Examples of Complex Sound Fields:

Example # 0: "Generic"3-D Monochromatic Traveling Wave:

Before we launch into discussing several specific examples of complex sound fields/sound propagation, it is useful/illuminating to first consider the more general case of a "generic" complex sound field associated with a 3-D monochromatic traveling wave. Again, we assume that we are working in the ***linear*** regime of "everyday" sound pressure levels $SPL \ll 134\,dB \ \left(|\tilde{p}| \ll 100\,Pa\right)$ and also can safely ignore/neglect any/all dissipative effects, such that the Euler equation for inviscid fluid flow is a valid/accurate description of the acoustical physics situation. Then:

The complex ***time-domain*** over-pressure amplitude $\tilde{p}(\vec{r},t)$ associated with a "generic" 3-D monochromatic traveling wave at the listener space-time point $(\vec{r},t)$ can be written as:

$$\tilde{p}(\vec{r},t) = \left|\tilde{p}_o(\vec{r},\omega)\right| e^{i(\omega t+\varphi_p(\vec{r},\omega))} = \underbrace{\left|\tilde{p}_o(\vec{r},\omega)\right| \cdot e^{i\varphi_p(\vec{r},\omega)}}_{\equiv \tilde{p}(\vec{r},\omega)} \cdot e^{i\omega t} = \tilde{p}(\vec{r},\omega)\cdot e^{i\omega t}$$

where: $\tilde{p}(\vec{r},\omega) = \left|\tilde{p}_o(\vec{r},\omega)\right| \cdot e^{i\varphi_p(\vec{r},\omega)}$ is the corresponding complex ***frequency-domain*** over-pressure amplitude associated with the "generic" 3-D monochromatic traveling wave at the listener space-time point $(\vec{r},t)$. Note that in general, both the magnitude of the complex over-pressure amplitude $\left|\tilde{p}_o(\vec{r},\omega)\right|$ and the phase $\varphi_p(\vec{r},\omega)$ are {listener} position-dependent and {angular} frequency-dependent quantities for a "generic" 3-D monochromatic traveling wave.

The {linearized} Euler equation for inviscid fluid flow (*i.e.* no dissipation) relates the complex ***time-domain*** 3-D particle velocity $\tilde{\vec{u}}(\vec{r},t)$ to the complex ***time-domain*** over-pressure amplitude $\tilde{p}(\vec{r},t)$:

$$\frac{\partial \tilde{\vec{u}}(\vec{r},t)}{\partial t} = -\frac{1}{\rho_o}\vec{\nabla}\tilde{p}(\vec{r},t)$$

In general, for "generic" 3-D monochromatic traveling wave, the complex ***time-domain*** 3-D particle velocity $\tilde{\vec{u}}(\vec{r},t)$ will be of the form: $\tilde{\vec{u}}(\vec{r},t) = \tilde{\vec{u}}(\vec{r},\omega)\cdot e^{i\omega t}$ where $\tilde{\vec{u}}(\vec{r},\omega)$ is the corresponding complex ***frequency-domain*** 3-D particle velocity.

On the LHS of the Euler equation, for a harmonic (*i.e.* monochromatic) complex sound field, since $\tilde{\vec{u}}(\vec{r},t) \propto e^{i\omega t}$, it is easy to show that $\partial\tilde{\vec{u}}(\vec{r},t)/\partial t = i\omega\tilde{\vec{u}}(\vec{r},t)$. Then on the RHS of the Euler equation:

$$\vec{\nabla}\tilde{p}(\vec{r},t) = \vec{\nabla}\tilde{p}(\vec{r},\omega)\cdot e^{i\omega t} = \vec{\nabla}\left[\left|\tilde{p}_o(\vec{r},\omega)\right|\cdot e^{i\varphi_p(\vec{r},\omega)}\right]\cdot e^{i\omega t}$$

Using the chain rule of differentiation, this relation becomes:

$$\vec{\nabla}\tilde{p}(\vec{r},t) = \left[\left\{\vec{\nabla}\left|\tilde{p}_o(\vec{r},\omega)\right|\right\}\cdot e^{i\varphi_p(\vec{r},\omega)} + \left|\tilde{p}_o(\vec{r},\omega)\right|\cdot\vec{\nabla}e^{i\varphi_p(\vec{r},\omega)}\right]\cdot e^{i\omega t}$$

$$= \left[\frac{\left\{\vec{\nabla}\left|\tilde{p}_o(\vec{r},\omega)\right|\right\}}{\left|\tilde{p}_o(\vec{r},\omega)\right|}\left|\tilde{p}_o(\vec{r},\omega)\right|\cdot e^{i\varphi_p(\vec{r},\omega)} + i\left|\tilde{p}_o(\vec{r},\omega)\right|\cdot\left\{\vec{\nabla}\varphi_p(\vec{r},\omega)\right\}e^{i\varphi_p(\vec{r},\omega)}\right]\cdot e^{i\omega t}$$

$$= \left[\frac{\vec{\nabla}\left|\tilde{p}_o(\vec{r},\omega)\right|}{\left|\tilde{p}_o(\vec{r},\omega)\right|} + i\vec{\nabla}\varphi_p(\vec{r},\omega)\right]\underbrace{\left|\tilde{p}_o(\vec{r},\omega)\right|\cdot e^{i\varphi_p(\vec{r},\omega)}\cdot e^{i\omega t}}_{=\tilde{p}(\vec{r},t)}$$

$$= \left[\frac{\vec{\nabla}\left|\tilde{p}_o(\vec{r},\omega)\right|}{\left|\tilde{p}_o(\vec{r},\omega)\right|} + i\vec{\nabla}\varphi_p(\vec{r},\omega)\right]\tilde{p}(\vec{r},t)$$

The Euler equation for this "generic" 3-D monochromatic traveling wave is:

$$i\omega\cdot\tilde{\vec{u}}(\vec{r},t) = -\frac{1}{\rho_o}\left[\frac{\vec{\nabla}\left|\tilde{p}_o(\vec{r},\omega)\right|}{\left|\tilde{p}_o(\vec{r},\omega)\right|} + i\vec{\nabla}\varphi_p(\vec{r},\omega)\right]\tilde{p}(\vec{r},t)$$

or:

$$\tilde{\vec{u}}(\vec{r},t) = -\frac{1}{i\rho_o\omega}\left[\frac{\vec{\nabla}\left|\tilde{p}_o(\vec{r},\omega)\right|}{\left|\tilde{p}_o(\vec{r},\omega)\right|} + i\vec{\nabla}\varphi_p(\vec{r},\omega)\right]\tilde{p}(\vec{r},t) = +\frac{i}{\rho_o\omega}\left[\frac{\vec{\nabla}\left|\tilde{p}_o(\vec{r},\omega)\right|}{\left|\tilde{p}_o(\vec{r},\omega)\right|} + i\vec{\nabla}\varphi_p(\vec{r},\omega)\right]\tilde{p}(\vec{r},t)$$

$$= +\frac{1}{\rho_o\omega}\left[i\frac{\vec{\nabla}\left|\tilde{p}_o(\vec{r},\omega)\right|}{\left|\tilde{p}_o(\vec{r},\omega)\right|} - \vec{\nabla}\varphi_p(\vec{r},\omega)\right]\tilde{p}(\vec{r},t) = -\frac{1}{\rho_o\omega}\left[\vec{\nabla}\varphi_p(\vec{r},\omega) - i\frac{\vec{\nabla}\left|\tilde{p}_o(\vec{r},\omega)\right|}{\left|\tilde{p}_o(\vec{r},\omega)\right|}\right]\tilde{p}(\vec{r},t)$$

Thus, for a "generic" 3-D monochromatic traveling wave, the complex ***time-domain*** 3-D particle velocity $\tilde{\vec{u}}(\vec{r},t)$ is related to the complex ***time-domain*** over-pressure amplitude $\tilde{p}(\vec{r},t)$ via the {linearized} Euler equation relation:

$$\boxed{\tilde{\vec{u}}(\vec{r},t) = -\frac{1}{\rho_o\omega}\left[\vec{\nabla}\varphi_p(\vec{r},\omega) - i\frac{\vec{\nabla}\left|\tilde{p}_o(\vec{r},\omega)\right|}{\left|\tilde{p}_o(\vec{r},\omega)\right|}\right]\tilde{p}(\vec{r},t)}$$

There are two different kinds of terms/contributions on the RHS of this equation. The first term, $-\vec{\nabla}\varphi_p(\vec{r},\omega)$ is the {negative of the} spatial gradient of the ***phase*** of the complex over-pressure amplitude – note that for this contribution, $\tilde{\vec{u}}(\vec{r},t)$ is ***in-phase*** with $\tilde{p}(\vec{r},t)$. The second term, $+i\vec{\nabla}\left|\tilde{p}_o(\vec{r},\omega)\right|/\left|\tilde{p}_o(\vec{r},\omega)\right|$ is the {normalized/fractional} spatial gradient of the complex over-pressure ***amplitude*** – note that for this contribution, $\tilde{\vec{u}}(\vec{r},t)$ is ***90°-out-of-phase*** with $\tilde{p}(\vec{r},t)$. Then *e.g.* for the specific case of a monochromatic 3-D traveling plane wave, $\varphi_p(\vec{r},\omega) = -\vec{k}\bullet\vec{r}$ and $\tilde{p}_o(\vec{r},\omega) = p_o \neq fcn(\vec{r},\omega)$, thus: $\vec{\nabla}\varphi_p(\vec{r},\omega) = -\vec{\nabla}(\vec{k}\bullet\vec{r}) = -\vec{k}$ and: $\vec{\nabla}\left|\tilde{p}_o(\vec{r},\omega)\right| = 0$, hence {here} $\tilde{\vec{u}}(\vec{r},t)$ is ***in-phase*** with $\tilde{p}(\vec{r},t)$ and using $\omega = ck$ we also see that: $\tilde{\vec{u}}(\vec{r},t) = \left(\tilde{p}(\vec{r},t)/\rho_o c\right)\hat{k}$.

Since $\tilde{p}(\vec{r},t) = \tilde{p}(\vec{r},\omega)\cdot e^{i\omega t}$ and $\tilde{\vec{u}}(\vec{r},t) = \tilde{\vec{u}}(\vec{r},\omega)\cdot e^{i\omega t}$, the complex 3-D vector ***specific*** acoustic impedance {here} is:

$$\tilde{\vec{z}}_a(\vec{r},\omega) = \frac{\tilde{p}(\vec{r},\omega)}{\tilde{\vec{u}}(\vec{r},\omega)} = \frac{\cancel{\tilde{p}(\vec{r},t)}}{-\frac{1}{\rho_o\omega}\left[\vec{\nabla}\varphi_p(\vec{r},\omega) - i\frac{\vec{\nabla}\left|\tilde{p}_o(\vec{r},\omega)\right|}{\left|\tilde{p}_o(\vec{r},\omega)\right|}\right]\cancel{\tilde{p}(\vec{r},t)}} = -\frac{\rho_o c(\omega/c)}{\left[\vec{\nabla}\varphi_p(\vec{r},\omega) - i\frac{\vec{\nabla}\left|\tilde{p}_o(\vec{r},\omega)\right|}{\left|\tilde{p}_o(\vec{r},\omega)\right|}\right]}$$

The ***purely real*** quantity $z_o \equiv \rho_o c = 1.204\left(kg/m^3\right)\cdot 343\left(m/s\right) \simeq 413\ \left(Pascal\text{-}sec/m \equiv Rayls = \Omega_a\right)$ @ NTP is known as the ***characteristic*** ***longitudinal*** ***specific*** ***acoustic*** ***impedance*** of ***free*** ***air***.

Its inverse is the ***purely real characteristic longitudinal specific acoustic admittance*** of ***free air***: $y_o = 1/z_o = 1/\rho_o c \simeq 1/413 \simeq 2.42\times 10^{-3}\ \left(\Omega_a^{-1}\right)$.

