{r},t)\hat{x} + \tilde{z}_{a_y}(\vec{r},t)\hat{y} + \tilde{z}_{a_z}(\vec{r},t)\hat{z} \\
&= \frac{\tilde{p}(\vec{r},t)}{\vec{\tilde{u}}(\vec{r},t)}\bullet\frac{\vec{\tilde{u}}^*(\vec{r},t)}{\vec{\tilde{u}}^*(\vec{r},t)} = \frac{\tilde{p}(\vec{r},t)\vec{\tilde{u}}^*(\vec{r},t)}{\vec{\tilde{u}}(\vec{r},t)\bullet\vec{\tilde{u}}^*(\vec{r},t)} = \frac{\tilde{p}(\vec{r},t)\vec{\tilde{u}}^*(\vec{r},t)}{\left|\vec{\tilde{u}}(\vec{r},t)\right|^2} \\
&= \frac{\tilde{p}(\vec{r},t)\left[\tilde{u}_x^*(\vec{r},t)\hat{x} + \tilde{u}_y^*(\vec{r},t)\hat{y} + \tilde{u}_z^*(\vec{r},t)\hat{z}\right]}{\left|\vec{\tilde{u}}(\vec{r},t)\right|^2} \\
&= \frac{\tilde{p}(\vec{r},t)\left[\tilde{u}_x^*(\vec{r},t)\hat{x} + \tilde{u}_y^*(\vec{r},t)\hat{y} + \tilde{u}_z^*(\vec{r},t)\hat{z}\right]}{\left|\vec{\tilde{u}}_x(\vec{r},t)\right|^2 + \left|\vec{\tilde{u}}_y(\vec{r},t)\right|^2 + \left|\vec{\tilde{u}}_z(\vec{r},t)\right|^2}
\end{aligned}$$

where: $\tilde{z}_{a_x}(\vec{r},t) = \frac{\tilde{p}(\vec{r},t)\tilde{u}_x^*(\vec{r},t)}{\left|\vec{\tilde{u}}(\vec{r},t)\right|^2}$, $\tilde{z}_{a_y}(\vec{r},t) = \frac{\tilde{p}(\vec{r},t)\tilde{u}_y^*(\vec{r},t)}{\left|\vec{\tilde{u}}(\vec{r},t)\right|^2}$, $\tilde{z}_{a_z}(\vec{r},t) = \frac{\tilde{p}(\vec{r},t)\tilde{u}_z^*(\vec{r},t)}{\left|\vec{\tilde{u}}(\vec{r},t)\right|^2}$

Hence, the technical/mathematical issue here is the rationalization of an arbitrary, "generic" complex reciprocal 3-D vector quantity:

$$\vec{\tilde{u}}^{-1} = \frac{1}{\vec{\tilde{u}}} = \frac{\vec{\tilde{u}}^*}{\vec{\tilde{u}}\bullet\vec{\tilde{u}}^*} = \frac{\vec{\tilde{u}}^*}{\left|\vec{\tilde{u}}^*\right|^2}$$

paralleling that which is done for an arbitrary, "generic" complex reciprocal scalar quantity:

$$\tilde{p}^{-1} = \frac{1}{\tilde{p}} = \frac{\tilde{p}^*}{\tilde{p}\bullet\tilde{p}^*} = \frac{\tilde{p}^*}{\left|\tilde{p}^*\right|^2}$$

It can be seen that indeed: $\left|\vec{\tilde{y}}_a(\vec{r},t)\right| = \left|\frac{\vec{\tilde{u}}(\vec{r},t)}{\tilde{p}(\vec{r},t)}\right| = \frac{\left|\vec{\tilde{u}}(\vec{r},t)\right|}{\left|\tilde{p}(\vec{r},t)\right|} = \frac{1}{\left|\vec{\tilde{z}}_a(\vec{r},t)\right|}$, and also that:


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$$\left|\vec{\tilde{z}}_a(\vec{r},t)\right| = \sqrt{\vec{\tilde{z}}_a(\vec{r},t)\bullet\vec{\tilde{z}}_a^*(\vec{r},t)} = \sqrt{\left|\tilde{z}_{a_x}(\vec{r},t)\right|^2 + \left|\tilde{z}_{a_y}(\vec{r},t)\right|^2 + \left|\tilde{z}_{a_z}(\vec{r},t)\right|^2}$$

$$= \sqrt{\frac{\left|\tilde{p}(\vec{r},t)\right|^2\left|\tilde{u}_x(\vec{r},t)\right|^2}{\left(\left|\vec{\tilde{u}}(\vec{r},t)\right|^2\right)^2} + \frac{\left|\tilde{p}(\vec{r},t)\right|^2\left|\tilde{u}_y(\vec{r},t)\right|^2}{\left(\left|\vec{\tilde{u}}(\vec{r},t)\right|^2\right)^2} + \frac{\left|\tilde{p}(\vec{r},t)\right|^2\left|\tilde{u}_z(\vec{r},t)\right|^2}{\left(\left|\vec{\tilde{u}}(\vec{r},t)\right|^2\right)^2}}$$

$$= \sqrt{\frac{\left|\tilde{p}(\vec{r},t)\right|^2\left[\left|\tilde{u}_x(\vec{r},t)\right|^2 + \left|\tilde{u}_y(\vec{r},t)\right|^2 + \left|\tilde{u}_z(\vec{r},t)\right|^2\right]}{\left(\left|\vec{\tilde{u}}(\vec{r},t)\right|^2\right)^2}} = \sqrt{\frac{\left|\tilde{p}(\vec{r},t)\right|^2\cancel{\left|\vec{\tilde{u}}(\vec{r},t)\right|^2}}{\left(\left|\vec{\tilde{u}}(\vec{r},t)\right|^2\right)^{\cancel{2}}}}$$

$$= \sqrt{\frac{\left|\tilde{p}(\vec{r},t)\right|^2}{\left|\vec{\tilde{u}}(\vec{r},t)\right|^2}} = \frac{\left|\tilde{p}(\vec{r},t)\right|}{\left|\vec{\tilde{u}}(\vec{r},t)\right|} = \frac{1}{\left|\vec{\tilde{y}}_a(\vec{r},t)\right|}$$

However, we also see for the individual *x*, *y*, *z* components of the complex 3-D vector specific acoustic immittances that:

$$\left\{\tilde{y}_{a_x}(\vec{r},t) \equiv \frac{\tilde{u}_x(\vec{r},t)}{\tilde{p}(\vec{r},t)}\right\} \neq \left\{\frac{1}{\tilde{z}_{a_x}(\vec{r},t)} \equiv \frac{\left|\vec{\tilde{u}}(\vec{r},t)\right|^2}{\tilde{p}(\vec{r},t)\tilde{u}_x^*(\vec{r},t)}\right\}$$

$$\left\{\tilde{y}_{a_y}(\vec{r},t) \equiv \frac{\tilde{u}_y(\vec{r},t)}{\tilde{p}(\vec{r},t)}\right\} \neq \left\{\frac{1}{\tilde{z}_{a_y}(\vec{r},t)} \equiv \frac{\left|\vec{\tilde{u}}(\vec{r},t)\right|^2}{\tilde{p}(\vec{r},t)\tilde{u}_y^*(\vec{r},t)}\right\}$$

$$\left\{\tilde{y}_{a_z}(\vec{r},t) \equiv \frac{\tilde{u}_z(\vec{r},t)}{\tilde{p}(\vec{r},t)}\right\} \neq \left\{\frac{1}{\tilde{z}_{a_z}(\vec{r},t)} \equiv \frac{\left|\vec{\tilde{u}}(\vec{r},t)\right|^2}{\tilde{p}(\vec{r},t)\tilde{u}_z^*(\vec{r},t)}\right\}$$

