Lecture II

Simple One-Dimensional Vibrating Systems

One method of producing a sound relies on a physical object (*e.g*. various types of musical instruments – stringed and wind instruments in particular) to be made to vibrate, by whatever means possible. This vibration is (clearly) mechanical in nature.

Mechanical vibration explicitly means a *displacement* of the (at least some portions of the) matter/material the object is comprised of *from its equilibrium position/configuration* – which requires the input of *energy* to the object in order to accomplish this – initially in the form of (static) *potential* energy (*P.E.*) , which as time progresses, is subsequently transformed into *kinetic* (motional) energy (*K.E.*). As time progresses further, the energy oscillates back and forth between potential and kinetic energy, the total energy, $E_{tot} = P.E.(t) + K.E.(t)$ remaining *constant* in time, if no energy losses (energy dissipation processes) are present in the mechanical system.

The mechanically vibrating object couples to the air surrounding it, transferring energy in this process - sound waves in the air are created, which propagate outwards from the source (the vibrating object) to an observer's ear(s). Thus a sound is heard (perceived). Thus, by *energy conservation*, some of the initial energy input to the mechanically vibrating system *is* radiated away in the form of sound energy. Eventually the mechanically vibrating system ceases to do so, because of this, and other (frictional) dissipative energy loss mechanisms present.

A simple example of a vibrating system is a mass on a spring (a crude model of a vibrating musical instrument) which undergoes so-called 1-D simple harmonic motion:

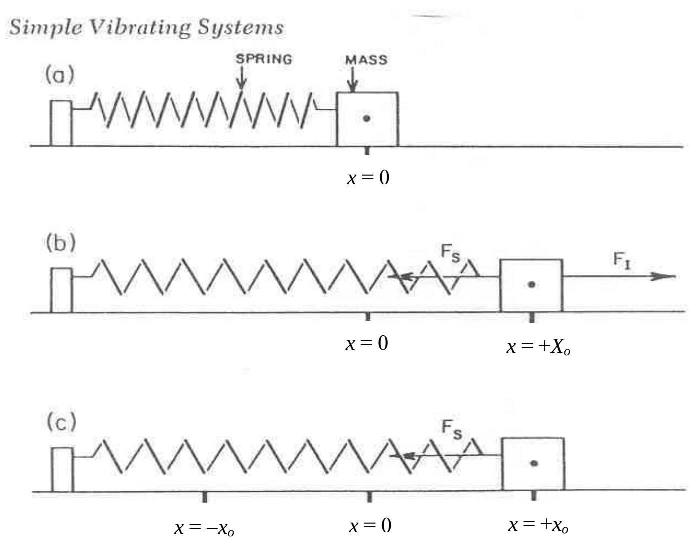


FIG. 1. A n to a spring and restin oth table. (a) Equilibrium position. (b) Mass displaced and held in new position. (c) Mass released.

If there is no friction, and the mass M is horizontally displaced from its equilibrium ($x = 0$) position by pulling on it to the right, as shown in the above figure, the force necessary to accomplish this is $\boxed{F_I = +kx_o}$ (Hooke's Law), where $k > 0$ is the so-called "spring constant" of the spring (k has metric units of Newtons/meter) and x_o = the initial displacement of the mass M from its $x = 0$ equilibrium position.

At time $t = 0$ the mass is released. At that instant, the only force acting on the mass is due to the {horizontal} restoring force of the spring: $\boxed{F_S(t=0) = -kx_o = -F_I}$. However, from Newton's 2nd Law: $F(t) = Ma(t)$, therefore at time $t = 0$: $\boxed{F_S(t=0) = -kx_o = Ma(t=0)}$.

As time progresses the mass M oscillates horizontally back and forth about its $x = 0$ equilibrium position, exhibiting sinusoidal/harmonic motion. Mathematically, the time-dependence of this horizontal sinusoidal/harmonic motion is described by:

Longitudinal displacement from equilibrium: $\boxed{x(t) = x_o \cos(2\pi ft) = x_o \cos(\omega t)}$ (m)

(meters) displacement amplitude (meters) — frequency of oscillation (cycles per second = Hertz) cps Hz

Omega: $\boxed{\omega \equiv 2\pi f = \text{angular frequency (units = radians per second)}}$

Period of oscillation: $\boxed{\tau \equiv \dfrac{1}{f} = \dfrac{2\pi}{\omega}}$ (seconds)