Note that c, ρ_0, z_o and y_o are ***not*** constants, they are dependent *e.g.* on the air temperature, T as shown in the table below, for an ambient pressure of P_{atm} = 1.0 atmosphere:

Temperature (°C)	c (m/s)	ρ_0 (kg/m^3)	z_o (Ω_a)	y_o (Ω_a^{-1})
–10	325.2	1.342	436.1	2.293×10^{-3}
–5	328.3	1.317	432.0	2.315×10^{-3}
0	331.3	1.292	428.4	2.334×10^{-3}
+5	334.3	1.269	424.3	2.357×10^{-3}
+10	337.3	1.247	420.6	2.378×10^{-3}
+15	340.3	1.225	416.8	2.399×10^{-3}
+20	343.2	1.204	413.2	2.420×10^{-3}
+25	346.1	1.184	409.8	2.440×10^{-3}
+30	349.0	1.165	406.3	2.461×10^{-3}

For the specific case of a monochromatic 3-D traveling plane wave propagating *e.g.* in "free air", using $k = \omega/c$, where $k = \left|\vec{k}\right| = \sqrt{k_x^2 + k_y^2 + k_z^2}\ \left(m^{-1}\right)$ and using the relation $z_o \equiv \rho_o c$, we can rewrite the above expression for the complex 3-D vector ***specific*** acoustic impedance for the specific case of a monochromatic 3-D traveling plane wave propagating *e.g.* in "free air" as:

$$\tilde{\vec{z}}_a(\vec{r},\omega) = \frac{\tilde{p}(\vec{r},\omega)}{\tilde{\vec{u}}(\vec{r},\omega)} = -\frac{k}{\left[\vec{\nabla}\varphi_p(\vec{r},\omega) - i\frac{\vec{\nabla}\left|\tilde{p}_o(\vec{r},\omega)\right|}{\left|\tilde{p}_o(\vec{r},\omega)\right|}\right]}\cdot z_o$$

We can also write this as a dimensionless relation, and since $\tilde{\vec{z}}_a(\vec{r},t) = \rho_o\tilde{\vec{c}}_a(\vec{r},t)$, we have:

$$\frac{\tilde{\vec{z}}_a(\vec{r},\omega)}{z_o} = \frac{\tilde{\vec{c}}_a(\vec{r},\omega)}{c} = -\frac{k}{\left[\vec{\nabla}\varphi_p(\vec{r},\omega) - i\frac{\vec{\nabla}\left|\tilde{p}_o(\vec{r},\omega)\right|}{\left|\tilde{p}_o(\vec{r},\omega)\right|}\right]}$$

<u>Example # 1: 1-D Plane Monochromatic Traveling Wave Propagating in "Free Air"</u>:

In "***free air***", the ***instantaneous <u>time</u>-<u>domain</u>*** pressure at a space-time point (x,t) associated with a 1-D plane monochromatic traveling wave propagating *e.g.* in the +*x*-direction is a ***purely real*** quantity: $p(x,t) = p_o \cos(\omega t - kx)$.

The 1-D ***instantaneous <u>time</u>-<u>domain</u>*** longitudinal particle velocity (*i.e.* in the +*x*-/propagation direction) at the space-time point (x,t) associated with a 1-D plane monochromatic traveling wave is obtained via the {linearized} 1-D Euler equation for inviscid fluid flow:

$$\frac{\partial u^{\|}(x,t)}{\partial t} = -\frac{1}{\rho_o}\frac{\partial p(x,t)}{\partial x} = -\frac{p_o}{\rho_o}\frac{\partial \cos(\omega t - kx)}{\partial x} = -\frac{kp_o}{\rho_o}\sin(\omega t - kx)$$

Then:

$$u^{\|}(x,t) = -\frac{kp_o}{\rho_o}\int \sin(\omega t - kx)\,dt = +\frac{kp_o}{\omega\rho_o}\cos(\omega t - kx) = \frac{p_o}{\rho_o c}\cos(\omega t - kx) = u_o^{\|}\cos(\omega t - kx)$$

where we have used the relation $c = \omega/k = 343\,m/s$ = speed of sound in {bone-dry} air @ NTP (obtained from the 1-D wave equation(s) for *p* or $u^{\|}$). Note also that: $u_o^{\|} = p_o/\rho_o c = p_o/z_o$.

Since $p(x,t) = p_o \cos(\omega t - kx)$ and $u^{\|}(x,t) = (p_o/\rho_o c)\cos(\omega t - kx) = u_o^{\|}\cos(\omega t - kx)$, we see that the ***instantaneous <u>time</u>-<u>domain</u>*** pressure and longitudinal particle velocity are ***<u>in-phase</u>*** with each other for a 1-D monochromatic plane wave propagating in "***free air***". This in turn implies that for ***harmonic*** (*i.e.* single-frequency) {*aka* monochromatic} plane waves, the longitudinal ***<u>specific</u>*** acoustic impedance, ***<u>specific</u>*** admittance and intensity will thus also be ***purely <u>real</u>*** quantities for a 1-D monochromatic plane wave propagating in "***free air***"

We then "complexify" the above ***instantaneous <u>time</u>-<u>domain</u>*** pressure and longitudinal particle velocity expressions to obtain their complex ***<u>time</u>-<u>domain</u>*** representations: $\tilde{p}(x,t) = p_o e^{i(\omega t - kx)}$ and $\tilde{u}^{\|}(x,t) = u_o^{\|} e^{i(\omega t - kx)}$. The longitudinal ***<u>specific</u>*** acoustic impedance associated with a 1-D monochromatic plane wave propagating *e.g.* in the +*x*-direction in "***free air***" is then easily seen to {also} be a ***purely <u>real</u>*** quantity:

$$\tilde{z}_a^{\|}(x) = \frac{\tilde{p}(x,t)}{\tilde{u}^{\|}(x,t)} = \frac{p_o \cancel{e^{i(\omega t - kx)}}}{u_o^{\|}\cancel{e^{i(\omega t - kx)}}} = \frac{p_o}{u_o^{\|}} = \frac{\cancel{p_o}}{\cancel{p_o}/\rho_o c} = \rho_o c \equiv z_o \;\; (\Omega_a)$$

Since {here} $\tilde{z}_a^{\|}(x) = \rho_o \tilde{c}_a^{\|}(x)$, we see that the longitudinal velocity of energy flow $\tilde{c}_a^{\|}(x) = c$ for a 1-D monochromatic plane wave propagating *e.g.* in the +*x*-direction in "***free air***".

Note that this acoustic sound field example is the electrical analog of a simple *AC* circuit, *e.g.* driven at constant voltage by a sine-wave generator with a ***purely real instantaneous*** *AC* voltage $V(t) = V_o \cos\omega t$ imposed across an ***ideal*** resistor of resistance $R \;(\Omega)$ (hence ***purely real*** impedance $\tilde{Z} = R + i0\,(\Omega_e)$) resulting in a ***purely real instantaneous*** *AC* current $I(t) = I_o \cos\omega t$ flowing through it.

Note also that the purely real longitudinal ***specific*** acoustic impedance $z_a^{\parallel}(x) = \rho_o c \equiv z_o \ (\Omega_a)$ and/or the longitudinal ***specific*** acoustic admittance $y_a^{\parallel}(x) = 1/z_a^{\parallel}(x) = 1/\rho_o c \equiv y_o^{\parallel} = 1/z_o^{\parallel} \ (\Omega_a^{-1})$ and also the longitudinal velocity of energy flow, $\tilde{c}_a^{\parallel}(x) = c$ associated with a 1-D monochromatic plane wave propagating *e.g.* in the +*x*-direction in "***free air***" have ***no*** spatial (*i.e.* *x*-) and/or frequency (*i.e.* *f* -) dependence.

The ***instantaneous time-domain*** longitudinal acoustic intensity associated with a 1-D monochromatic plane traveling wave propagating in the +*x*-direction in "***free air***" is also a ***purely real*** quantity – *i.e.* plane wave acoustic energy is entirely in the form of ***pure sound radiation*** – no acoustic energy is {temporarily} stored "locally" at the point *x*. The ***instantaneous time-domain*** complex longitudinal acoustic intensity is:

$$I_a^{\parallel}(x,t) \equiv p(x,t)\cdot u^{\parallel}(x,t) = p_o u_o^{\parallel} \cos^2(\omega t - kx)$$

For an observer's/listener's position *e.g.* at $x = 0$:

$$I_a^{\parallel}(x=0,t) = p(x=0,t)\, u^{\parallel}(x=0,t) = p_o u_o^{\parallel} \cos^2 \omega t$$

Noting that the time-averaged $\left\langle \cos^2 \omega t \right\rangle_t \equiv \frac{1}{\tau}\int_{t=0}^{t=\tau} \cos^2 \omega t \, dt = \frac{1}{2}$, the ***time-averaged instantaneous time-domain*** complex longitudinal sound intensity at the listener's position $x = 0$ associated with a 1-D monochromatic plane traveling wave propagating in the +*x*-direction in "***free air***" is:

$$\left\langle I_a^{\parallel}(x=0,t) \right\rangle_t = p_o u_o^{\parallel} \left\langle \cos^2 \omega t \right\rangle_t = \tfrac{1}{2} p_o u_o^{\parallel}$$

We can also define ***RMS amplitudes*** of over-pressure and particle velocity in terms of their respective ***peak amplitudes***: $p_o^{rms} \equiv \tfrac{1}{\sqrt{2}} p_o$ and $u_o^{\parallel rms} \equiv \tfrac{1}{\sqrt{2}} u_o^{\parallel}$. Thus, we see that the ***RMS*** value of the ***instantaneous time-domain*** longitudinal sound intensity at the listener's position $x = 0$ associated with a 1-D monochromatic plane traveling wave propagating in the +*x*-direction in "***free air***" is equal to the ***time-averaged*** longitudinal sound intensity at that point, *i.e.*:

$$I_a^{\parallel rms}(x=0) = \left\langle I_a^{\parallel}(x=0) \right\rangle_t = \tfrac{1}{2} p_o u_o^{\parallel} = p_o^{rms} u_o^{\parallel rms}$$

The reader can also easily verify for this example that the ***frequency domain*** active (*i.e.* real) and reactive (*i.e.* imaginary/quadrature) components of the ***complex*** longitudinal acoustic intensity associated with a 1-D monochromatic traveling plane wave propagating in the +*x*-direction in "***free air***" are given by:

$$\tilde{I}_a^{\parallel}(x,\omega) \equiv \tfrac{1}{2}\tilde{p}(x,\omega)\tilde{u}^{\parallel *}(x,\omega) = \tfrac{1}{2} p_o \cancel{e^{i(\omega t - kx)}} u_o^{\parallel} \cancel{e^{-i(\omega t - kx)}} = \tfrac{1}{2} p_o u_o^{\parallel} = \tfrac{1}{2} p_o u_o^{\parallel} + 0i = \left\langle \tilde{I}_a^{\parallel}(x,t) \right\rangle_t$$

Here in ***this*** problem, note that: $\left\langle \tilde{I}_a^{\parallel}(x,t) \right\rangle_t = \left\langle \tilde{I}_{a\,r}^{\parallel}(x,t) \right\rangle_t + i\left\langle \tilde{I}_{a\,i}^{\parallel}(x,t) \right\rangle_t = p_o u_o^{\parallel} + 0i = p_o u_o^{\parallel}$ has ***no*** position (*i.e.* *x*-) dependence!

The ***instantaneous*** potential, kinetic and total energy densities associated with a 1-D monochromatic traveling plane wave propagating in the +x-direction in "***free air***" at $x = 0$ are:

$$w_{potl}^{inst}\left(x=0,t\right) \equiv \frac{1}{2}\frac{1}{\rho_o c^2}\, p^2\left(x=0,t\right) = \frac{1}{2}\frac{1}{\rho_o c^2}\, p_o^2 \cos^2 \omega t = \frac{1}{\rho_o c^2}\, p_o^{rms\,2} \cos^2 \omega t$$

$$w_{kin}^{inst}\left(x=0,t\right) \equiv \frac{1}{2}\rho_o \vec{u}_{\|}\left(x=0,t\right) \bullet \vec{u}_{\|}\left(x=0,t\right) = \frac{1}{2}\rho_o u_o^{\|2} \cos^2 \omega t = \rho_o u_o^{\|rms\,2} \cos^2 \omega t$$

$$w_{tot}^{inst}\left(x=0,t\right) \equiv w_{potl}^{inst}\left(x=0,t\right) + w_{kin}^{inst}\left(x=0,t\right)$$

$$= \frac{1}{2}\frac{1}{\rho_o c^2}\, p_o^2 \cos^2 \omega t + \frac{1}{2}\rho_o u_o^{\|2} \cos^2 \omega t = \frac{1}{\rho_o c^2}\, p_o^{rms\,2} \cos^2 \omega t + \rho_o u_o^{\|rms\,2} \cos^2 \omega t$$

For this situation with a 1-D monochromatic traveling plane wave, we obtained the relation

$$z_a^{\|}\left(x\right) = \frac{p\left(x,t\right)}{u^{\|}\left(x,t\right)} = \frac{p_o}{u_o^{\|}} = \rho_o c \equiv z_o \quad \left(\Omega_a\right)$$

Thus we see again here that: $p_o = \rho_o c u_o^{\|} = z_o u_o^{\|}$. Using the <u>square</u> of this relation in the above instantaneous total energy density expression, we also see that {***here***}:

$$w_{tot}^{inst}\left(x=0,t\right) \equiv w_{potl}^{inst}\left(x=0,t\right) + w_{kin}^{inst}\left(x=0,t\right) = \frac{1}{\rho_o c^2}\, p_o^2 \cos^2 \omega t = \rho_o u_o^{\|2} \cos^2 \omega t$$