Additionally, the expressions for the complex 3-D vector specific acoustic immittances:

$$\vec{\tilde{y}}_a(\vec{r},t) \equiv \frac{\vec{\tilde{u}}(\vec{r},t)}{\tilde{p}(\vec{r},t)} = \tilde{y}_{a_x}(\vec{r},t)\hat{x} + \tilde{y}_{a_y}(\vec{r},t)\hat{y} + \tilde{y}_{a_z}(\vec{r},t)\hat{z} = \frac{\tilde{u}_x(\vec{r},t)}{\tilde{p}(\vec{r},t)}\hat{x} + \frac{\tilde{u}_y(\vec{r},t)}{\tilde{p}(\vec{r},t)}\hat{y} + \frac{\tilde{u}_z(\vec{r},t)}{\tilde{p}(\vec{r},t)}\hat{z}$$

and:

$$\vec{\tilde{z}}_a(\vec{r},t) \equiv \frac{\tilde{p}(\vec{r},t)}{\vec{\tilde{u}}(\vec{r},t)} = \tilde{z}_{a_x}(\vec{r},t)\hat{x} + \tilde{z}_{a_y}(\vec{r},t)\hat{y} + \tilde{z}_{a_z}(\vec{r},t)\hat{z} = \frac{\tilde{p}(\vec{r},t)\left[\tilde{u}_x^*(\vec{r},t)\hat{x} + \tilde{u}_y^*(\vec{r},t)\hat{y} + \tilde{u}_z^*(\vec{r},t)\hat{z}\right]}{\left|\vec{\tilde{u}}(\vec{r},t)\right|^2}$$

can be seen to mathematically behave properly *e.g*. under arbitrary rotations of the local 3-D coordinate system, as well as for rotations of 3-D sound sources, and also for complex 3-D sound fields composed of *e.g*. an arbitrary superposition/linear combination of three mutually-orthogonal propagating monochromatic plane traveling waves – propagating in the $+\hat{x}$, $+\hat{y}$ and $+\hat{z}$ directions, with common scalar complex pressure, $\tilde{p}_{tot}(\vec{r},t) = \tilde{p}_1(\vec{r},t) + \tilde{p}_2(\vec{r},t) + \tilde{p}_3(\vec{r},t)$.


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Note also that ***both*** the ***time-domain*** complex pressure $\tilde{p}(\vec{r},t)$ and the ***time-domain*** complex 3-D particle velocity $\tilde{\vec{u}}(\vec{r},t)$ associated *e.g.* with a single frequency (*aka* harmonic) sound field will in general have time dependence of the form $e^{i\omega t}$. Thus, since the 3-D specific acoustic immittances are defined as ***ratios*** of these two quantities, the $e^{i\omega t}$ factor in the both the numerator and the denominator of the ratios $\tilde{\vec{y}}_a(\vec{r},t) = \tilde{\vec{u}}(\vec{r},t)/\tilde{p}(\vec{r},t)$ and $\tilde{\vec{z}}_a(\vec{r},t) = \tilde{p}(\vec{r},t)/\tilde{\vec{u}}(\vec{r},t)$ ***cancels*** for harmonic/single-frequency complex sound fields, thus we see that the complex 3-D vector specific acoustic immittances are in fact ***time-independent*** quantities… In fact, they are manifestly ***frequency domain*** quantities!

$$\text{Time Domain: } \tilde{\vec{y}}_a(\vec{r},t) \equiv \frac{\tilde{\vec{u}}(\vec{r},t)}{\tilde{p}(\vec{r},t)} = \frac{\tilde{\vec{u}}(\vec{r},\omega)\cancel{e^{i\omega t}}}{\tilde{p}(\vec{r},\omega)\cancel{e^{i\omega t}}} = \frac{\tilde{\vec{u}}(\vec{r},\omega)}{\tilde{p}(\vec{r},\omega)} \equiv \tilde{\vec{y}}_a(\vec{r},\omega) \text{ Frequency Domain}$$

$$\text{Time Domain: } \tilde{\vec{z}}_a(\vec{r},t) \equiv \frac{\tilde{p}(\vec{r},t)}{\tilde{\vec{u}}(\vec{r},t)} = \frac{\tilde{p}(\vec{r},\omega)\cancel{e^{i\omega t}}}{\tilde{\vec{u}}(\vec{r},\omega)\cancel{e^{i\omega t}}} = \frac{\tilde{p}(\vec{r},\omega)}{\tilde{\vec{u}}(\vec{r},\omega)} \equiv \tilde{\vec{z}}_a(\vec{r},\omega) \text{ Frequency Domain}$$

**Complex 3-D *Specific* Acoustic Immittances (for Harmonic Sound Fields):**

Complex *Specific* Acoustic Impedance: $$\tilde{\vec{z}}_a(\vec{r},\omega) \equiv \frac{\tilde{p}(\vec{r},\omega)}{\tilde{\vec{u}}(\vec{r},\omega)} = \frac{1}{\tilde{\vec{y}}_a(\vec{r},\omega)} \quad (\Omega_a = Rayl)$$

**Time-independent quantity!**
**⇒ Frequency-domain quantity!**

Complex *Specific* Acoustic Admittance: $$\tilde{\vec{y}}_a(\vec{r},\omega) \equiv \frac{\tilde{\vec{u}}(\vec{r},\omega)}{\tilde{p}(\vec{r},\omega)} = \frac{1}{\tilde{\vec{z}}_a(\vec{r},\omega)} \quad (\Omega_a^{-1} = Rayl^{-1})$$

**Time-independent quantity!**
**⇒ Frequency-domain quantity!**

The time-independent complex ***specific*** acoustic immittances are 3-D ***vector frequency-domain*** quantities. Their 3-D *x*-*y*-*z* Cartesian ***frequency-domain*** components can be explicitly written out as:

$$\begin{aligned}\tilde{\vec{y}}_a(\vec{r},\omega) &= \tilde{y}_{a_x}(\vec{r},\omega)\hat{x} + \tilde{y}_{a_y}(\vec{r},\omega)\hat{y} + \tilde{y}_{a_z}(\vec{r},\omega)\hat{z} \\ &= \frac{\tilde{u}_x(\vec{r},\omega)}{\tilde{p}(\vec{r},\omega)}\hat{x} + \frac{\tilde{u}_y(\vec{r},\omega)}{\tilde{p}(\vec{r},\omega)}\hat{y} + \frac{\tilde{u}_z(\vec{r},\omega)}{\tilde{p}(\vec{r},\omega)}\hat{z} = \frac{\tilde{\vec{u}}(\vec{r},\omega)}{\tilde{p}(\vec{r},\omega)} = \frac{1}{\tilde{\vec{z}}_a(\vec{r},\omega)}\end{aligned}$$

$$\begin{aligned}\tilde{\vec{z}}_a(\vec{r},\omega) &= \tilde{z}_{a_x}(\vec{r},\omega)\hat{x} + \tilde{z}_{a_y}(\vec{r},\omega)\hat{y} + \tilde{z}_{a_z}(\vec{r},\omega)\hat{z} = \frac{\tilde{p}(\vec{r},\omega)}{\tilde{\vec{u}}(\vec{r},\omega)} = \frac{1}{\tilde{\vec{y}}_a(\vec{r},\omega)} \\ &= \frac{\tilde{p}(\vec{r},\omega)\tilde{u}_x^*(\vec{r},\omega)}{\left|\tilde{\vec{u}}(\vec{r},\omega)\right|^2}\hat{x} + \frac{\tilde{p}(\vec{r},\omega)\tilde{u}_y^*(\vec{r},\omega)}{\left|\tilde{\vec{u}}(\vec{r},\omega)\right|^2}\hat{y} + \frac{\tilde{p}(\vec{r},\omega)\tilde{u}_z^*(\vec{r},\omega)}{\left|\tilde{\vec{u}}(\vec{r},\omega)\right|^2}\hat{z} = \frac{\tilde{p}(\vec{r},\omega)\tilde{\vec{u}}^*(\vec{r},\omega)}{\left|\tilde{\vec{u}}(\vec{r},\omega)\right|^2}\end{aligned}$$