The instantaneous horizontal speed of the moving mass $v(t)$ with time t is defined as the time rate of change of the horizontal position (longitudinal displacement) of the moving mass with time t, physically, $v(t)$ is the instantaneous local slope of the $x(t)$ vs. t graph at time t:

$$v(t) = \frac{\Delta x(t)}{\Delta t} = \frac{dx(t)}{dt} = \text{ total derivative of } x \text{ with respect to time, } t \text{ \{since 1-D partial => total\}.}$$

$$v(t) = \frac{d}{dt}(x(t)) = \frac{d}{dt}\left[x_o \cos(2\pi ft)\right] = -2\pi f\ x_o \sin(2\pi ft) = -\omega x_o \sin(\omega t) \equiv v_o \sin(\omega t)$$

We see that: $\boxed{v_o = -\omega x_o = -2\pi f\ x_o}$

– sign defines the ***phase relation*** between velocity $v(t)$ ***relative*** to displacement $x(t)$.

i.e. the speed "amplitude", v_o = max speed is related to the displacement amplitude, x_o by this formula for ***harmonic*** motion.

Instantaneous Horizontal Speed of the Moving Mass: $\boxed{v(t) = v_o \sin(2\pi ft) = v_o \sin(\omega t)}$ (m/s)

(meters/sec) speed amplitude (m/s) — frequency of oscillation (cycles per second = Hertz)

The instantaneous horizontal acceleration of the moving mass $a(t)$ with time t is defined as the time rate of change of the horizontal speed of the moving mass with time t, physically, $a(t)$ is the instantaneous local slope of the $v(t)$ vs. t graph at time t:

$$a(t) = \frac{\Delta v(t)}{\Delta t} = \frac{dv(t)}{dt} = \text{total derivative of } v(t) \text{ with respect to time, } t.$$

$$a(t) = \frac{d}{dt}\left(v(t)\right) = \frac{d}{dt}\left[v_o \sin\left(2\pi ft\right)\right] = 2\pi f\, v_o \cos\left(2\pi ft\right) = \omega v_o \cos\left(\omega t\right) \equiv a_o \cos\left(\omega t\right)$$

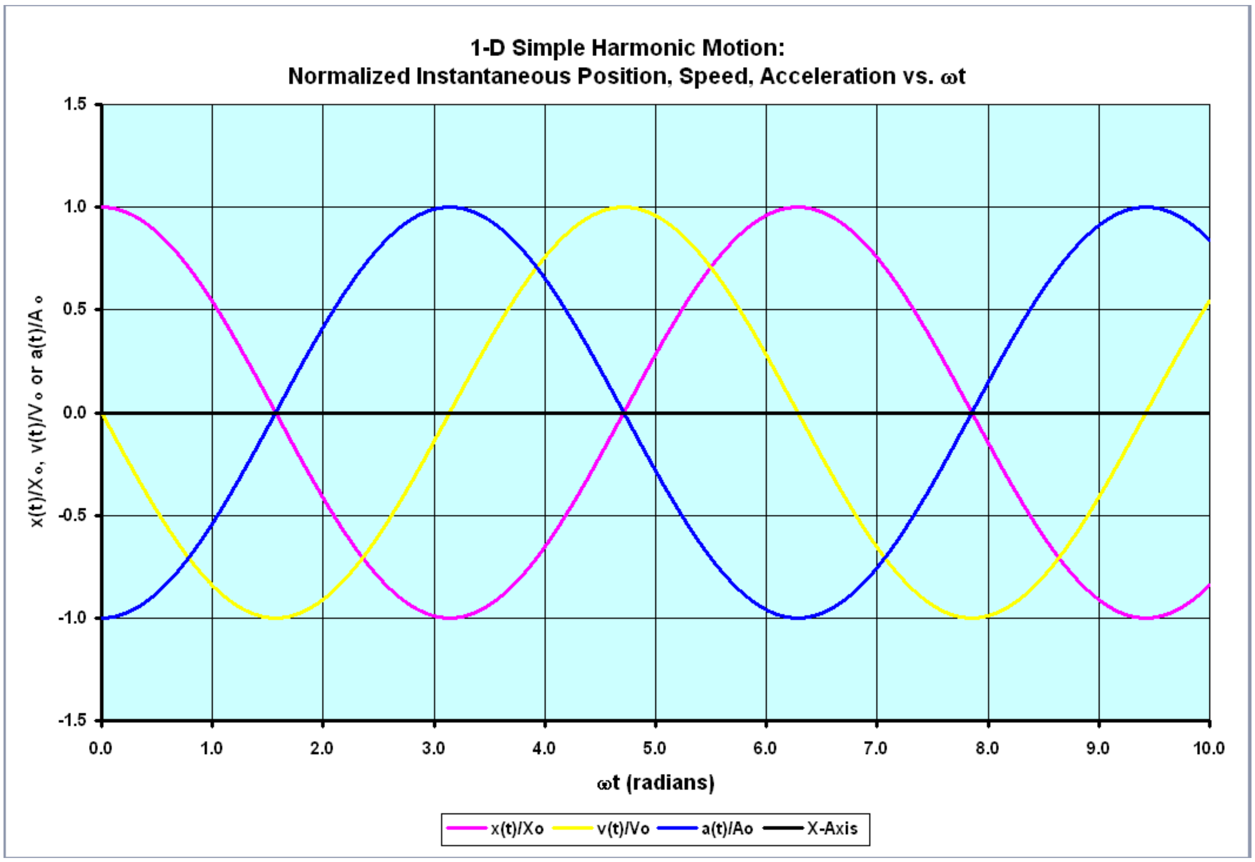
<u>We see that</u>: $a_o = \omega v_o = 2\pi f\, v_o$ <u>but</u>: $v_o = -\omega x_o = -2\pi f\, x_o$ $\therefore$ $a_o = -\omega^2 x_o = -\left(2\pi f\right)^2 x_o$