The ***time-averages*** of the ***instantaneous*** potential, kinetic and total energy densities associated with a 1-D monochromatic traveling plane wave propagating in the +x-direction in "***free air***" at $x = 0$ are:

$$\left\langle w_{potl}^{inst}\left(x=0,t\right)\right\rangle_t = \frac{1}{2}\frac{1}{\rho_o c^2}\, p_o^2 \underbrace{\left\langle \cos^2 \omega t\right\rangle_t}_{=1/2} = \frac{1}{4}\frac{1}{\rho_o c^2}\, p_o^2 = \frac{1}{2}\frac{1}{\rho_o c^2}\, p_o^{rms\,2} \quad \left(Joules/m^3\right)$$

$$\left\langle w_{kin}^{inst}\left(x=0,t\right)\right\rangle_t = \frac{1}{2}\rho_o u_o^{\|2} \underbrace{\left\langle \cos^2 \omega t\right\rangle_t}_{=1/2} = \frac{1}{4}\rho_o u_o^{\|2} = \frac{1}{2}\rho_o u_o^{\|rms\,2} \qquad \left(Joules/m^3\right)$$

$$\left\langle w_{tot}^{inst}\left(x=0,t\right)\right\rangle_t \equiv \left\langle w_{potl}^{inst}\left(x=0,t\right)\right\rangle_t + \left\langle w_{kin}^{inst}\left(x=0,t\right)\right\rangle_t = \frac{1}{4}\frac{p_o^2}{\rho_o c^2} + \frac{1}{4}\rho_o u_o^{\|2} = \frac{1}{2}\frac{p_o^{rms\,2}}{\rho_o c^2} + \frac{1}{2}\rho_o u_o^{\|rms\,2} \quad \left(Joules/m^3\right)$$

Again, using the square of the relation $p_o = \rho_o c u_o^{\|} = z_o u_o^{\|}$ in the above expression, we see that:

$$\left\langle w_{tot}^{inst}\left(x=0,t\right)\right\rangle_t \equiv \left\langle w_{potl}^{inst}\left(x=0,t\right)\right\rangle_t + \left\langle w_{kin}^{inst}\left(x=0,t\right)\right\rangle_t = \frac{1}{2}\frac{p_o^2}{\rho_o c^2} = \frac{1}{2}\rho_o u_o^{\|2} = \frac{1}{2c}\, p_o u_o^{\|} \quad \left(Joules/m^3\right)$$

Note that the ratio of the ***time-averaged*** potential energy density to the ***time-averaged*** kinetic energy density *e.g.* at $x = 0$ is equal to <u>unity</u> for a 1-D monochromatic traveling wave:

$$\frac{\left\langle w_{potl}^{inst}\left(x=0,t\right)\right\rangle_t}{\left\langle w_{kin}^{inst}\left(x=0,t\right)\right\rangle_t} = \frac{\frac{1}{4}\frac{p_o^2}{\rho_o c^2}}{\frac{1}{4}\rho_o u_o^{\|2}} = \frac{p_o^2}{\rho_o^2 c^2 u_o^{\|2}} = \frac{p_o^2}{z_o^2 u_o^{\|2}} = \frac{z_o^2}{z_o^2} = 1$$

Note further that:

$$\left\langle \tilde{I}_a^{\parallel}\left(x=0,t\right)\right\rangle_t = \frac{1}{2}p_o u_o^{\parallel} = \frac{1}{2}\frac{p_o^2}{\rho_o c} = \frac{1}{2}\rho_o c u_o^{\parallel 2} = \frac{1}{2}u_o^{\parallel 2} z_o \quad \left(Watts/m^2\right)$$

and again using the relation $p_o = \rho_o c u_o^{\parallel} = z_o u_o^{\parallel}$, that:

$$\tilde{I}_a^{\parallel}\left(x=0,\omega\right) = \left\langle \tilde{I}_a^{\parallel}\left(x=0,t\right)\right\rangle_t = c\left\langle w_{tot}^{inst}\left(x=0,t\right)\right\rangle_t = \frac{1}{2}p_o u_o^{\parallel} = \frac{1}{2}\frac{p_o^2}{\rho_o c} = \frac{1}{2}\rho_o c u_o^{\parallel 2} = \frac{1}{2}u_o^{\parallel 2} z_o \quad \left(Watts/m^2\right)$$

Example # 2: Two Counter-Propagating 1-D Plane Monochromatic Traveling Waves in "Free Air":

In this example, we imagine two ***un-equal*** strength harmonic (*i.e.* single-frequency) sound sources located at $x = \pm\infty$, with an observer/listener located near/at the origin *x* = 0. At the observer's location there will therefore be two 1-D monochromatic plane traveling waves propagating in opposite directions in "***free air***" (*i.e.* the Great Wide-Open).

The ***physical***, ***instantaneous time-domain*** over-pressure amplitudes associated with the right- and left-going 1-D monochromatic plane waves are individually ***purely real*** quantities:

$$p_A\left(x,t\right) = A\cos\left(\omega t - kx + \varphi_A^o\right) \text{ and } p_B\left(x,t\right) = B\cos\left(\omega t + kx + \varphi_B^o\right) \text{ with } A \neq B \text{ \{necessarily\}}$$

Note here that the frequency and position-independent phases φ_A^o and φ_B^o are explicitly included here to generalize the {relative} phase relation between the two counter-propagating 1-D monochromatic traveling waves, *e.g.* consider their phase relation at *x* = 0 and *t* = 0: $p_A\left(x=0,t=0\right) = A\cos\varphi_A^o$ and $p_B\left(x=0,t=0\right) = B\cos\varphi_B^o$.

The corresponding complex ***time-domain*** over-pressure amplitudes are:

$$\tilde{p}_A\left(x,t\right) = \tilde{A}e^{i(\omega t - kx)} \text{ and } \tilde{p}_B\left(x,t\right) = \tilde{B}e^{i(\omega t + kx)} \text{ with } \tilde{A} \neq \tilde{B} \text{ \{necessarily\}}$$

where $\tilde{A} = \left|\tilde{A}\right|e^{i\varphi_A^o} \equiv Ae^{i\varphi_A^o}$ and $\tilde{B} = \left|\tilde{B}\right|e^{i\varphi_B^o} \equiv Be^{i\varphi_B^o}$.

Each individual complex ***time-domain*** over-pressure amplitude satisfies its own Euler's equation:

$$\frac{\partial u_{A,B}^{\parallel}\left(x,t\right)}{\partial t} = -\frac{1}{\rho_o}\frac{\partial p_{A,B}\left(x,t\right)}{\partial x}$$

The corresponding right- and left-going complex ***time-domain*** longitudinal particle velocities are:

$$\tilde{u}_A^{\parallel}\left(x,t\right) = \frac{\tilde{A}}{\rho_o c}e^{i(\omega t - kx)} \equiv \tilde{u}_{A_o}^{\parallel}e^{i(\omega t - kx)} \text{ and: } \tilde{u}_B^{\parallel}\left(x,t\right) = -\frac{\tilde{B}}{\rho_o c}e^{i(\omega t + kx)} \equiv -\tilde{u}_{B_o}^{\parallel}e^{i(\omega t + kx)} \text{ (using } c = \omega/k\text{)}$$

Note the –*ve* sign in the left-going complex longitudinal particle velocity amplitude, which simple reflects the fact that it is propagating in the –*ve x*-direction.

For "everyday" sound pressure levels $SPL = L_p = 20\log_{10}\left(p_{atm}/p_o\right) \ll 134\,dB$, corresponding to sound over-pressure amplitudes in "***free air***" at NTP of $\left|\tilde{p}\left(\vec{r},t\right)\right| \ll 100\ RMS\ Pascals$, the ***principle*** ***of*** ***linear*** ***superposition*** holds, such that the total/resultant complex over-pressure and longitudinal particle velocity amplitudes respectively are:

$$\tilde{p}_{tot}\left(x,t\right) = \tilde{p}_A\left(x,t\right) + \tilde{p}_B\left(x,t\right) \;=\; \tilde{A}e^{i(\omega t - kx)} + \tilde{B}e^{i(\omega t + kx)} \quad (Pascals)$$

and:

$$\tilde{u}_{tot}^{\|}\left(x,t\right) = \tilde{u}_A^{\|}\left(x,t\right) + \tilde{u}_B^{\|}\left(x,t\right) \;=\; \tilde{u}_{A_o}^{\|}e^{i(\omega t - kx)} + \tilde{u}_{B_o}^{\|}e^{i(\omega t + kx)} = \frac{\tilde{A}}{\rho_o c}e^{i(\omega t - kx)} - \frac{\tilde{B}}{\rho_o c}e^{i(\omega t + kx)}\,(m/s)$$

We can recast the above equations in terms of the dimensionless complex variable:

$$\tilde{R} \equiv \frac{\tilde{B}}{\tilde{A}} = \frac{\left|\tilde{B}\right|e^{i\varphi_B^o}}{\left|\tilde{A}\right|e^{i\varphi_A^o}} = \frac{\left|\tilde{B}\right|}{\left|\tilde{A}\right|}e^{i\left(\varphi_B^o - \varphi_A^o\right)} = \left|\tilde{R}\right|e^{i\left(\varphi_B^o - \varphi_A^o\right)} = \left|\tilde{R}\right|e^{i\Delta\varphi_{BA}^o} \text{ where: } \Delta\varphi_{BA}^o \equiv \varphi_B^o - \varphi_A^o$$

Thus:

$$\tilde{p}_{tot}\left(x,t\right) = \tilde{A}\left[e^{i(\omega t - kx)} + \left|\tilde{R}\right|e^{i(\omega t + kx)}\cdot e^{i\Delta\varphi_{BA}^o}\right] = \tilde{A}\left[1 + \left|\tilde{R}\right|e^{i\left(2kx + \Delta\varphi_{BA}^o\right)}\right]e^{i(\omega t - kx)}$$

and:

$$\tilde{u}_{tot}^{\|}\left(x,t\right) = \frac{\tilde{A}}{\rho_o c}\left[e^{i(\omega t - kx)} - \left|\tilde{R}\right|e^{i(\omega t + kx)}\cdot e^{i\Delta\varphi_{BA}^o}\right] = \frac{\tilde{A}}{\rho_o c}\left[1 - \left|\tilde{R}\right|e^{i\left(2kx + \Delta\varphi_{BA}^o\right)}\right]e^{i(\omega t - kx)}$$

We first calculate the ***magnitudes*** of the complex total/resultant over-pressure $\left|\tilde{p}_{tot}\left(x,t\right)\right|$ and longitudinal particle velocity $\left|\tilde{u}_{tot}^{\|}\left(x,t\right)\right|$:

$$\begin{aligned}
\left|\tilde{p}_{tot}\left(x,t\right)\right| &\equiv \sqrt{\tilde{p}_{tot}\left(x,t\right)\cdot\tilde{p}_{tot}^*\left(x,t\right)} \\
&= \left|\tilde{A}\right|\sqrt{\left(1 + \left|\tilde{R}\right|e^{i\left(2kx + \Delta\varphi_{BA}^o\right)}\right)\cdot\left(1 + \left|\tilde{R}\right|e^{i\left(2kx + \Delta\varphi_{BA}^o\right)}\right)^*} \\
&= \left|\tilde{A}\right|\sqrt{\left(1 + \left|\tilde{R}\right|e^{i\left(2kx + \Delta\varphi_{BA}^o\right)}\right)\cdot\left(1 + \left|\tilde{R}\right|e^{-i\left(2kx + \Delta\varphi_{BA}^o\right)}\right)} \\
&= \left|\tilde{A}\right|\sqrt{1 + \left|\tilde{R}\right|e^{i\left(2kx + \Delta\varphi_{BA}^o\right)} + \left|\tilde{R}\right|e^{-i\left(2kx + \Delta\varphi_{BA}^o\right)} + \left|\tilde{R}\right|^2} \\
&= \left|\tilde{A}\right|\sqrt{1 + \left|\tilde{R}\right|\left\{e^{i\left(2kx + \Delta\varphi_{BA}^o\right)} + e^{-i\left(2kx + \Delta\varphi_{BA}^o\right)}\right\} + \left|\tilde{R}\right|^2} \\
&= \left|\tilde{A}\right|\sqrt{1 + 2\left|\tilde{R}\right|\cos\left(2kx + \Delta\varphi_{BA}^o\right) + \left|\tilde{R}\right|^2}
\end{aligned}$$

and:

$$\begin{aligned}\left|\tilde{u}_{tot}^{\parallel}(x,t)\right| &\equiv \sqrt{\tilde{u}_{tot}^{\parallel}(x,t)\cdot\tilde{u}_{tot}^{\parallel *}(x,t)} \\ &= \frac{\left|\tilde{A}\right|}{\rho_o c}\sqrt{\left(1-\left|\tilde{R}\right|e^{i\left(2kx+\Delta\varphi_{BA}^o\right)}\right)\cdot\left(1-\left|\tilde{R}\right|e^{i\left(2kx+\Delta\varphi_{BA}^o\right)}\right)^*} \\ &= \frac{\left|\tilde{A}\right|}{\rho_o c}\sqrt{\left(1-\left|\tilde{R}\right|e^{i\left(2kx+\Delta\varphi_{BA}^o\right)}\right)\cdot\left(1-\left|\tilde{R}\right|e^{-i\left(2kx+\Delta\varphi_{BA}^o\right)}\right)} \\ &= \frac{\left|\tilde{A}\right|}{\rho_o c}\sqrt{1-\left|\tilde{R}\right|e^{i\left(2kx+\Delta\varphi_{BA}^o\right)}-\left|\tilde{R}\right|e^{-i\left(2kx+\Delta\varphi_{BA}^o\right)}+\left|\tilde{R}\right|^2} \\ &= \frac{\left|\tilde{A}\right|}{\rho_o c}\sqrt{1-\left|\tilde{R}\right|\left\{e^{i\left(2kx+\Delta\varphi_{BA}^o\right)}+e^{-i\left(2kx+\Delta\varphi_{BA}^o\right)}\right\}+\left|\tilde{R}\right|^2} \\ &= \frac{\left|\tilde{A}\right|}{\rho_o c}\sqrt{1-2\left|\tilde{R}\right|\cos\left(2kx+\Delta\varphi_{BA}^o\right)+\left|\tilde{R}\right|^2}\end{aligned}$$