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Next, we explain why $\tilde{\vec{z}}_a(\vec{r},\omega)$ and $\tilde{\vec{y}}_a(\vec{r},\omega) = 1/\tilde{\vec{z}}_a(\vec{r},\omega)$ are called complex ⇒ ***specific*** ⇐ acoustic impedance and admittance, respectively. As mentioned above, $\tilde{\vec{z}}_a(\vec{r},\omega)$ and $\tilde{\vec{y}}_a(\vec{r},\omega) = 1/\tilde{\vec{z}}_a(\vec{r},\omega)$ are immittances ***specifically*** associated with the propagation medium. And, in order to avoid confusion, there {already} exists two ***other*** acoustic immittance quantities, known as the complex 3-D vector acoustic impedance $\tilde{\vec{\mathbb{Z}}}_a(\vec{r},\omega)$ and the complex 3-D vector acoustic admittance $\tilde{\vec{Y}}_a(\vec{r},\omega) = 1/\tilde{\vec{\mathbb{Z}}}_a(\vec{r},\omega)$, which are associated with the acoustics of sound waves propagating inside ***ducts*** (*i.e.* pipes) with cross-sectional area $S_\perp$ as defined below:

**Complex 3-D Acoustic Immittances (for Harmonic Sound Fields):**

**Complex 3-D Acoustic Impedance:**

$$\tilde{\vec{\mathbb{Z}}}_a(\vec{r},\omega) \equiv \frac{\tilde{p}(\vec{r},\omega)}{\tilde{\vec{u}}(\vec{r},\omega)S_\perp} = \frac{1}{\tilde{\vec{Y}}_a(\vec{r},\omega)} \quad \left(\begin{matrix} Pa\text{-}s/m^3 \\ = N\text{-}s/m^5 \end{matrix} = Rayl/m^2\right)$$

**Complex 3-D Acoustic Admittance:**

$$\tilde{\vec{Y}}_a(\vec{r},\omega) \equiv \frac{\tilde{\vec{u}}(\vec{r},\omega)S_\perp}{\tilde{p}(\vec{r},\omega)} = \frac{1}{\tilde{\vec{\mathbb{Z}}}_a(\vec{r},\omega)} \quad \left(\begin{matrix} m/Pa\text{-}s \\ = m^3/N\text{-}s \end{matrix} = Rayl^{-1}\text{-}m^2\right)$$

Note that the quantity $\tilde{\vec{U}}(\vec{r},\omega) \equiv \tilde{\vec{u}}(\vec{r},\omega)S_\perp$ $\left(m/s \cdot m^2 = m^3/s\right)$ is known as the ***volume*** ***velocity***, because of its dimensions $\left(m^3/s\right)$.

Inside a duct of cross sectional area $S_\perp$, the complex 3-D vector ***specific*** acoustic immittances $\tilde{\vec{z}}_a(\vec{r},\omega)$ and $\tilde{\vec{y}}_a(\vec{r},\omega) = 1/\tilde{\vec{z}}_a(\vec{r},\omega)$ are thus related to the complex 3-D vector immittances $\tilde{\vec{\mathbb{Z}}}_a(\vec{r},\omega)$ and $\tilde{\vec{Y}}_a(\vec{r},\omega) = 1/\tilde{\vec{\mathbb{Z}}}_a(\vec{r},\omega)$ by the relations:

$$\tilde{\vec{z}}_a(\vec{r},\omega) = \tilde{\vec{\mathbb{Z}}}_a(\vec{r},\omega)S_\perp \quad \text{and} \quad \tilde{\vec{y}}_a(\vec{r},\omega) = \tilde{\vec{Y}}_a(\vec{r},\omega)/S_\perp$$

or:

$$\tilde{\vec{\mathbb{Z}}}_a(\vec{r},\omega) = \tilde{\vec{z}}_a(\vec{r},\omega)/S_\perp \quad \text{and} \quad \tilde{\vec{Y}}_a(\vec{r},\omega) = \tilde{\vec{y}}_a(\vec{r},\omega)S_\perp$$

From the above relations, since the complex 3-D vector ***specific*** acoustic immittances $\tilde{\vec{z}}_a(\vec{r},\omega)$ and $\tilde{\vec{y}}_a(\vec{r},\omega) = 1/\tilde{\vec{z}}_a(\vec{r},\omega)$ are manifestly ***frequency*** ***domain*** quantities, we see that the complex 3-D vector acoustic immittances $\tilde{\vec{\mathbb{Z}}}_a(\vec{r},\omega)$ and $\tilde{\vec{Y}}_a(\vec{r},\omega) = 1/\tilde{\vec{\mathbb{Z}}}_a(\vec{r},\omega)$ are also manifestly ***frequency*** ***domain*** quantities.


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Physically, just as the complex scalar electrical ***impedance*** $\tilde{Z}_e$ is a measure of an electrical device to ***impede*** the ***flow*** of a complex scalar *AC* electrical current $\tilde{I} = \vec{\tilde{J}}_e \bullet \vec{S}_\perp \left(C/s\right)$ when a complex scalar *AC* voltage $\tilde{V}$ is applied across the terminals of the electrical device, the complex 3-D vector acoustic impedance $\vec{\tilde{\mathbb{Z}}}_a\left(\vec{r},\omega\right)$ is a measure of the acoustical medium's ability to ***impede*** the ***flow*** of a complex acoustic mass current $\tilde{I}_a\left(\vec{r},\omega\right) = \vec{\tilde{J}}_a\left(\vec{r},\omega\right) \bullet \vec{S}_\perp = \rho_o \vec{\tilde{u}}\left(\vec{r},\omega\right) \bullet \vec{S}_\perp \left(kg/s\right)$ for a complex over-pressure $\tilde{p}\left(\vec{r},\omega\right)$ at point $\vec{r}$ .

Similarly, just as complex scalar electrical ***admittance*** $\tilde{Y}_e = 1/\tilde{Z}_e$ is a measure of the ***ease*** with which an electrical device ***admits*** the ***flow*** of a complex scalar *AC* electrical current $\tilde{I}_e$ when a complex scalar *AC* voltage $\tilde{V}$ is applied across the terminals of the electrical device, the complex 3-D vector acoustic admittance $\vec{\tilde{Y}}_a\left(\vec{r},\omega\right) = 1/\vec{\tilde{\mathbb{Z}}}_a\left(\vec{r},\omega\right)$ is a measure of the ***ease*** with which an acoustical medium's ***admits*** the ***flow*** of a complex scalar acoustic mass current $\tilde{I}_a\left(\vec{r},\omega\right) = \vec{\tilde{J}}_a\left(\vec{r},\omega\right) \bullet \vec{S}_\perp = \rho_o \vec{\tilde{u}}\left(\vec{r},\omega\right) \bullet \vec{S}_\perp \ \left(kg/s\right)$ in the presence of a complex over-pressure $\tilde{p}\left(\vec{r},\omega\right)$ at the point $\vec{r}$ .

Another way to gain some physical insight into the nature of complex 3-D vector ***specific*** acoustic impedance $\vec{\tilde{z}}_a\left(\vec{r},\omega\right) = \tilde{p}\left(\vec{r},\omega\right)/\vec{\tilde{u}}\left(\vec{r},\omega\right)$ and complex 3-D vector ***specific*** acoustic admittance $\vec{\tilde{y}}_a\left(\vec{r},\omega\right) = \vec{\tilde{u}}\left(\vec{r},\omega\right)/\tilde{p}\left(\vec{r},\omega\right) = 1/\vec{\tilde{z}}_a\left(\vec{r},\omega\right)$ of a medium associated with a harmonic sound field is to imagine a physical situation where we set the {magnitude} of the complex scalar over-pressure $\tilde{p}\left(\vec{r},\omega\right)$ to be a constant/fixed value, *e.g.* $\left|\tilde{p}\left(\vec{r},\omega\right)\right| = 1.0\ Pascal$ .

Then, for a harmonic sound field, if the complex 3-D vector ***specific*** acoustic impedance $\vec{\tilde{z}}_a\left(\vec{r},\omega\right) = \tilde{p}\left(\vec{r},\omega\right)/\vec{\tilde{u}}\left(\vec{r},\omega\right)$ at the point $\vec{r}$ happens to be very ***high***, for a fixed complex scalar over-pressure $\tilde{p}\left(\vec{r},\omega\right)$, this tells us that the complex 3-D vector particle velocity $\vec{\tilde{u}}\left(\vec{r},\omega\right)$ at that point must therefore be very ***small***, and hence the corresponding complex 3-D vector acoustic mass current density $\vec{\tilde{J}}_a\left(\vec{r},\omega\right) = \rho_o \vec{\tilde{u}}\left(\vec{r},\omega\right)$ at that point must also be very ***small***.

Conversely, if for a harmonic sound field the complex 3-D vector ***specific*** acoustic impedance $\vec{\tilde{z}}_a\left(\vec{r},\omega\right) = \tilde{p}\left(\vec{r},\omega\right)/\vec{\tilde{u}}\left(\vec{r},\omega\right)$ at the point $\vec{r}$ happens to be very ***low***, for a fixed complex scalar over-pressure $\tilde{p}\left(\vec{r},\omega\right)$, this tells us that the complex 3-D vector particle velocity $\vec{\tilde{u}}\left(\vec{r},\omega\right)$ at that point must therefore be very ***large***, and hence the corresponding complex 3-D vector acoustic mass current density $\vec{\tilde{J}}_a\left(\vec{r},\omega\right) = \rho_o \vec{\tilde{u}}\left(\vec{r},\omega\right)$ at that point must also be very ***large***.


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For a harmonic sound field, if the complex 3-D vector ***specific*** admittance $\tilde{\vec{y}}_a(\vec{r},\omega) = \tilde{\vec{u}}(\vec{r},\omega)/\tilde{p}(\vec{r},\omega) = 1/\tilde{\vec{z}}_a(\vec{r},\omega)$ at the point $\vec{r}$ happens to be very ***high***, for a fixed complex scalar over-pressure $\tilde{p}(\vec{r},\omega)$, this tells us that the complex 3-D vector particle velocity $\tilde{\vec{u}}(\vec{r},\omega)$ at that point must therefore be very ***large***, and hence the corresponding complex 3-D vector acoustic mass current density $\tilde{\vec{J}}_a(\vec{r},\omega) = \rho_o\tilde{\vec{u}}(\vec{r},\omega)$ at that point must also be very ***large***.

Conversely, if for a harmonic sound field the complex 3-D vector ***specific*** acoustic admittance $\tilde{\vec{y}}_a(\vec{r},\omega) = \tilde{\vec{u}}(\vec{r},\omega)/\tilde{p}(\vec{r},\omega) = 1/\tilde{\vec{z}}_a(\vec{r},\omega)$ at the point $\vec{r}$ happens to be very ***low***, for a fixed complex scalar over-pressure $\tilde{p}(\vec{r},\omega)$, this tells us that the complex 3-D vector particle velocity $\tilde{\vec{u}}(\vec{r},\omega)$ at that point must therefore be very ***small***, and hence the corresponding complex 3-D vector acoustic mass current density $\tilde{\vec{J}}_a(\vec{r},\omega) = \rho_o\tilde{\vec{u}}(\vec{r},\omega)$ at that point must also be very ***small***.

**The Real and Imaginary Components of Complex 3-D Vector *Specific* Acoustic Immittances:**

As in the case for *AC* electrical circuits, the complex scalar electrical impedance $\tilde{Z}_e$ and complex scalar electrical admittance $\tilde{Y}_e = 1/\tilde{Z}_e$ can be written out explicitly in terms of their real and imaginary components:

$\tilde{Z}_e \equiv R_e + iX_e\ (\Omega)$ where $R_e = \text{Re}\{\tilde{Z}_e\}$ is the ***resistance*** and $X_e = \text{Im}\{\tilde{Z}_e\}$ is the ***reactance***.
$\tilde{Y}_e \equiv G_e + iB_e\ (\Omega^{-1})$ where $G_e = \text{Re}\{\tilde{Y}_e\}$ is the ***conductance*** and $B_e = \text{Im}\{\tilde{Y}_e\}$ is the ***susceptance***.

Similarly, for the case a complex harmonic sound field $\tilde{S}(\vec{r})$, the complex 3-D vector ***specific*** acoustic impedance $\tilde{\vec{z}}_a(\vec{r})$ and complex 3-D ***specific*** acoustic admittance $\tilde{\vec{y}}_a(\vec{r}) = 1/\tilde{\vec{z}}_a(\vec{r})$ can be written out explicitly in terms of their real and imaginary components:

$$\tilde{\vec{z}}_a(\vec{r},\omega) \equiv \vec{r}_a(\vec{r},\omega) + i\vec{\chi}_a(\vec{r},\omega)\ \ (\Omega_a) \text{ where:}$$

$\vec{r}_a(\vec{r},\omega) = \text{Re}\{\tilde{\vec{z}}_a(\vec{r},\omega)\}$ is the 3-D ***specific*** acoustic ***resistance*** at the point $\vec{r}$ and:
$\chi_a(\vec{r},\omega) = \text{Im}\{\tilde{z}_a(\vec{r},\omega)\}$ is the 3-D ***specific*** acoustic ***reactance*** at the point $\vec{r}$.

$$\tilde{\vec{y}}_a(\vec{r},\omega) \equiv \vec{g}_a(\vec{r},\omega) + i\vec{b}_a(\vec{r},\omega)\ \ (\Omega_a^{-1}) \text{ where:}$$

$\vec{g}_a(\vec{r},\omega) = \text{Re}\{\tilde{\vec{y}}_a(\vec{r},\omega)\}$ is the 3-D ***specific*** acoustic ***conductance*** at the point $\vec{r}$ and:
$\vec{b}_a(\vec{r},\omega) = \text{Im}\{\tilde{\vec{y}}_a(\vec{r},\omega)\}$ is the 3-D ***specific*** acoustic ***susceptance*** at the point $\vec{r}$.


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For harmonic/single-frequency sound fields, we can obtain expressions for the real and imaginary parts of frequency-domain complex 3-D vector ***specific*** acoustic impedance $\vec{\tilde{z}}_a(\vec{r},\omega)$ and admittance $\vec{\tilde{y}}_a(\vec{r},\omega)$ in terms of the real and imaginary parts of complex scalar over-pressure $\tilde{p}(\vec{r},\omega)$ and complex 3-D vector particle velocity $\vec{\tilde{u}}(\vec{r},\omega)$ from their respective definitions $\vec{\tilde{z}}_a(\vec{r},\omega)=\tilde{p}(\vec{r},\omega)/\vec{\tilde{u}}(\vec{r},\omega)$ and $\vec{\tilde{y}}_a(\vec{r},\omega)=\vec{\tilde{u}}(\vec{r},\omega)/\tilde{p}(\vec{r},\omega)=1/\vec{\tilde{z}}_a(\vec{r},\omega)$.