Thus, *e.g*. for an observer/listener's position $x = 0$, **.and.** for ***equal*-*strength*** over-pressure amplitudes $\left|\tilde{A}\right|=\left|\tilde{B}\right| \Rightarrow \left|\tilde{R}\right|\equiv\left|\tilde{B}\right|/\left|\tilde{A}\right|=1$ (*i.e*. a ***pure*** standing wave!) these formulae simplify to:

$$\left|\tilde{p}_{tot}(x=0,t)\right|=\sqrt{2}\left|\tilde{A}\right|\sqrt{1+\cos\Delta\varphi_{BA}^o} \text{ and: } \left|\tilde{u}_{tot}^{\parallel}(x=0,t)\right|=\frac{\sqrt{2}\left|\tilde{A}\right|}{\rho_o c}\sqrt{1-\cos\Delta\varphi_{BA}^o}$$

Thus, we see that when: $\Delta\varphi_{BA}^o = 0,\pm 2\pi,\pm 4\pi,\ldots=\pm n_{even}\pi$ that: $\cos\Delta\varphi_{BA}^o=+1$ and thus:

$$\left|\tilde{p}_{tot}(x=0,t)\right|=2\left|\tilde{A}\right| \text{ and: } \left|\tilde{u}_{tot}^{\parallel}(x=0,t)\right|=0$$

i.e. we have complete constructive (destructive) interference associated with the two individual complex over-pressure (longitudinal particle velocity) amplitudes, respectively.

We also see that when: $\Delta\varphi_{BA}^o=\pm 1\pi,\pm 3\pi,\pm 5\pi,\ldots=\pm n_{odd}\pi$ that: $\cos\Delta\varphi_{BA}^o=-1$ and thus:

$$\left|\tilde{p}_{tot}(x=0,t)\right|=0 \text{ and: } \left|\tilde{u}_{tot}^{\parallel}(x=0,t)\right|=2\left|\tilde{A}\right|/\rho_o c$$

i.e. we have complete destructive (constructive) interference associated with the two individual complex over-pressure (longitudinal particle velocity) amplitudes, respectively.

Hence, we can also now see that when $\left|\tilde{R}\right|\equiv\left|\tilde{B}\right|/\left|\tilde{A}\right|\neq 1$, it is ***not*** possible to ever achieve ***complete*** constructive/destructive interference effects between the two individual right- and left-moving complex over-pressure and/or longitudinal particle velocity amplitudes.

Since: $\tilde{p}_{tot}(x,t)=\tilde{A}\left[1+\left|\tilde{R}\right|e^{i\left(2kx+\Delta\varphi_{BA}^o\right)}\right]e^{i(\omega t-kx)}$ and: $\tilde{u}_{tot}^{\parallel}(x,t)=\frac{\tilde{A}}{\rho_o c}\left[1-\left|\tilde{R}\right|e^{i\left(2kx+\Delta\varphi_{BA}^o\right)}\right]e^{i(\omega t-kx)}$,

the phases of the complex total/resultant pressure and longitudinal particle velocity associated with the two counter-propagating 1-D monochromatic plane waves are given by:

$$\varphi_{p_{tot}}(x) \equiv \tan^{-1}\left(\frac{\operatorname{Im}\{\tilde{p}_{tot}(x,t)\}}{\operatorname{Re}\{\tilde{p}_{tot}(x,t)\}}\right) = \tan^{-1}\left(\frac{\operatorname{Im}\left\{\left[1 + |\tilde{R}|e^{i(2kx+\Delta\varphi_{BA}^{o})}\right]\right\}}{\operatorname{Re}\left\{\left[1 + |\tilde{R}|e^{i(2kx+\Delta\varphi_{BA}^{o})}\right]\right\}}\right)$$

$$= \tan^{-1}\left[\frac{\sin kx\left(1+|\tilde{R}|\cos\left(2kx+\Delta\varphi_{BA}^{o}\right)\right)+|\tilde{R}|\cos kx\cdot\sin\left(2kx+\Delta\varphi_{BA}^{o}\right)}{\cos kx\left(1+|\tilde{R}|\cos\left(2kx+\Delta\varphi_{BA}^{o}\right)\right)-|\tilde{R}|\sin kx\cdot\sin\left(2kx+\Delta\varphi_{BA}^{o}\right)}\right]$$

and:

$$\varphi_{u_{tot}}(x) \equiv \tan^{-1}\left(\frac{\operatorname{Im}\{\tilde{u}_{tot}^{\parallel}(x,t)\}}{\operatorname{Re}\{\tilde{u}_{tot}^{\parallel}(x,t)\}}\right) = \tan^{-1}\left(\frac{\operatorname{Im}\left\{\left[1-|\tilde{R}|e^{i(2kx+\Delta\varphi_{BA}^{o})}\right]\right\}}{\operatorname{Re}\left\{\left[1-|\tilde{R}|e^{i(2kx+\Delta\varphi_{BA}^{o})}\right]\right\}}\right)$$

$$= \tan^{-1}\left[\frac{\sin kx\left(1-|\tilde{R}|\cos\left(2kx+\Delta\varphi_{BA}^{o}\right)\right)-|\tilde{R}|\cos kx\cdot\sin\left(2kx+\Delta\varphi_{BA}^{o}\right)}{\cos kx\left(1-|\tilde{R}|\cos\left(2kx+\Delta\varphi_{BA}^{o}\right)\right)+|\tilde{R}|\sin kx\cdot\sin\left(2kx+\Delta\varphi_{BA}^{o}\right)}\right]$$

The complex longitudinal ***specific*** acoustic impedance associated with the two counter-propagating 1-D monochromatic plane waves is:

$$\tilde{z}_{a\,tot}^{\parallel}(x) \equiv \frac{\tilde{p}_{tot}(x,t)}{\tilde{u}_{tot}^{\parallel}(x,t)} = \frac{\cancel{\tilde{A}}\left[1 + |\tilde{R}|e^{i(2kx+\Delta\varphi_{BA}^{o})}\right]\cdot\cancel{e^{i(\omega t-kx)}}}{\frac{\cancel{\tilde{A}}}{\rho_o c}\left[1-|\tilde{R}|e^{i(2kx+\Delta\varphi_{BA}^{o})}\right]\cdot\cancel{e^{i(\omega t-kx)}}} = \rho_o c\,\frac{\left[1 + |\tilde{R}|e^{i(2kx+\Delta\varphi_{BA}^{o})}\right]}{\left[1-|\tilde{R}|e^{i(2kx+\Delta\varphi_{BA}^{o})}\right]}$$

Since the ***characteristic*** ***longitudinal*** ***specific*** acoustic impedance of "***free air***" is $z_o \equiv \rho_o c$, then:

$$\tilde{z}_{a\,tot}^{\parallel}(x) = z_o\,\frac{\left[1 + |\tilde{R}|e^{i(2kx+\Delta\varphi_{BA}^{o})}\right]}{\left[1-|\tilde{R}|e^{i(2kx+\Delta\varphi_{BA}^{o})}\right]} = z_o\,\frac{\left[1 + |\tilde{R}|e^{i(2kx+\Delta\varphi_{BA}^{o})}\right]}{\left[1-|\tilde{R}|e^{i(2kx+\Delta\varphi_{BA}^{o})}\right]}\cdot\frac{\left[1-|\tilde{R}|e^{i(2kx+\Delta\varphi_{BA}^{o})}\right]^{*}}{\left[1-|\tilde{R}|e^{i(2kx+\Delta\varphi_{BA}^{o})}\right]^{*}}$$

$$= z_o\,\frac{\left[1 + |\tilde{R}|e^{i(2kx+\Delta\varphi_{BA}^{o})}\right]}{\left[1-|\tilde{R}|e^{i(2kx+\Delta\varphi_{BA}^{o})}\right]}\cdot\frac{\left[1-|\tilde{R}|e^{-i(2kx+\Delta\varphi_{BA}^{o})}\right]}{\left[1-|\tilde{R}|e^{-i(2kx+\Delta\varphi_{BA}^{o})}\right]} = z_o\,\frac{\left[1+|\tilde{R}|e^{i(2kx+\Delta\varphi_{BA})}-|\tilde{R}|e^{-i(2kx+\Delta\varphi_{BA}^{o})}-|\tilde{R}|^{2}\right]}{\left[1-|\tilde{R}|e^{i(2kx+\Delta\varphi_{BA})}-|\tilde{R}|e^{-i(2kx+\Delta\varphi_{BA}^{o})}+|\tilde{R}|^{2}\right]}$$

$$= z_o\,\frac{\left[1+|\tilde{R}|\left\{e^{i(2kx+\Delta\varphi_{BA}^{o})}-e^{-i(2kx+\Delta\varphi\varphi_{BA}^{o})}\right\}-|\tilde{R}|^{2}\right]}{\left[1-|\tilde{R}|\left\{e^{i(2kx+\Delta\varphi_{BA}^{o})}+e^{-i(2kx+\Delta\varphi_{BA}^{o})}\right\}+|\tilde{R}|^{2}\right]} = z_o\,\frac{\left[\left\{1-|\tilde{R}|^{2}\right\}+|\tilde{R}|\left\{e^{i(2kx+\Delta\varphi_{BA}^{o})}-e^{-i(2kx+\Delta\varphi_{BA}^{o})}\right\}\right]}{\left[\left\{1+|\tilde{R}|^{2}\right\}-|\tilde{R}|\left\{e^{i(2kx+\varphi_{BA}^{o})}+e^{-i(2kx+\Delta\varphi_{BA}^{o})}\right\}\right]}$$

Using the Euler relations: $\cos\theta \equiv \frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)$ and: $\sin\theta \equiv \frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)$:

$$\tilde{z}^{\parallel}_{a\,tot}(x) = z_o \frac{\left[\left\{1-\left|\tilde{R}\right|^2\right\}+2i\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi^o_{BA}\right)\right]}{\left[\left\{1+\left|\tilde{R}\right|^2\right\}-2\left|\tilde{R}\right|\cos\left(2kx+\Delta\varphi^o_{BA}\right)\right]}$$

We can also write this as a dimensionless quantity:

$$\frac{\tilde{z}^{\parallel}_{a\,tot}(x)}{z_o} = \frac{\left[\left\{1-\left|\tilde{R}\right|^2\right\}+2i\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi^o_{BA}\right)\right]}{\left[\left\{1+\left|\tilde{R}\right|^2\right\}-2\left|\tilde{R}\right|\cos\left(2kx+\Delta\varphi^o_{BA}\right)\right]} = \frac{\tilde{c}^{\parallel}_{a\,tot}(x)}{c}$$

Note that $\tilde{z}^{\parallel}_{tot}(x)$ and $\tilde{c}^{\parallel}_{a\,tot}(x)$ have ***no*** explicit time dependence, but both have spatial/position (*x*-) and frequency (*f*-) dependence (via the wavenumber $k = 2\pi/\lambda = 2\pi f/c = \omega/c$)!