Suppressing the frequency-domain argument $(\vec{r},\omega)$ for notational clarity's sake, and working with only one of the three vectorial components $k = x, y, \text{or } z$, for complex 3-D vector ***specific*** acoustic admittance:

$$\tilde{y}_{a_k} = y^{\mathrm{r}}_{a_k} + iy^{\mathrm{i}}_{a_k} = \frac{\tilde{u}_k}{\tilde{p}} = \frac{u_{\mathrm{r}_k}+iu_{\mathrm{i}_k}}{p_{\mathrm{r}}+ip_{\mathrm{i}}} = \left(\frac{u_{\mathrm{r}_k}+iu_{\mathrm{i}_k}}{p_{\mathrm{r}}+ip_{\mathrm{i}}}\right)\cdot\left(\frac{p_{\mathrm{r}}-ip_{\mathrm{i}}}{p_{\mathrm{r}}-ip_{\mathrm{i}}}\right) = \left(\frac{p_{\mathrm{r}}u_{\mathrm{r}_k}+p_{\mathrm{i}}u_{\mathrm{i}_k}}{|\tilde{p}|^2}\right) + i\left(\frac{p_{\mathrm{r}}u_{\mathrm{i}_k}-p_{\mathrm{i}}u_{\mathrm{r}_k}}{|\tilde{p}|^2}\right)$$

Thus we see that for $k = x, y, \text{or } z$:

$$y^{\mathrm{r}}_{a_k} = \mathrm{Re}\{\tilde{y}_{a_k}\} = \frac{p_{\mathrm{r}}u_{\mathrm{r}_k}+p_{\mathrm{i}}u_{\mathrm{i}_k}}{|\tilde{p}|^2} \text{ and: } y^{\mathrm{i}}_{a_k} = \mathrm{Im}\{\tilde{y}_{a_k}\} = \frac{p_{\mathrm{r}}u_{\mathrm{i}_k}-p_{\mathrm{i}}u_{\mathrm{r}_k}}{|\tilde{p}|^2} = -\frac{p_{\mathrm{i}}u_{\mathrm{r}_k}-p_{\mathrm{r}}u_{\mathrm{i}_k}}{|\tilde{p}|^2}$$

Likewise, for complex 3-D vector ***specific*** acoustic impedance:

$$\tilde{z}_{a_k} = z^{\mathrm{r}}_{a_k} + iz^{\mathrm{i}}_{a_k} = \frac{\tilde{p}\cdot\tilde{u}^*_k}{|\vec{\tilde{u}}|^2} = \frac{(p_{\mathrm{r}}+ip_{\mathrm{i}})(u_{\mathrm{r}_k}+iu_{\mathrm{i}_k})^*}{|\vec{\tilde{u}}|^2} = \frac{(p_{\mathrm{r}}+ip_{\mathrm{i}})(u_{\mathrm{r}_k}-iu_{\mathrm{i}_k})}{|\vec{\tilde{u}}|^2} = \left(\frac{p_{\mathrm{r}}u_{\mathrm{r}_k}+p_{\mathrm{i}}u_{\mathrm{i}_k}}{|\vec{\tilde{u}}|^2}\right) + i\left(\frac{p_{\mathrm{i}}u_{\mathrm{r}_k}-p_{\mathrm{r}}u_{\mathrm{i}_k}}{|\vec{\tilde{u}}|^2}\right)$$

Thus, we see that for $k = x, y, \text{or } z$:

$$z^{\mathrm{r}}_{a_k} = \mathrm{Re}\{\tilde{z}_{a_k}\} = \frac{p_{\mathrm{r}}u_{\mathrm{r}_k}+p_{\mathrm{i}}u_{\mathrm{i}_k}}{|\vec{\tilde{u}}|^2} \text{ and: } z^{\mathrm{i}}_{a_k} = \mathrm{Im}\{\tilde{z}_{a_k}\} = \frac{p_{\mathrm{i}}u_{\mathrm{r}_k}-p_{\mathrm{r}}u_{\mathrm{i}_k}}{|\vec{\tilde{u}}|^2}$$

Noting that: $|\tilde{y}_{a_k}|^2 = \tilde{y}_{a_k}\bullet\tilde{y}^*_{a_k} = \frac{\tilde{u}_k}{\tilde{p}}\bullet\frac{\tilde{u}^*_k}{\tilde{p}^*} = \frac{|\tilde{u}_k|^2}{|\tilde{p}|^2}$ and that: $|\tilde{z}_{a_k}|^2 = \tilde{z}_{a_k}\bullet\tilde{z}^*_{a_k} = \frac{\tilde{p}\tilde{u}^*_k}{|\vec{\tilde{u}}|^2}\bullet\frac{\tilde{p}^*\tilde{u}_k}{|\vec{\tilde{u}}|^2} = \frac{|\tilde{p}|^2|\tilde{u}_k|^2}{\left(|\vec{\tilde{u}}|^2\right)^2}$

We see that: $|\vec{\tilde{u}}|^2 z^{\mathrm{r}}_{a_k} = p_{\mathrm{r}}u_{\mathrm{r}_k}+p_{\mathrm{i}}u_{\mathrm{i}_k} = |\tilde{p}|^2 y^{\mathrm{r}}_{a_k}$ and that: $|\vec{\tilde{u}}|^2 z^{\mathrm{i}}_{a_k} = p_{\mathrm{i}}u_{\mathrm{r}_k}-p_{\mathrm{r}}u_{\mathrm{i}_k} = -|\tilde{p}|^2 y^{\mathrm{i}}_{a_k}$

or equivalently that: $z^{\mathrm{r}}_{a_k} = |\vec{\tilde{z}}_a|^2 y^{\mathrm{r}}_{a_k}$ or: $y^{\mathrm{r}}_{a_k} = |\vec{\tilde{y}}_a|^2 z^{\mathrm{r}}_{a_k}$ and that: $z^{\mathrm{i}}_{a_k} = -|\vec{\tilde{z}}_a|^2 y^{\mathrm{i}}_{a_k}$ or: $y^{\mathrm{i}}_{a_k} = -|\vec{\tilde{y}}_a|^2 z^{\mathrm{i}}_{a_k}$


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Thus, we see that for a given $k = x, y,$ or $z$ component of $\vec{\tilde{z}}_a(\vec{r},\omega)$:

$$\boxed{z_{a_k}^{\mathrm{r}} = \mathrm{Re}\{\tilde{z}_{a_k}\} = \frac{p_{\mathrm{r}}u_{\mathrm{r}_k} + p_{\mathrm{i}}u_{\mathrm{i}_k}}{|\vec{\tilde{u}}|^2}} \text{ and: } \boxed{z_{a_k}^{\mathrm{i}} = \mathrm{Im}\{\tilde{z}_{a_k}\} = \frac{p_{\mathrm{i}}u_{\mathrm{r}_k} - p_{\mathrm{r}}u_{\mathrm{i}_k}}{|\vec{\tilde{u}}|^2}}$$

and we see that for a given $k = x, y,$ or $z$ component of $\vec{\tilde{y}}_a(\vec{r},\omega)$:

$$\boxed{y_{a_k}^{\mathrm{r}} = \mathrm{Re}\{\tilde{y}_{a_k}\} = \frac{p_{\mathrm{r}}u_{\mathrm{r}_k} + p_{\mathrm{i}}u_{\mathrm{i}_k}}{|\tilde{p}|^2}} \text{ and: } \boxed{y_{a_k}^{\mathrm{i}} = \mathrm{Im}\{\tilde{y}_{a_k}\} = \frac{p_{\mathrm{r}}u_{\mathrm{i}_k} - p_{\mathrm{i}}u_{\mathrm{r}_k}}{|\tilde{p}|^2} = -\frac{p_{\mathrm{i}}u_{\mathrm{r}_k} - p_{\mathrm{r}}u_{\mathrm{i}_k}}{|\tilde{p}|^2}}$$

as well as: $\tilde{z}_{a_k} = \dfrac{\tilde{y}_{a_k}^*}{|\vec{\tilde{y}}_a|^2}$ and: $\tilde{y}_{a_k} = \dfrac{\tilde{z}_{a_k}^*}{|\vec{\tilde{z}}_a|^2}$ or equivalently: $\tilde{y}_{a_k} = |\vec{\tilde{y}}_a|^2\,\tilde{z}_{a_k}^*$ and: $\tilde{z}_{a_k} = |\vec{\tilde{z}}_a|^2\,\tilde{y}_{a_k}^*$.

It can be seen from these definitions that in **<u>general</u>** the individual vectorial components $k = x, y,$ or $z$ that: $\tilde{z}_{a_k}(\vec{r},\omega)$ and $\tilde{y}_{a_k}(\vec{r},\omega)$ do ***<u>not</u>*** point in the same direction in space.