The ***magnitude*** of the complex longitudinal specific acoustic impedance associated with the two counter-propagating 1-D monochromatic plane waves is:

$$\begin{aligned}
\left|\tilde{z}^{\parallel}_{a\,tot}(x)\right| &\equiv \sqrt{\tilde{z}^{\parallel}_{tot}(x)\cdot\tilde{z}^{\parallel *}_{tot}(x)} \\
&= z_o \frac{\sqrt{\left[\left\{1-\left|\tilde{R}\right|^2\right\}+2i\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi^o_{BA}\right)\right]\cdot\left[\left\{1-\left|\tilde{R}\right|^2\right\}+2i\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi^o_{BA}\right)\right]^*}}{\left[\left\{1+\left|\tilde{R}\right|^2\right\}-2\left|\tilde{R}\right|\cos\left(2kx+\Delta\varphi^o_{BA}\right)\right]} \\
&= z_o \frac{\sqrt{\left[\left\{1-\left|\tilde{R}\right|^2\right\}+2i\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi^o_{BA}\right)\right]\cdot\left[\left\{1-\left|\tilde{R}\right|^2\right\}-2i\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi^o_{BA}\right)\right]}}{\left[\left\{1+\left|\tilde{R}\right|^2\right\}-2\left|\tilde{R}\right|\cos\left(2kx+\Delta\varphi^o_{BA}\right)\right]} \\
&= z_o \frac{\sqrt{\left\{1-\left|\tilde{R}\right|^2\right\}^2+4\left|\tilde{R}\right|^2\sin^2\left(2kx+\Delta\varphi^o_{BA}\right)}}{\left[\left\{1+\left|\tilde{R}\right|^2\right\}-2\left|\tilde{R}\right|\cos\left(2kx+\Delta\varphi^o_{BA}\right)\right]}
\end{aligned}$$

Again, we can write this as a dimensionless quantity:

$$\frac{\left|\tilde{z}^{\parallel}_{a\,tot}(x)\right|}{z_o} = \frac{\sqrt{\left\{1-\left|\tilde{R}\right|^2\right\}^2+4\left|\tilde{R}\right|^2\sin^2\left(2kx+\Delta\varphi^o_{BA}\right)}}{\left[\left\{1+\left|\tilde{R}\right|^2\right\}-2\left|\tilde{R}\right|\cos\left(2kx+\Delta\varphi^o_{BA}\right)\right]} = \frac{\left|\tilde{c}^{\parallel}_{a\,tot}(x)\right|}{c}$$

The ***phase*** of the complex longitudinal ***specific*** acoustic impedance and longitudinal energy flow velocity associated with the two counter-propagating 1-D monochromatic plane waves is:

$$\varphi_{z_{a\,tot}}(x) \equiv \tan^{-1}\left(\frac{\operatorname{Im}\{\tilde{z}^{\parallel}_{a\,tot}(x)\}}{\operatorname{Re}\{\tilde{z}^{\parallel}_{a\,tot}(x)\}}\right) = \tan^{-1}\left(\frac{\cancel{z_o}\left[2|\tilde{R}|\sin(2kx+\Delta\varphi^o_{BA}) \Big/ \cancel{\left[\{1+|\tilde{R}|^2\}-2|\tilde{R}|\cos(2kx+\Delta\varphi^o_{BA})\right]}\right]}{\cancel{z_o}\left[\{1-|\tilde{R}|^2\} \Big/ \cancel{\left[\{1+|\tilde{R}|^2\}-2|\tilde{R}|\cos(2kx+\Delta\varphi^o_{BA})\right]}\right]}\right)$$

$$= \tan^{-1}\left(\frac{2|\tilde{R}|\sin(2kx+\Delta\varphi^o_{BA})}{\{1-|\tilde{R}|^2\}}\right) = \Delta\varphi_{p_{tot}-u^{\parallel}_{tot}}(x) = \varphi_{p_{tot}}(x) - \varphi_{u^{\parallel}_{tot}}(x) = \varphi_{c_{a\,tot}}(x)$$

Thus, *e.g.* for an observer/listener's position $x = 0$, **.and.** for ***equal-strength*** pressure amplitudes $|\tilde{A}| = |\tilde{B}| \;\Rightarrow\; |\tilde{R}| \equiv |\tilde{B}|/|\tilde{A}| = 1$ (*i.e.* a "***pure***" standing wave) these two formulae simplify to:

$$\tilde{z}^{\parallel}_{a\,tot}(x=0) = z_o\frac{i\sin\Delta\varphi^o_{BA}}{[1-\cos\Delta\varphi^o_{BA}]}, \text{ or: } \tilde{c}^{\parallel}_{a\,tot}(x=0) = c\frac{i\sin\Delta\varphi^o_{BA}}{[1-\cos\Delta\varphi^o_{BA}]}$$

and:

$$\varphi_{z_a}(x=0) = \varphi_{c_a}(x=0) = \tan^{-1}\left(\frac{2\sin\Delta\varphi^o_{BA}}{0}\right) = \tan^{-1}(\pm\infty)$$
$$= \Delta\varphi_{p_{tot}-u_{tot}}(x=0) = \varphi_{p_{tot}}(x=0) - \varphi_{u_{tot}}(x=0)$$
$$= \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, ... \;\; = \pm n_{odd}\,\pi/2$$

i.e. for an observer/listener's position $x = 0$, **.and.** for ***equal-strength*** pressure amplitudes $|\tilde{A}| = |\tilde{B}|$ $\Rightarrow |\tilde{R}| \equiv |\tilde{B}|/|\tilde{A}| = 1$ the complex longitudinal ***specific*** acoustic impedance $\tilde{z}^{\parallel}_{tot}(x=0)$ is ***purely imaginary***; its phase $\varphi_z(x=0)$ is an ***odd*** integer multiple of $\pm\pi/2 = \pm 90^{\rm o}$ – which in turn also tells us that in this situation, the complex pressure $\tilde{p}_{tot}(x=0,t)$ and longitudinal particle velocity $\tilde{u}^{\parallel}_{tot}(x=0,t)$ differ in phase by an ***odd*** integer multiple of $\pm\pi/2 = \pm 90^{\rm o}$.

Note that in general, for arbitrary values of x, ***maxima*** of the complex longitudinal ***specific*** acoustic impedance $\tilde{z}^{\parallel}_{tot}(x)$ occur whenever $(2kx+\Delta\varphi^o_{BA}) = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi ... = \pm n_{even}\pi$, *i.e.* whenever $\cos(2kx+\Delta\varphi^o_{BA}) = +1$, and thus $\sin(2kx+\Delta\varphi^o_{BA}) = 0$, then:

$$\frac{|\tilde{z}^{\parallel}_{a\,tot}(x)|_{\rm maxima}}{z_o} = \frac{\sqrt{\{1-|\tilde{R}|^2\}^2 + 4|\tilde{R}|^2\sin^2(2kx+\Delta\varphi^o_{BA})}}{\left[\{1+|\tilde{R}|^2\}-2|\tilde{R}|\cos(2kx+\Delta\varphi^o_{BA})\right]} = \frac{\sqrt{\{1-|\tilde{R}|^2\}^2}}{\left[\{1+|\tilde{R}|^2\}-2|\tilde{R}|\right]}$$

$$= \left(\frac{1-|\tilde{R}|^2}{1-2|\tilde{R}|+|\tilde{R}|^2}\right) = \left(\frac{1-|\tilde{R}|^2}{(1-|\tilde{R}|)^2}\right) = \frac{\cancel{(1-|\tilde{R}|)}\cdot(1+|\tilde{R}|)}{\cancel{(1-|\tilde{R}|)}\cdot(1-|\tilde{R}|)} = \frac{(1+|\tilde{R}|)}{(1-|\tilde{R}|)} = \frac{|\tilde{c}^{\parallel}_{a\,tot}(x)|_{\rm maxima}}{c}$$

The phase(s) associated with the complex longitudinal ***specific*** acoustic impedance and complex longitudinal energy flow velocity ***maxima*** occur when:

$$\varphi_{z_a}(x)\Big|_{\text{maxima}} = \tan^{-1}\left(\frac{2\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi_{BA}^{o}\right)}{\left\{1-\left|\tilde{R}\right|^2\right\}}\right) = \tan^{-1}(0) = 0$$

$$= \Delta\varphi_{p_{tot}-u_{tot}^{\parallel}}(x)\Big|_{\text{maxima}} = \left(\varphi_{p_{tot}}(x) - \varphi_{u_{tot}^{\parallel}}(x)\right)_{\text{maxima}} = \varphi_{c_a}(x)\Big|_{\text{maxima}}$$

Thus, for longitudinal ***specific*** acoustic impedance and longitudinal energy flow velocity ***maxima*** associated with this situation, we see that the total/resultant complex pressure $\tilde{p}_{tot}(x,t)$ and longitudinal particle velocity $\tilde{u}_{tot}^{\parallel}(x,t)$ are precisely ***in-phase*** with each other, or at least by ± ***even*** integer multiples of π.

Since $\left|\tilde{z}_{tot}^{\parallel}(x)\right|_{\text{maxima}} = \left|\tilde{p}_{tot}(x,t)\right| / \left|\tilde{u}_{tot}^{\parallel}(x,t)\right|_{\text{maxima}}$ this also tells us that whenever $\left(2kx+\Delta\varphi_{BA}^{o}\right) = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi ... = \pm n_{even}\pi$, the ***magnitude*** of the total/resultant complex pressure $\left|\tilde{p}_{tot}(x,t)\right|$ will also be a ***maxima***, whereas the ***magnitude*** of the total/resultant complex longitudinal particle velocity $\left|\tilde{u}_{tot}^{\parallel}(x,t)\right|$ will simultaneously be a ***minima***:

$$\tilde{p}_{tot}(x,t) = \tilde{A}\left[1 + \left|\tilde{R}\right| e^{i\left(2kx+\Delta\varphi_{BA}^{o}\right)}\right]\cdot e^{i(\omega t - kx)} = \tilde{A}\left[1 + \left|\tilde{R}\right| e^{\pm i n_{even}\pi}\right]\cdot e^{i(\omega t-kx)}$$

$$= \tilde{A}\left[1 + \left|\tilde{R}\right|\left\{\cos\left(\pm n_{even}\pi\right) + \cancel{i\sin\left(\pm n_{even}\pi\right)}\right\}\right]\cdot e^{i(\omega t-kx)} = \tilde{A}\left[1 + \left|\tilde{R}\right|\right]\cdot e^{i(\omega t-kx)}$$

and:

$$\tilde{u}_{tot}^{\parallel}(x,t) = \frac{\tilde{A}}{\rho_o c}\left[1 - \left|\tilde{R}\right| e^{i\left(2kx+\Delta\varphi_{BA}^{o}\right)}\right]\cdot e^{i(\omega t-kx)} = \frac{\tilde{A}}{\rho_o c}\left[1 - \left|\tilde{R}\right| e^{\pm i n_{even}\pi}\right]\cdot e^{i(\omega t-kx)}$$

$$= \frac{\tilde{A}}{\rho_o c}\left[1 - \left|\tilde{R}\right|\left\{\cos\left(\pm n_{even}\pi\right) + \cancel{i\sin\left(\pm n_{even}\pi\right)}\right\}\right]\cdot e^{i(\omega t-kx)} = \frac{\tilde{A}}{\rho_o c}\left[1 - \left|\tilde{R}\right|\right]\cdot e^{i(\omega t-kx)}$$

$$\Rightarrow \quad \left|\tilde{p}_{tot}(x,t)\right| \equiv \sqrt{\tilde{p}_{tot}(x,t)\cdot\tilde{p}_{tot}^{*}(x,t)} = \left|\tilde{A}\right|\left[1 + \left|\tilde{R}\right|\right] \qquad \text{for } \left|\tilde{R}\right| = 1: \left|\tilde{p}_{tot}(x,t)\right| = 2\left|\tilde{A}\right|$$

$$\Rightarrow \quad \left|\tilde{u}_{tot}^{\parallel}(x,t)\right| \equiv \sqrt{\tilde{u}_{tot}^{\parallel}(x,t)\cdot\tilde{u}_{tot}^{\parallel *}(x,t)} = \frac{\left|\tilde{A}\right|}{\rho_o c}\left[1 - \left|\tilde{R}\right|\right] \qquad \text{for } \left|\tilde{R}\right| = 1: \left|\tilde{u}_{tot}^{\parallel}(x,t)\right| = 0$$

"Pure" standing wave!!!