Since $\vec{\tilde{z}}_a(\vec{r},\omega) = \tilde{p}(\vec{r},\omega)/\vec{\tilde{u}}(\vec{r},\omega)$, another useful relation is: $\vec{\tilde{z}}_a(\vec{r},\omega)\bullet\vec{\tilde{u}}(\vec{r},\omega) = \tilde{p}(\vec{r},\omega)$:

$$\vec{\tilde{z}}_a(\vec{r},\omega)\bullet\vec{\tilde{u}}(\vec{r},\omega) = \left[\frac{\tilde{p}(\vec{r},\omega)}{\vec{\tilde{u}}(\vec{r},\omega)}\right]\bullet\vec{\tilde{u}}(\vec{r},\omega) = \left[\frac{\tilde{p}(\vec{r},\omega)\vec{\tilde{u}}^*(\vec{r},\omega)}{|\vec{\tilde{u}}(\vec{r},\omega)|^2}\right]\bullet\vec{\tilde{u}}(\vec{r},\omega) = \frac{\tilde{p}(\vec{r},\omega)\,\cancel{|\vec{\tilde{u}}(\vec{r},\omega)|^2}}{\cancel{|\vec{\tilde{u}}(\vec{r},\omega)|^2}}$$

$$= \tilde{p}(\vec{r},\omega)$$

Similarly, since $\vec{\tilde{y}}_a(\vec{r},\omega) = \vec{\tilde{u}}(\vec{r},\omega)/\tilde{p}(\vec{r},\omega)$, then: $\vec{\tilde{y}}_a(\vec{r},\omega)\,\tilde{p}(\vec{r},\omega) = \vec{\tilde{u}}(\vec{r},\omega)$.

Note that the above expressions for the real and imaginary components of complex acoustic specific impedance and/or admittance given in terms of linear combinations of the real and imaginary components of complex scalar acoustic over-pressure and complex vector particle velocity. As we have discussed previously, the physical meaning of the real and imaginary components of complex scalar acoustic over-pressure and complex vector particle velocity are respectively the in-phase and 90$^{\mathrm{o}}$ (quadrature) components relative to the driving sound source. However, this is ***<u>not</u>*** the physical meaning of the real and imaginary components of complex acoustic specific immittances, because of the above-defined linear combinations of complex scalar acoustic over-pressure and complex vector particle velocity. We shall see/learn that the physical meaning of the real and imaginary components of complex acoustic immittances – properties of the physical medium in which acoustic disturbances propagate – are respectively associated with the ***<u>propagating</u>*** and ***<u>non</u>-<u>propagating</u>*** components of acoustic energy density.

The real and imaginary components of the acoustic specific immittances are often called the ***<u>active</u>*** and ***<u>reactive</u>*** components of the complex sound field, respectively, since (see above):

$$\boxed{\vec{\tilde{z}}_a(\vec{r},\omega) \equiv \vec{r}_a(\vec{r},\omega) + i\vec{\chi}_a(\vec{r},\omega) \quad (\Omega_a)} \text{ and: } \boxed{\vec{\tilde{y}}_a(\vec{r},\omega) \equiv \vec{g}_a(\vec{r},\omega) + i\vec{b}_a(\vec{r},\omega) \quad (\Omega_a^{-1})}$$


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We can gain further/additional insight into the nature of complex $\vec{\tilde{z}}_a(\vec{r},\omega)$ and $\vec{\tilde{y}}_a(\vec{r},\omega)$ by writing our primary acoustic ***frequency-domain*** variables in complex ***polar*** notation form:

**Complex scalar pressure:**

$$\tilde{p}(\vec{r},\omega) = p_{\mathrm{r}}(\vec{r},\omega) + ip_{\mathrm{i}}(\vec{r},\omega) = |\tilde{p}(\vec{r})| e^{i\varphi_p(\vec{r},\omega)}$$

**Complex 3-D vector particle velocity:**

$$\begin{aligned}\vec{\tilde{u}}(\vec{r},\omega) &= \vec{u}_{\mathrm{r}}(\vec{r},\omega) + i\vec{u}_{\mathrm{i}}(\vec{r},\omega)\\ &= \left[u_{\mathrm{r}_x}(\vec{r},\omega) + iu_{\mathrm{i}_x}(\vec{r},\omega)\right]\hat{x} + \left[u_{\mathrm{r}_y}(\vec{r},\omega) + iu_{\mathrm{i}_y}(\vec{r},\omega)\right]\hat{y} + \left[u_{\mathrm{r}_z}(\vec{r},\omega) + iu_{\mathrm{i}_z}(\vec{r},\omega)\right]\hat{z}\\ &= |\tilde{u}_x(\vec{r},\omega)| e^{i\varphi_{u_x}(\vec{r},\omega)}\hat{x} + |\tilde{u}_y(\vec{r},\omega)| e^{i\varphi_{u_y}(\vec{r},\omega)}\hat{y} + |\tilde{u}_z(\vec{r},\omega)| e^{i\varphi_{u_z}(\vec{r},\omega)}\hat{z}\end{aligned}$$

**Complex 3-D vector *specific* acoustic admittance:**

$$\begin{aligned}\vec{\tilde{y}}_a(\vec{r},\omega) &= \vec{y}_{\mathrm{r}}(\vec{r},\omega) + i\vec{y}_{\mathrm{i}}(\vec{r},\omega)\\ &= \left[y_{\mathrm{r}_x}(\vec{r},\omega) + iy_{\mathrm{i}_x}(\vec{r},\omega)\right]\hat{x} + \left[y_{\mathrm{r}_y}(\vec{r},\omega) + iy_{\mathrm{i}_y}(\vec{r},\omega)\right]\hat{y} + \left[y_{\mathrm{r}_z}(\vec{r},\omega) + iy_{\mathrm{i}_z}(\vec{r},\omega)\right]\hat{z}\\ &= |\tilde{y}_x(\vec{r},\omega)| e^{i\varphi_{y_x}(\vec{r},\omega)}\hat{x} + |\tilde{y}_y(\vec{r},\omega)| e^{i\varphi_{y_y}(\vec{r},\omega)}\hat{y} + |\tilde{y}_z(\vec{r},\omega)| e^{i\varphi_{y_z}(\vec{r},\omega)}\hat{z}\end{aligned}$$

**Complex 3-D vector *specific* acoustic impedance:**

$$\begin{aligned}\vec{\tilde{z}}_a(\vec{r},\omega) &= \vec{z}_{\mathrm{r}}(\vec{r},\omega) + i\vec{z}_{\mathrm{i}}(\vec{r},\omega)\\ &= \left[z_{\mathrm{r}_x}(\vec{r},\omega) + iz_{\mathrm{i}_x}(\vec{r},\omega)\right]\hat{x} + \left[z_{\mathrm{r}_y}(\vec{r},\omega) + iz_{\mathrm{i}_y}(\vec{r},\omega)\right]\hat{y} + \left[z_{\mathrm{r}_z}(\vec{r},\omega) + iz_{\mathrm{i}_z}(\vec{r},\omega)\right]\hat{z}\\ &= |\tilde{z}_x(\vec{r},\omega)| e^{i\varphi_{z_x}(\vec{r},\omega)}\hat{x} + |\tilde{z}_y(\vec{r},\omega)| e^{i\varphi_{z_y}(\vec{r},\omega)}\hat{y} + |\tilde{z}_z(\vec{r},\omega)| e^{i\varphi_{z_z}(\vec{r},\omega)}\hat{z}\end{aligned}$$