In general, for arbitrary values of *x*, ***minima*** of the complex longitudinal ***specific*** acoustic impedance $\tilde{z}_{a_{tot}}^{\parallel}(x)$ and the complex longitudinal energy flow velocity $\tilde{c}_{a_{tot}}^{\parallel}(x)$ will occur whenever $\left(2kx+\Delta\varphi_{BA}^{o}\right) = \pm 1\pi, \pm 3\pi, \pm 5\pi ... = \pm n_{odd}\pi$, *i.e.* whenever $\cos\left(2kx+\Delta\varphi_{BA}^{o}\right) = -1$, and $\sin\left(2kx+\Delta\varphi_{BA}^{o}\right) = 0$, then:

$$\frac{\left|\tilde{z}^{\|}_{a\,tot}(x)\right|_{\text{minima}}}{z_o} = \frac{\sqrt{\left\{1-\left|\tilde{R}\right|^2\right\}^2 + 4\left|\tilde{R}\right|^2 \sin^2\left(2kx+\Delta\varphi^o_{BA}\right)}}{\left[\left\{1+\left|\tilde{R}\right|^2\right\} - 2\left|\tilde{R}\right|\cos\left(2kx+\Delta\varphi^o_{BA}\right)\right]} = \frac{\sqrt{\left\{1-\left|\tilde{R}\right|^2\right\}^2}}{\left[\left\{1+\left|\tilde{R}\right|^2\right\}+2\left|\tilde{R}\right|\right]}$$

$$= \left(\frac{1-\left|\tilde{R}\right|^2}{1+2\left|\tilde{R}\right|+\left|\tilde{R}\right|^2}\right) = \left(\frac{1-\left|\tilde{R}\right|^2}{\left(1+\left|\tilde{R}\right|\right)^2}\right) = \frac{\left(1-\left|\tilde{R}\right|\right)\cdot\cancel{\left(1+\left|\tilde{R}\right|\right)}}{\left(1+\left|\tilde{R}\right|\right)\cdot\cancel{\left(1+\left|\tilde{R}\right|\right)}} = \frac{\left(1-\left|\tilde{R}\right|\right)}{\left(1+\left|\tilde{R}\right|\right)} = \frac{\left|\tilde{c}^{\|}_{a\,tot}(x)\right|_{\text{minima}}}{c}$$

The phase(s) associated with the complex longitudinal ***specific*** acoustic impedance and complex energy flow velocity ***minima*** are:

$$\varphi_{z_a}(x)\Big|_{\text{minima}} = \tan^{-1}\left(\frac{2\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi^o_{BA}\right)}{\left\{1-\left|\tilde{R}\right|^2\right\}}\right) = \tan^{-1}(0) = 0$$

$$= \Delta\varphi_{p_{tot}-u_{tot}}(x)\Big|_{\text{minima}} = \left(\varphi_{p_{tot}}(x) - \varphi_{u_{tot}}(x)\right)_{\text{minima}} = \varphi_{c_a}(x)\Big|_{\text{minima}}$$

Thus, for longitudinal ***specific*** acoustic impedance and longitudinal energy flow velocity ***minima*** associated with this situation, we see that the total/resultant complex pressure $\tilde{p}_{tot}(x,t)$ and longitudinal particle velocity $\tilde{u}^{\|}_{tot}(x,t)$ are precisely ***out-of-phase*** with each other, or at least by ± ***odd*** integer multiples of π. Since $\left|\tilde{z}^{\|}_{tot}(x)\right|_{\text{minima}} = \left|\tilde{p}_{tot}(x,t)\right| / \left|\tilde{u}^{\|}_{tot}(x,t)\right|_{\text{minima}}$ this also tells us that whenever $(2kx+\Delta\varphi_{BA}) = \pm 1\pi, \pm 3\pi, \pm 5\pi ... = \pm n_{odd}\pi$, the magnitude of the total/resultant complex pressure $\left|\tilde{p}_{tot}(x,t)\right|$ will also be a ***minima***, whereas the magnitude of the total/resultant complex longitudinal particle velocity $\left|\tilde{u}^{\|}_{tot}(x,t)\right|$ will simultaneously be a ***maxima***:

$$\tilde{p}_{tot}(x,t) = \tilde{A}\left[1 + \left|\tilde{R}\right| e^{i\left(2kx+\Delta\varphi^o_{BA}\right)}\right]\cdot e^{i(\omega t - kx)} = \tilde{A}\left[1 + \left|\tilde{R}\right| e^{\pm i n_{odd}\pi}\right]\cdot e^{i(\omega t-kx)}$$

$$= \tilde{A}\left[1 + \left|\tilde{R}\right|\left\{\cos(\pm n_{odd}\pi) + \cancel{i\sin(\pm n_{odd}\pi)}\right\}\right]\cdot e^{i(\omega t-kx)} = \tilde{A}\left[1 - \left|\tilde{R}\right|\right]\cdot e^{i(\omega t-kx)}$$

and:

$$\tilde{u}^{\|}_{tot}(x,t) = \frac{\tilde{A}}{\rho_o c}\left[1 - \left|\tilde{R}\right| e^{i\left(2kx+\Delta\varphi^o_{BA}\right)}\right]\cdot e^{i(\omega t-kx)} = \frac{\tilde{A}}{\rho_o c}\left[1 - \left|\tilde{R}\right| e^{\pm i n_{odd}\pi}\right]\cdot e^{i(\omega t-kx)}$$

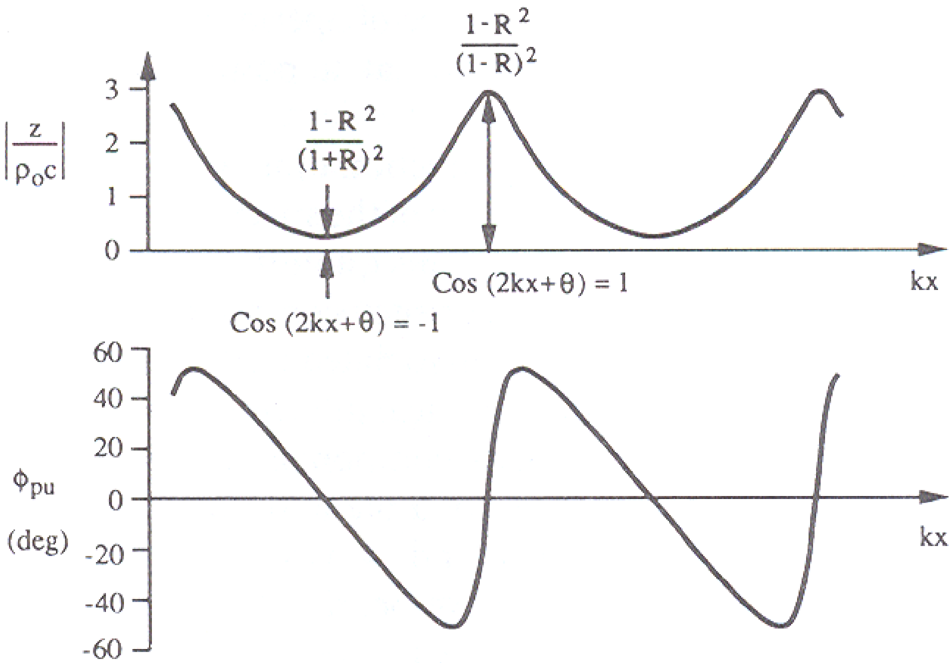
$$= \frac{\tilde{A}}{\rho_o c}\left[1 - \left|\tilde{R}\right|\left\{\cos(\pm n_{odd}\pi) + \cancel{i\sin(\pm n_{odd}\pi)}\right\}\right]\cdot e^{i(\omega t-kx)} = \frac{\tilde{A}}{\rho_o c}\left[1 + \left|\tilde{R}\right|\right]\cdot e^{i(\omega t-kx)}$$

$$\Rightarrow \quad \left|\tilde{p}_{tot}(x,t)\right| \equiv \sqrt{\tilde{p}_{tot}(x,t)\cdot\tilde{p}^*_{tot}(x,t)} = \left|\tilde{A}\right|\left[1 - \left|\tilde{R}\right|\right] \qquad \text{for } \left|\tilde{R}\right| = 1: \ \left|\tilde{p}_{tot}(x,t)\right| = 0$$

$$\Rightarrow \quad \left|\tilde{u}^{\|}_{tot}(x,t)\right| \equiv \sqrt{\tilde{u}^{\|}_{tot}(x,t)\cdot\tilde{u}^{\|*}_{tot}(x,t)} = \frac{\left|\tilde{A}\right|}{\rho_o c}\left[1 + \left|\tilde{R}\right|\right] \qquad \text{for } \left|\tilde{R}\right| = 1: \ \left|\tilde{u}^{\|}_{tot}(x,t)\right| = \frac{2\left|\tilde{A}\right|}{\rho_o c}$$

"Pure" standing wave!!!

A perhaps somewhat more general situation associated with two counter-propagating monochromatic plane waves in "***free air***", is *e.g*. the case when $|\tilde{R}| \equiv |\tilde{B}|/|\tilde{A}| = 0.5$ and $\Delta\varphi_{BA}^{o} = 0.5$; the {normalized} magnitude of the complex longitudinal ***specific*** acoustic impedance $|\tilde{z}_{tot}^{\parallel}(x)|/z_o = |\tilde{z}_{tot}^{\parallel}(x)|/\rho_o c$ and its phase $\varphi_z(x) = \Delta\varphi_{p_{tot}-u_{tot}^{\parallel}}(x) = \varphi_{p_{tot}}(x) - \varphi_{u_{tot}^{\parallel}}(x)$ *vs.* kx are shown in the figure(s) below.



The complex ***frequency-domain*** total/resultant complex longitudinal sound intensity associated with two counter-propagating monochromatic plane waves in "***free air***" , with $k = \omega/c$ is:

$$\tilde{I}_{a_{tot}}^{\parallel}(x,\omega) \equiv \frac{1}{2}\tilde{p}_{tot}(x,\omega)\cdot\tilde{u}_{tot}^{\parallel *}(x,\omega)$$

$$= \frac{1}{2}\tilde{A}\left[1+|\tilde{R}|e^{i(2kx+\Delta\varphi_{BA}^{o})}\right]\cancel{e^{i(\omega t-kx)}}\cdot\frac{\tilde{A}^{*}}{\rho_o c}\left[1-|\tilde{R}|e^{-i(2kx+\Delta\varphi_{BA}^{o})}\right]\cancel{e^{-i(\omega t-kx)}}$$

$$= \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c}\left[1+|\tilde{R}|e^{i(2kx+\Delta\varphi_{BA}^{o})}\right]\cdot\left[1-|\tilde{R}|e^{-i(2kx+\Delta\varphi_{BA}^{o})}\right]$$

$$= \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c}\left[1+|\tilde{R}|\left\{e^{i(2kx+\Delta\varphi_{BA}^{o})}-e^{-i(2kx+\Delta\varphi_{BA}^{o})}\right\}-|\tilde{R}|^2\right] \quad \text{and using: } z_o \equiv \rho_o c$$

$$= \frac{1}{2}\frac{|\tilde{A}|^2}{z_o}\left[1+2i|\tilde{R}|\sin\left(2kx+\Delta\varphi_{BA}^{o}\right)-|\tilde{R}|^2\right] = \frac{1}{2}\frac{|\tilde{A}|^2}{z_o}\left[\left\{1-|\tilde{R}|^2\right\}+2i|\tilde{R}|\sin\left(2kx+\Delta\varphi_{BA}^{o}\right)\right]$$

The ***phase*** associated with the complex ***frequency-domain*** $\tilde{I}^{\parallel}_{a_{tot}}(x,\omega)$ is:

$$\varphi_{I_{a\,tot}}(x,\omega) \equiv \tan^{-1}\left(\frac{\operatorname{Im}\left\{\tilde{I}^{\parallel}_{a\,tot}(x,\omega)\right\}}{\operatorname{Re}\left\{\tilde{I}^{\parallel}_{a\,tot}(x,\omega)\right\}}\right) = \tan^{-1}\left(\frac{\frac{1}{2}\frac{|\tilde{A}|^2}{z_o}\left[2|\tilde{R}|\sin\left(2kx+\Delta\varphi^o_{BA}\right)\right]}{\frac{1}{2}\frac{|\tilde{A}|^2}{z_o}\left[\left\{1-|\tilde{R}|^2\right\}\right]}\right) = \tan^{-1}\left(\frac{\left[2|\tilde{R}|\sin\left(2kx+\Delta\varphi^o_{BA}\right)\right]}{\left[\left\{1-|\tilde{R}|^2\right\}\right]}\right)$$

Compare the above ***frequency-domain*** total/resultant complex longitudinal acoustic intensity expressions to those associated with the complex longitudinal ***specific*** acoustic impedance and complex longitudinal energy flow velocity:

$$\frac{\tilde{z}^{\parallel}_{a_{tot}}(x,\omega)}{z_o} = \frac{\left[\left\{1-|\tilde{R}|^2\right\}+2i|\tilde{R}|\sin\left(2kx+\Delta\varphi^o_{BA}\right)\right]}{\left[\left\{1+|\tilde{R}|^2\right\}-2|\tilde{R}|\cos\left(2kx+\Delta\varphi^o_{BA}\right)\right]} = \frac{\tilde{c}^{\parallel}_{a_{tot}}(x,\omega)}{c}$$

Since: $\vec{\tilde{I}}_a \equiv \frac{1}{2}\tilde{p}\vec{\tilde{u}}^*$ and $\vec{\tilde{z}}_a \equiv \frac{\tilde{p}}{\vec{\tilde{u}}} = \frac{\tilde{p}}{\vec{\tilde{u}}}\frac{\bullet\vec{\tilde{u}}^*}{\bullet\vec{\tilde{u}}^*} = \frac{\tilde{p}\vec{\tilde{u}}^*}{|\vec{\tilde{u}}|^2} = \frac{2\vec{\tilde{I}}_a}{|\vec{\tilde{u}}|^2}$, or: $\vec{\tilde{I}}_a(x,\omega) = \frac{1}{2}|\vec{\tilde{u}}(x,\omega)|^2\,\vec{\tilde{z}}_a(x,\omega)$,

For the situation ***here*** with counter-propagating 1-D monochromatic traveling plane waves, and using $k=\omega/c$:

$$|\tilde{u}^{\parallel}_{tot}(x,\omega)|^2 = \frac{|\tilde{A}|^2}{z_o^2}\left[1-2|\tilde{R}|\cos\left(2kx+\Delta\varphi_{BA}\right)+|\tilde{R}|^2\right] = \frac{|\tilde{A}|^2}{z_o^2}\left[\left\{1+|\tilde{R}|^2\right\}-2|\tilde{R}|\cos\left(2kx+\Delta\varphi_{BA}\right)\right]$$

Thus we see that, indeed:

$$\frac{2\tilde{I}^{\parallel}_{a_{tot}}(x,\omega)}{|\tilde{u}^{\parallel}_{tot}(x,\omega)|^2} = \frac{2\cdot\frac{1}{2}\frac{|\tilde{A}|^2}{z_o}\left[\left\{1-|\tilde{R}|^2\right\}+2i|\tilde{R}|\sin\left(2kx+\Delta\varphi^o_{BA}\right)\right]}{\frac{|\tilde{A}|^2}{z_o^2}\left[\left\{1+|\tilde{R}|^2\right\}-2|\tilde{R}|\cos\left(2kx+\Delta\varphi^o_{BA}\right)\right]} = z_o\frac{\left[\left\{1-|\tilde{R}|^2\right\}+2i|\tilde{R}|\sin\left(2kx+\Delta\varphi^o_{BA}\right)\right]}{\left[\left\{1+|\tilde{R}|^2\right\}-2|\tilde{R}|\cos\left(2kx+\Delta\varphi^o_{BA}\right)\right]} = \tilde{z}^{\parallel}_{a_{tot}}(x,\omega)$$

i.e. that:

$$\frac{2\tilde{I}^{\parallel}_{a_{tot}}(x,\omega)}{|\tilde{u}^{\parallel}_{tot}(x,\omega)|^2 z_o} = \frac{\left[\left\{1-|\tilde{R}|^2\right\}+2i|\tilde{R}|\sin\left(2kx+\Delta\varphi^o_{BA}\right)\right]}{\left[\left\{1+|\tilde{R}|^2\right\}-2|\tilde{R}|\cos\left(2kx+\Delta\varphi^o_{BA}\right)\right]} = \frac{\tilde{z}^{\parallel}_{a_{tot}}(x,\omega)}{z_o} = \frac{\tilde{c}^{\parallel}_{a_{tot}}(x,\omega)}{c}$$

and we see that:

$$\varphi_{z_{a\,tot}}(x,\omega) \equiv \tan^{-1}\left(\frac{\mathrm{Im}\left\{\tilde{z}^{\parallel}_{a\,tot}(x,\omega)\right\}}{\mathrm{Re}\left\{\tilde{z}^{\parallel}_{a\,tot}(x,\omega)\right\}}\right) = \tan^{-1}\left(\frac{2\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi^{o}_{BA}\right)}{\left\{1-\left|\tilde{R}\right|^2\right\}}\right)$$

$$= \Delta\varphi_{p_{tot}-u^{\parallel}_{tot}}(x,\omega) = \varphi_{p_{tot}}(x,\omega) - \varphi_{u^{\parallel}_{tot}}(x,\omega) =$$

$$\varphi_{c_{a\,tot}}(x,\omega) \equiv \tan^{-1}\left(\frac{\mathrm{Im}\left\{\tilde{c}^{\parallel}_{a\,tot}(x,\omega)\right\}}{\mathrm{Re}\left\{\tilde{c}^{\parallel}_{a\,tot}(x,\omega)\right\}}\right)$$

Hence, we also see that:

$$\varphi_{I_{a\,tot}}(x,\omega) = \varphi_{z_{a\,tot}}(x,\omega) = \varphi_{c_{a\,tot}}(x,\omega) = \Delta\varphi_{p_{tot}-u^{\parallel}_{tot}}(x,\omega) = \varphi_{p_{tot}}(x,\omega) - \varphi_{u^{\parallel}_{tot}}(x,\omega)$$

$$= \tan^{-1}\left(\frac{2\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi^{o}_{BA}\right)}{\left\{1-\left|\tilde{R}\right|^2\right\}}\right)$$

When $\left|\tilde{R}\right| \equiv \left|\tilde{B}\right| / \left|\tilde{A}\right| = 1$ (*i.e.* a ***pure*** standing wave!), then:

$$\tilde{I}^{\parallel}_{a_{tot}}(x,\omega) \equiv \tfrac{1}{2}\,\tilde{p}_{tot}(x,\omega)\cdot\tilde{u}^{\parallel *}_{tot}(x,\omega) = \frac{1}{2}\frac{\left|\tilde{A}\right|^2}{z_o}\left[\left\{1-\left|\tilde{R}\right|^2\right\} + 2i\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi^{o}_{BA}\right)\right] = \tilde{I}^{\parallel}_{a_{tot}\,\mathrm{r}}(x,\omega) + i\tilde{I}^{\parallel}_{a_{tot}\,\mathrm{i}}(x,\omega)$$

For an observer/listener's position at $x = 0$ **.and.** $\left|\tilde{R}\right| = 1$, this reduces to:

$$\tilde{I}^{\parallel}_{a_{tot}}(x=0,\omega) \equiv \tfrac{1}{2}\,\tilde{p}_{tot}(x=0,\omega)\cdot\tilde{u}^{\parallel *}_{tot}(x=0,\omega) = i\frac{\left|\tilde{A}\right|^2}{z_o}\sin\Delta\varphi^{o}_{BA}$$ (*n.b.* ***purely imaginary quantity***!)

We see again that when additionally: $\Delta\varphi^{o}_{BA} = 0, \pm1\pi, \pm2\pi, \pm3\pi, \ldots = \pm n\pi$ that: $\tilde{I}^{\parallel}_{a_{tot}}(x=0,t) = 0$!!!

Similarly, we see that $\tilde{I}^{\parallel}_{a_{tot}}(x=0,\omega)$ has a purely ***imaginary*** ***extremum*** amplitude of $\pm\left|\tilde{A}\right|^2/\rho_o c = \pm\left|\tilde{A}\right|^2/z_o$ when $\Delta\varphi_{AB} = \pm\pi/2, \pm3\pi/2, \pm5\pi/2, \ldots = \pm m_{odd}\pi/2$.

Physically, the ***real*** part of the complex ***frequency-domain*** longitudinal acoustic intensity $\tilde{I}^{\parallel}_{a_{tot}}(x,\omega)$ represents the ***frequency-domain*** "amplitude" of the ***net flux/flow*** of acoustic energy crossing unit area per unit time (SI units $Watts/m^2$) – *i.e.* the ***real*** part of the complex acoustic intensity is physically associated with ***propagating*** sound/sound ***radiation***. The ***imaginary*** part of the complex ***frequency-domain*** longitudinal acoustic intensity is physically associated with ***non-propagating*** acoustic energy, *i.e.* energy sloshing back and forth each cycle of oscillation.

The ***instantaneous*** potential, kinetic and total energy densities (*n.b.* ***always purely real, additive quantities***!) associated with two counter-propagating 1-D monochromatic traveling plane waves are:

$$w_{potl}^{inst}(x,t) \equiv \frac{1}{2}\frac{1}{\rho_o c^2} p_{tot}^2(x,t) \qquad = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos\left(\omega t - kx + \varphi_A^o\right) + |\tilde{R}|\cos\left(\omega t + kx + \varphi_B^o\right)\right]^2$$

$$= \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos^2\left(\omega t - kx + \varphi_A^o\right) + 2|\tilde{R}|\cos\left(\omega t - kx + \varphi_A^o\right)\cos\left(\omega t + kx + \varphi_B^o\right) + |\tilde{R}|^2\cos^2\left(\omega t + kx + \varphi_B^o\right)\right]$$

$$w_{kin}^{inst}(x,t) \equiv \frac{1}{2}\rho_o \vec{u}_{tot}^{\parallel}(x,t)\bullet\vec{u}_{tot}^{\parallel}(x,t) = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos\left(\omega t - kx + \varphi_A^o\right) - |\tilde{R}|\cos\left(\omega t + kx + \varphi_B^o\right)\right]^2$$

$$= \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos^2\left(\omega t - kx + \varphi_A^o\right) - 2|\tilde{R}|\cos\left(\omega t - kx + \varphi_A^o\right)\cos\left(\omega t + kx + \varphi_B^o\right) + |\tilde{R}|^2\cos^2\left(\omega t + kx + \varphi_B^o\right)\right]$$

$$w_{tot}^{inst}(x,t) \equiv w_{potl}^{inst}(x,t) + w_{kin}^{inst}(x,t) = \frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos^2\left(\omega t - kx + \varphi_A^o\right) + |\tilde{R}|^2\cos^2\left(\omega t + kx + \varphi_B^o\right)\right]$$

Again, for an observer/listener's position at $x = 0$ **.and.** $|\tilde{R}| = 1$ (*i.e.* a "***pure***" standing wave), these quantities reduce to:

$$w_{potl}^{inst}(x=0,t) = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos\omega t + \cos\left(\omega t + \Delta\varphi_{BA}^o\right)\right]^2$$

$$w_{kin}^{inst}(x=0,t) = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos\omega t - \cos\left(\omega t + \Delta\varphi_{BA}^o\right)\right]^2$$

$$w_{tot}^{inst}(x,t) \equiv w_{potl}^{inst}(x,t) + w_{kin}^{inst}(x,t) = \frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos^2\omega t + |\tilde{R}|^2\cos^2\left(\omega t + \Delta\varphi_{BA}^o\right)\right]$$

We see that when: $\Delta\varphi_{BA}^o = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots = \pm n_{even}\pi$

since: $\cos\left(\theta \pm n_{even}\pi\right) = \left(\cos\theta\cdot\cos n_{even}\pi\right) \mp \left(\sin\theta\cdot \cancel{\sin n_{even}\pi}\right) = \cos\theta$, the ***total*** energy density is ***all*** in the form of ***potential*** energy density:

$$w_{potl}^{inst}(x=0,t) = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos\omega t + \cos\left(\omega t \pm n_{even}\pi\right)\right]^2 = \frac{1}{2}\frac{4|\tilde{A}|^2}{\rho_o c^2}\cos^2\omega t = \frac{2|\tilde{A}|^2}{\rho_o c^2}\cos^2\omega t = \frac{2|\tilde{A}|^2}{z_o c}\cos^2\omega t$$

$$w_{kin}^{inst}(x=0,t) = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos\omega t - \cos\left(\omega t + \Delta\varphi_{BA}^o\right)\right]^2 = 0$$

$$w_{tot}^{inst}(x=0,t) \equiv w_{potl}^{inst}(x=0,t) + w_{kin}^{inst}(x=0,t) = \frac{2|\tilde{A}|^2}{\rho_o c^2}\cos^2\omega t = \frac{2|\tilde{A}|^2}{z_o c}\cos^2\omega t$$