Thus, for harmonic/single-frequency sound fields we see that for a given $k = x, y,$ or $z$ component of $\vec{\tilde{y}}_a(\vec{r},\omega)$, that:

$$\tilde{y}_{a_k}(\vec{r},\omega) = \frac{\tilde{u}_k(\vec{r},\omega)}{\tilde{p}(\vec{r},\omega)} \;\Rightarrow\; |\tilde{y}_{a_k}| e^{i\varphi_{y_k}} = \frac{|\tilde{u}_k| e^{i\varphi_{u_k}}}{|\tilde{p}| e^{i\varphi_p}} = \frac{|\tilde{u}_k|}{|\tilde{p}|} e^{-i\varphi_p} \cdot e^{i\varphi_{u_k}} = |\tilde{y}_{a_k}| e^{i\left[\varphi_{u_k} - \varphi_p\right]} = |\tilde{y}_{a_k}| e^{-i\Delta\varphi_{p-u_k}}$$

Similarly, for a given $k = x, y,$ or $z$ component of $\vec{\tilde{z}}_a(\vec{r},\omega)$:

$$\tilde{z}_{a_k}(\vec{r},\omega) = \frac{\tilde{p}(\vec{r},\omega)\tilde{u}_k^*(\vec{r},\omega)}{|\tilde{u}(\vec{r},\omega)|^2} \;\Rightarrow\; |\tilde{z}_{a_k}| e^{i\varphi_{z_k}} = \frac{|\tilde{p}| e^{i\varphi_p} |\tilde{u}_k| e^{-i\varphi_{u_k}}}{|\vec{\tilde{u}}(\vec{r},\omega)|^2} = \frac{|\tilde{p}||\tilde{u}_k|}{|\vec{\tilde{u}}(\vec{r},\omega)|^2} e^{i\left[\varphi_p - \varphi_{u_k}\right]} = |\tilde{z}_{a_k}| e^{i\Delta\varphi_{p-u_k}}$$


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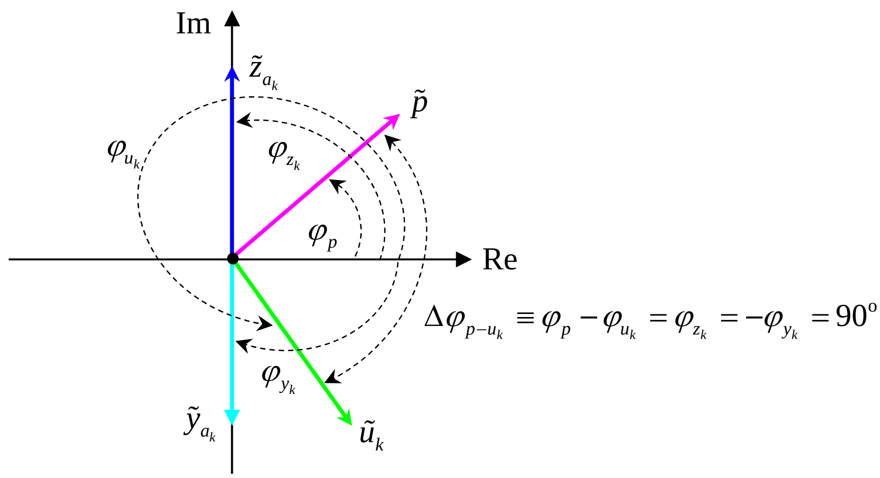
We also see that for harmonic/single-frequency sound fields the $z_k$ -phase: $\varphi_{z_k} = \Delta\varphi_{p-u_k} \equiv \varphi_p - \varphi_{u_k}$ whereas the $y_k$ -phase: $\varphi_{y_k} = \Delta\varphi_{u_k-p} \equiv \varphi_{u_k} - \varphi_p = -\left(\varphi_p - \varphi_{u_k}\right) = -\varphi_{z_k}$ , in analogy to similar relations obtained *e.g*. for complex *AC* electrical circuits!

The phasor relation(s) in the complex plane for $\tilde{p} = p_r + ip_i = |\tilde{p}|e^{i\varphi_p}$ , $\tilde{u}_k = u_{r_k} + iu_{i_k} = |\tilde{u}_k|e^{i\varphi_{u_k}}$ , $\tilde{z}_{a_k} = z^r_{a_k} + iz^i_{a_k} = |\tilde{z}_{a_k}|e^{i\varphi_{z_k}}$ and $\tilde{y}_{a_k} = y^r_{a_k} + iy^i_{a_k} = |\tilde{y}_{a_k}|e^{i\varphi_{y_k}}$ are shown in the figure below, for the special/limiting case of $\Delta\varphi_{p-u_k} \equiv \varphi_p - \varphi_{u_k} = \varphi_{z_k} = -\varphi_{y_k} = 90^o$ , where the impedance phasor component $\tilde{z}_{a_k}$ is back-to-back with the admittance phasor component $\tilde{y}_{a_k}$ {*n.b.* for the more general case where $\Delta\varphi_{p-u_k} \equiv \varphi_p - \varphi_{u_k} = \varphi_{z_k} = -\varphi_{y_k} \neq 90^o$ , then $\tilde{z}_{a_k}$ and $\tilde{y}_{a_k}$ are ***not*** back-to-back}:




If we now take the *cosine* of the two phases $\varphi_{z_k}$ and $\varphi_{y_k}$ :

$$\cos\varphi_{z_k} = \cos\Delta\varphi_{p-u_k} \equiv \cos\left(\varphi_p - \varphi_{u_k}\right) \quad \text{and:}$$

$$\cos\varphi_{y_k} = \cos\Delta\varphi_{u_k-p} \equiv \cos\left(\varphi_{u_k} - \varphi_p\right) = \cos\left[-\varphi_{z_k}\right] = \cos\varphi_{z_k} \ (\cos(x) \text{ even fcn}(x))$$

We see that when: $\cos\varphi_{z_k} = \cos\varphi_{y_k} = +1$ that: $\Delta\varphi_{p-u_k} = -\Delta\varphi_{u_k-p} = 0^o$ , *i.e*. that: $\varphi_p = \varphi_{u_k}$ .

When: $\cos\varphi_{z_k} = \cos\varphi_{y_k} = 0$ that: $\Delta\varphi_{p-u_k} = -\Delta\varphi_{u_k-p} = \pm 90^o$ , *i.e*. that: $\varphi_p = \varphi_{u_k} \pm 90^o$ .

When: $\cos\varphi_{z_k} = \cos\varphi_{y_k} = -1$ that: $\Delta\varphi_{p-u_k} = -\Delta\varphi_{u_k-p} = \pm 180^o$ , *i.e*. that: $\varphi_p = \varphi_{u_k} \pm 180^o$ .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**Summary of Various Frequency-Domain Sound Field Physical Quantities:**

**Complex scalar pressure:**

$$\tilde{p}(\vec{r},\omega) = p_{\mathrm{r}}(\vec{r},\omega) + i p_{\mathrm{i}}(\vec{r},\omega) = \left|\tilde{p}(\vec{r})\right| e^{i\varphi_p(\vec{r},\omega)}$$

**Complex 3-D vector particle displacement:**

$$\begin{aligned}\tilde{\vec{\xi}}(\vec{r},\omega) &= \vec{\xi}_{\mathrm{r}}(\vec{r},\omega) + i\vec{\xi}_{\mathrm{i}}(\vec{r},\omega) \\ &= \left[\xi_{\mathrm{r}_x}(\vec{r},\omega) + i\xi_{\mathrm{i}_x}(\vec{r},\omega)\right]\hat{x} + \left[\xi_{\mathrm{r}_y}(\vec{r},\omega) + i\xi_{\mathrm{i}_y}(\vec{r},\omega)\right]\hat{y} + \left[\xi_{\mathrm{r}_z}(\vec{r},\omega) + i\xi_{\mathrm{i}_z}(\vec{r},\omega)\right]\hat{z} \\ &= \left|\tilde{\xi}_x(\vec{r},\omega)\right| e^{i\varphi_{\xi_x}(\vec{r},\omega)}\hat{x} + \left|\tilde{\xi}_y(\vec{r},\omega)\right| e^{i\varphi_{\xi_y}(\vec{r},\omega)}\hat{y} + \left|\tilde{\xi}_z(\vec{r},\omega)\right| e^{i\varphi_{\xi_z}(\vec{r},\omega)}\hat{z}\end{aligned}$$