We also see that when: $\Delta\varphi_{BA} = \pm 1\pi, \pm 3\pi, \pm 5\pi, ... = \pm n_{odd}\pi$

since: $\cos(\theta \pm n_{odd}\pi) = (\cos\theta \cdot \cos n_{odd}\pi) \mp (\sin\theta \cdot \cancel{\sin n_{odd}\pi}) = -\cos\theta$, the ***total*** energy density is ***all*** in the form of ***kinetic*** energy density:

$$w_{potl}^{inst}(x=0,t) = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos\omega t + \cos(\omega t \pm n_{odd}\pi)\right]^2 = 0$$

$$w_{kin}^{inst}(x=0,t) = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[\cos\omega t - \cos(\omega t + \Delta\varphi_{BA}^o)\right]^2 = \frac{1}{2}\frac{4|\tilde{A}|^2}{\rho_o c^2}\cos^2\omega t = \frac{2|\tilde{A}|^2}{\rho_o c^2}\cos^2\omega t$$

$$w_{tot}^{inst}(x=0,t) \equiv w_{potl}^{inst}(x=0,t) + w_{kin}^{inst}(x=0,t) = \frac{2|\tilde{A}|^2}{\rho_o c^2}\cos^2\omega t = \frac{2|\tilde{A}|^2}{z_o c}\cos^2\omega t$$

The ***time***-***averaged*** potential, kinetic and total energy densities associated with two counter-propagating 1-D monochromatic traveling plane waves are:

$$\langle w_{potl}(x,t)\rangle_t = \frac{1}{2}\frac{\langle p_{tot}^2(x,t)\rangle_t}{\rho_o c^2} = \frac{1}{4}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[1 + |\tilde{R}|^2 + 2|\tilde{R}|\cos(2kx + \Delta\varphi_{BA}^o)\right]$$

$$\langle w_{kin}(x,t)\rangle_t = \frac{1}{2}\rho_o\langle u_{tot}^{\parallel 2}(x,t)\rangle_t = \frac{1}{4}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[1 + |\tilde{R}|^2 - 2|\tilde{R}|\cos(2kx + \Delta\varphi_{BA}^o)\right]$$

$$\langle w_{tot}(x,t)\rangle_t \equiv \langle w_{potl}(x,t)\rangle_t + \langle w_{kin}(x,t)\rangle_t = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[1 + |\tilde{R}|^2\right] = \frac{1}{2}\frac{|\tilde{A}|^2}{z_o c}\left[1 + |\tilde{R}|^2\right]$$

Note ***here***, that the ratio of the ***time-averaged potential*** energy density to the ***time-averaged kinetic*** energy density is ***not*** equal to unity for counter-propagating monochromatic plane waves:

$$\frac{\langle w_{potl}(x,t)\rangle_t}{\langle w_{kin}(x,t)\rangle_t} = \frac{\frac{1}{2}\frac{\langle p_{tot}^2(x,t)\rangle_t}{\rho_o c^2}}{\frac{1}{2}\rho_o\langle u_{tot}^{\parallel 2}(x,t)\rangle_t} = \frac{\left[1 + |\tilde{R}|^2 + 2|\tilde{R}|\cos(2kx + \Delta\varphi_{BA}^o)\right]}{\left[1 + |\tilde{R}|^2 - 2|\tilde{R}|\cos(2kx + \Delta\varphi_{BA}^o)\right]} \neq 1$$

Again, for an observer's position at $x = 0$ **.and.** $|\tilde{R}| \equiv |\tilde{B}| / |\tilde{A}| = 1$ (*i.e.* a "***pure***" standing wave), these quantities reduce to:

$$\langle w_{potl}(x=0,t)\rangle_t = \frac{1}{2}\frac{\langle p_{tot}^2(x=0,t)\rangle_t}{\rho_o c^2} = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[1 + \cos\Delta\varphi_{BA}^o\right]$$

$$\langle w_{kin}(x=0,t)\rangle_t = \frac{1}{2}\rho_o\langle u_{tot}^{\parallel 2}(x=0,t)\rangle_t = \frac{1}{2}\frac{|\tilde{A}|^2}{\rho_o c^2}\left[1 - \cos\Delta\varphi_{BA}^o\right]$$

$$\left\langle w_{tot}\left(x=0,t\right)\right\rangle_t \equiv \left\langle w_{potl}\left(x=0\right)\right\rangle_t + \left\langle w_{kin}\left(x=0\right)\right\rangle_t$$

$$= \frac{1}{2}\frac{\left|\tilde{A}\right|^2}{\rho_o c^2}\left[1+\cancel{\cos\Delta\varphi_{BA}^o}\right] + \frac{1}{2}\frac{\left|\tilde{A}\right|^2}{\rho_o c^2}\left[1-\cancel{\cos\Delta\varphi_{BA}^o}\right] = \frac{\left|\tilde{A}\right|^2}{\rho_o c^2} = \frac{\left|\tilde{A}\right|^2}{z_o c}$$

When: $\Delta\varphi_{BA} = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi, ... = \pm n_{even}\pi$ the energy density is all ***potential*** energy density:

$$\left\langle w_{potl}\left(x=0,t\right)\right\rangle_t = \frac{1}{2}\frac{\left\langle p_{tot}^2\left(x=0,t\right)\right\rangle_t}{\rho_o c^2} \quad = \frac{1}{2}\frac{\left|\tilde{A}\right|^2}{\rho_o c^2}\left[1+\cos\left(\pm n_{even}\pi\right)\right] = \frac{\left|\tilde{A}\right|^2}{\rho_o c^2}$$

$$\left\langle w_{kin}\left(x=0,t\right)\right\rangle_t = \frac{1}{2}\rho_o\left\langle u_{tot}^{\parallel 2}\left(x=0,t\right)\right\rangle_t = \frac{1}{2}\frac{\left|\tilde{A}\right|^2}{\rho_o c^2}\left[1-\cos\left(\pm n_{even}\pi\right)\right] = 0$$

$$\left\langle w_{tot}\left(x=0,t\right)\right\rangle_t \equiv \left\langle w_{potl}\left(x=0,t\right)\right\rangle_t + \left\langle w_{kin}\left(x=0,t\right)\right\rangle_t = \frac{\left|\tilde{A}\right|^2}{\rho_o c^2} + 0 = \frac{\left|\tilde{A}\right|^2}{\rho_o c^2} = \frac{\left|\tilde{A}\right|^2}{z_o c}$$

When: $\Delta\varphi_{AB} = \pm 1\pi, \pm 3\pi, \pm 5\pi, ... = \pm n_{odd}\pi$ the energy density is all ***kinetic*** energy density:

$$\left\langle w_{potl}\left(x=0,t\right)\right\rangle_t = \frac{1}{2}\frac{\left\langle p_{tot}^2\left(x=0,t\right)\right\rangle_t}{\rho_o c^2} \quad = \frac{1}{2}\frac{\left|\tilde{A}\right|^2}{\rho_o c^2}\left[1+\cos\left(\pm n_{odd}\pi\right)\right] = 0$$

$$\left\langle w_{kin}\left(x=0,t\right)\right\rangle_t = \frac{1}{2}\rho_o\left\langle u_{tot}^{\parallel 2}\left(x=0,t\right)\right\rangle_t = \frac{1}{2}\frac{\left|\tilde{A}\right|^2}{\rho_o c^2}\left[1-\cos\left(\pm n_{odd}\pi\right)\right] = \frac{\left|\tilde{A}\right|^2}{\rho_o c^2}$$

$$\left\langle w_{tot}\left(x=0,t\right)\right\rangle_t \equiv \left\langle w_{potl}\left(x=0\right)\right\rangle_t + \left\langle w_{kin}\left(x=0\right)\right\rangle_t = 0 + \frac{\left|\tilde{A}\right|^2}{\rho_o c^2} = \frac{\left|\tilde{A}\right|^2}{\rho_o c^2} = \frac{\left|\tilde{A}\right|^2}{z_o c}$$

We coded up the above acoustic expressions in Matlab to obtain plots of them *vs*. dimensionless position, $\theta = kx$ for various values of $0 \le \left|\tilde{R}\right| \le 1$ for two counter-propagating 1-D monochromatic traveling plane waves and posted a write-up along with the Matlab *.m script on the Physics 406 Software web-page: http://courses.physics.illinois.edu/phys406/406pom_sw.html

Acoustic Reflectance/Transmittance/Absorbance and Acoustic Reflection/Transmission/Absorption Coefficients:

The physical meaning of the complex quantity $\tilde{R} \equiv \tilde{B}/\tilde{A} = |\tilde{R}| e^{i\Delta\varphi^o_{BA}}$ used in (all of) the above formulae for this two counter-propagating monochromatic plane waves problem can also be used to describe various other types of acoustical physics situations, *e.g.* by interpreting $\tilde{R}$ as the ***complex acoustic reflectance*** associated with a sound wave reflecting off of a surface. The {purely real} ***reflection coefficient*** associated with the surface is then defined as: $0 \le \mathbb{R} \equiv |\tilde{R}|^2 = \tilde{R} \cdot \tilde{R}^* \le 1$.

If a sound wave is only partially reflected from a surface, then it is either partially transmitted (with complex acoustic ***transmittance*** $\tilde{T}$ and corresponding {purely real}***transmission coefficient*** $0 \le \mathbb{T} \equiv |\tilde{T}|^2 = \tilde{T} \cdot \tilde{T}^* \le 1$) and/or is absorbed by the surface (with complex acoustic ***absorbance*** $\tilde{A}$ and corresponding {purely real} ***absorption coefficient*** $0 \le \mathbb{A} \equiv |\tilde{A}|^2 = \tilde{A} \cdot \tilde{A}^* \le 1$), since we ***must*** have (by conservation of energy at the surface/interface): $\mathbb{R} + \mathbb{T} + \mathbb{A} = 1$.

Limiting/Special Cases of Interest:

1.) A single monochromatic traveling plane wave (emitted from a sound source *e.g.* located at $x = -\infty$) propagating in the +*ve* *x*-direction and reflects, at normal incidence, off of a ***rigid***, ***perfectly reflecting*** infinite plane (*e.g.* located at $x = x_o > 0$), thereby producing a reflected wave (of ***equal*** amplitude) that propagates in the –*ve* *x*-direction. This situation corresponds to $\tilde{R} = |\tilde{R}| e^0 = +1$ at $x = x_o > 0$, which has the associated boundary condition $\tilde{p}_{refl}(x = x_o, t) = \tilde{p}_{inc}(x = x_o, t)$, *i.e.* **no** phase change occurs upon reflection, such that an over-pressure ***anti-node*** exists at $x = x_o > 0$:

$$\tilde{p}_{tot}(x = x_o, t) = \tilde{p}_{inc}(x = x_o, t) + \tilde{p}_{refl}(x = x_o, t) = 2\tilde{p}_{inc}(x = x_o, t).$$

2.) A single monochromatic traveling plane wave (emitted from a sound source *e.g.* located at $x = -\infty$) propagating in the +*ve* *x*-direction and reflects, at normal incidence, off of an infinite ***pressure-release*** plane consisting of an air-water interface (located at $x = x_o > 0$), thereby producing a reflected wave (of equal amplitude) that propagates in the –*ve* *x*-direction.

This situation corresponds to $\tilde{R} = |\tilde{R}| e^{i\pi} = -1$. An air-water interface (*n.b.* "viewed" from the water side) closely approximates an ***ideal pressure-release surface***, for which the boundary condition at the pressure-release surface is $\tilde{p}_{refl}(x = x_o, t) = -\tilde{p}_{inc}(x = x_o, t)$ (*i.e.* a phase change of 180° occurs upon reflection), such that an over-pressure ***node*** exists at $x = x_o > 0$:

$$\tilde{p}_{tot}(x = x_o, t) = \tilde{p}_{inc}(x = x_o, t) - \tilde{p}_{refl}(x = x_o, t) = 0.$$

3.) **<u>The most general case</u>:** A single monochromatic traveling plane wave (emitted from a sound source *e.g.* located at $x=-\infty$) propagating in the +*ve* *x*-direction and reflects, at normal incidence off of an infinite plane (located at $x=x_o>0$) of ***<u>arbitrary</u>*** characteristics – *e.g.* it could be a "***<u>passive</u>***" surface that is only partially ***<u>reflecting</u>***/***<u>partially</u>*** ***<u>absorbing</u>*** (hence $|\tilde{R}|<1$) and in principle could have associated with it *e.g.* a frequency-dependent phase shift upon reflection $-\pi\le\Delta\varphi_{BA}^{o}\left(x=x_o,\omega\right)\le\pi$, thereby producing a reflected wave that propagates in the –*ve* *x*-direction. This situation physically corresponds to the most general $\tilde{R}=|\tilde{R}|e^{i\Delta\varphi_{BA}^{o}}$. If the reflecting surface were "***<u>active</u>***", it is also possible that $|\tilde{R}|>1$ (!), and depending on the details of the response of the "***active***" reflecting surface, the phase shift could be $-\pi\le\Delta\varphi_{BA}^{o}\left(x=x_o,\omega\right)\le\pi$.

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