**Complex 3-D vector particle velocity:**

$$\begin{aligned}\tilde{\vec{u}}(\vec{r},\omega) &= \vec{u}_{\mathrm{r}}(\vec{r},\omega) + i\vec{u}_{\mathrm{i}}(\vec{r},\omega) \\ &= \left[u_{\mathrm{r}_x}(\vec{r},\omega) + iu_{\mathrm{i}_x}(\vec{r},\omega)\right]\hat{x} + \left[u_{\mathrm{r}_y}(\vec{r},\omega) + iu_{\mathrm{i}_y}(\vec{r},\omega)\right]\hat{y} + \left[u_{\mathrm{r}_z}(\vec{r},\omega) + iu_{\mathrm{i}_z}(\vec{r},\omega)\right]\hat{z} \\ &= \left|\tilde{u}_x(\vec{r},\omega)\right| e^{i\varphi_{u_x}(\vec{r},\omega)}\hat{x} + \left|\tilde{u}_y(\vec{r},\omega)\right| e^{i\varphi_{u_y}(\vec{r},\omega)}\hat{y} + \left|\tilde{u}_z(\vec{r},\omega)\right| e^{i\varphi_{u_z}(\vec{r},\omega)}\hat{z}\end{aligned}$$

**Complex 3-D vector particle acceleration:**

$$\begin{aligned}\tilde{\vec{a}}(\vec{r},\omega) &= \vec{a}_{\mathrm{r}}(\vec{r},\omega) + i\vec{a}_{\mathrm{i}}(\vec{r},\omega) \\ &= \left[a_{\mathrm{r}_x}(\vec{r},\omega) + ia_{\mathrm{i}_x}(\vec{r},\omega)\right]\hat{x} + \left[a_{\mathrm{r}_y}(\vec{r},\omega) + ia_{\mathrm{i}_y}(\vec{r},\omega)\right]\hat{y} + \left[a_{\mathrm{r}_x}(\vec{r},\omega) + ia_{\mathrm{i}_x}(\vec{r},\omega)\right]\hat{z} \\ &= \left|\tilde{a}_x(\vec{r},\omega)\right| e^{i\varphi_{a_x}(\vec{r},\omega)}\hat{x} + \left|\tilde{a}_y(\vec{r},\omega)\right| e^{i\varphi_{a_y}(\vec{r},\omega)}\hat{y} + \left|\tilde{a}_z(\vec{r},\omega)\right| e^{i\varphi_{a_z}(\vec{r},\omega)}\hat{z}\end{aligned}$$

**Complex 3-D vector *specific* acoustic admittance:**

$$\begin{aligned}\tilde{\vec{y}}_a(\vec{r},\omega) &= \vec{y}_{\mathrm{r}}(\vec{r},\omega) + i\vec{y}_{\mathrm{i}}(\vec{r},\omega) \\ &= \left[y_{\mathrm{r}_x}(\vec{r},\omega) + iy_{\mathrm{i}_x}(\vec{r},\omega)\right]\hat{x} + \left[y_{\mathrm{r}_y}(\vec{r},\omega) + iy_{\mathrm{i}_y}(\vec{r},\omega)\right]\hat{y} + \left[y_{\mathrm{r}_z}(\vec{r},\omega) + iy_{\mathrm{i}_z}(\vec{r},\omega)\right]\hat{z} \\ &= \left|\tilde{y}_x(\vec{r},\omega)\right| e^{i\varphi_{y_x}(\vec{r},\omega)}\hat{x} + \left|\tilde{y}_y(\vec{r},\omega)\right| e^{i\varphi_{y_y}(\vec{r},\omega)}\hat{y} + \left|\tilde{y}_z(\vec{r},\omega)\right| e^{i\varphi_{y_z}(\vec{r},\omega)}\hat{z}\end{aligned}$$

**Complex 3-D vector *specific* acoustic impedance:**

$$\begin{aligned}\tilde{\vec{z}}_a(\vec{r},\omega) &= \vec{z}_{\mathrm{r}}(\vec{r},\omega) + i\vec{z}_{\mathrm{i}}(\vec{r},\omega) \\ &= \left[z_{\mathrm{r}_x}(\vec{r},\omega) + iz_{\mathrm{i}_x}(\vec{r},\omega)\right]\hat{x} + \left[z_{\mathrm{r}_y}(\vec{r},\omega) + iz_{\mathrm{i}_y}(\vec{r},\omega)\right]\hat{y} + \left[z_{\mathrm{r}_z}(\vec{r},\omega) + iz_{\mathrm{i}_z}(\vec{r},\omega)\right]\hat{z} \\ &= \left|\tilde{z}_x(\vec{r},\omega)\right| e^{i\varphi_{z_x}(\vec{r},\omega)}\hat{x} + \left|\tilde{z}_y(\vec{r},\omega)\right| e^{i\varphi_{z_y}(\vec{r},\omega)}\hat{y} + \left|\tilde{z}_z(\vec{r},\omega)\right| e^{i\varphi_{z_z}(\vec{r},\omega)}\hat{z}\end{aligned}$$


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For "everyday" harmonic/single-frequency sound fields, if the 3-D vector complex ***frequency-domain*** particle velocity amplitude $\vec{\tilde{u}}(\vec{r},\omega)$ is known/measured, then since the 3-D vector complex ***time-domain*** particle velocity $\vec{\tilde{u}}(\vec{r},t)=\vec{\tilde{u}}(\vec{r},\omega)\cdot e^{i\omega t}$, and the 3-D vector complex ***time-domain*** particle displacement $\vec{\tilde{\xi}}(\vec{r},t)=\vec{\tilde{\xi}}(\vec{r},\omega)\cdot e^{i\omega t}$, where: $\vec{\tilde{\xi}}(\vec{r},\omega)$ is the 3-D vector complex ***frequency-domain*** particle displacement amplitude, and since $\vec{\tilde{u}}(\vec{r},t)=\partial\vec{\tilde{\xi}}(\vec{r},t)/\partial t$, then:

$$\vec{\tilde{\xi}}(\vec{r},t)=\int\vec{\tilde{u}}(\vec{r},t)\,dt=\int\vec{\tilde{u}}(\vec{r},\omega)\cdot e^{i\omega t}dt=\vec{\tilde{u}}(\vec{r},\omega)\int\cdot e^{i\omega t}dt=\frac{1}{i\omega}\vec{\tilde{u}}(\vec{r},\omega)\cdot e^{i\omega t}$$

But since: $\vec{\tilde{\xi}}(\vec{r},t)=\vec{\tilde{\xi}}(\vec{r},\omega)\cdot e^{i\omega t}$, we see that: $\boxed{\vec{\tilde{\xi}}(\vec{r},\omega)=\frac{1}{i\omega}\vec{\tilde{u}}(\vec{r},\omega)=-i\frac{1}{\omega}\vec{\tilde{u}}(\vec{r},\omega)}$

Likewise, since: $\vec{\tilde{a}}(\vec{r},t)=\frac{\partial\vec{\tilde{u}}(\vec{r},t)}{\partial t}=\frac{\partial\vec{\tilde{u}}(\vec{r},\omega)\cdot e^{i\omega t}}{\partial t}=\vec{\tilde{u}}(\vec{r},\omega)\cdot\frac{\partial e^{i\omega t}}{\partial t}=i\omega\cdot\vec{\tilde{u}}(\vec{r},\omega)\cdot e^{i\omega t}$

But since: $\vec{\tilde{a}}(\vec{r},t)=\vec{\tilde{a}}(\vec{r},\omega)\cdot e^{i\omega t}$, we also see that: $\boxed{\vec{\tilde{a}}(\vec{r},\omega)=i\omega\cdot\vec{\tilde{u}}(\vec{r},\omega)}$


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