i.e. the acceleration amplitude, A_o = max acceleration is related to the displacement amplitude, X_o by this formula for ***harmonic*** motion.

<u>Instantaneous Horizontal Accel. of the Moving Mass</u>: $a(t) = a_o \cos\left(2\pi ft\right) = a_o \cos\left(\omega t\right)$ $\left(m/s^2\right)$

(meters/sec^2) acceleration <u>amplitude</u> (m/s^2) frequency of oscillation (cycles per second = Hertz)

The time dependence of the longitudinal position, $x(t)$ (*i.e.* displacement of the mass from its equilibrium position) *vs*. time, t and longitudinal speed of the mass, $v(t)$ *vs*. time, t and longitudinal acceleration $a(t)$ vs. time, t are shown in the figure below; note that each has been normalized to their respective amplitudes (note also the phase relation between $x(t)$, $v(t)$ and $a(t)$):



Once the mass M has been set in motion, Newton's 2nd Law tells us that: $F(t) = -kx(t) = Ma(t)$

However: $x(t) = x_o \cos(2\pi ft) = x_o \cos(\omega t)$ and: $a(t) = -a_o \cos(2\pi ft) = -a_o \cos(\omega t)$

And from above, we also know that: $a_o = \omega^2 x_o = (2\pi f)^2 x_o$ $\therefore$ $-kx_o = -\omega^2 M x_o$

Thus, the frequency f and angular frequency ω of oscillation of the mass M on the spring are:

$f = \frac{1}{2\pi}\sqrt{\frac{k}{M}}$ Cycles per second, or *Hz* and $\omega = 2\pi f = \sqrt{\frac{k}{M}}$ (*radians/sec*)

The period of oscillation τ of the mass M on the spring is: $\tau = \frac{1}{f} = 2\pi\sqrt{\frac{M}{k}}$ (*seconds*)

Note also that since the instantaneous acceleration $a(t) = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2}$, then we can write Newton's 2nd law for this system as a differential equation:

$F(t) = -kx(t) = Ma(t) \Rightarrow -kx(t) = M\frac{d^2x(t)}{dt^2}$ or: $M\frac{d^2x(t)}{dt^2} + kx(t) = 0$ or: $M\ddot{x}(t) + kx(t) = 0$

which is a linear, homogenous 2nd-order differential equation, and where: $\ddot{x}(t) \equiv \frac{d^2x(t)}{dt^2}$.

The instantaneous potential energy *stored* in the stretched/compressed spring is:

$P.E.(t) = \frac{1}{2}kx^2(t)$ (*Joules*)

The instantaneous kinetic energy associated with the *moving* mass, M is:

$K.E.(t) = \frac{1}{2}Mv^2(t)$ (*Joules*)

The potential energy of the spring and the kinetic energy of the moving mass are both time dependent:

$$P.E.(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kx_o^2\cos^2(\omega t) \geq 0$$

$$K.E.(t) = \frac{1}{2}Mv^2(t) = \frac{1}{2}Mv_o^2\sin^2(\omega t) \geq 0$$

However, since: $v_o = -\omega x_o$ and: $\omega = \sqrt{\frac{k}{M}}$ thus: $k = M\omega^2$ Hence, we see that:

$$P.E.(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kx_o^2\cos^2(\omega t)$$

$$K.E.(t) = \frac{1}{2}Mv^2(t) = \frac{1}{2}Mv_o^2\sin^2(\omega t) = \frac{1}{2}M\omega^2x_o^2\sin^2(\omega t) = \frac{1}{2}kx_o^2\sin^2(\omega t)$$

Let us define: $E_o \equiv \frac{1}{2}kx_o^2 = \frac{1}{2}M\omega^2x_o^2$ Then: $P.E.(t) = E_o\cos^2(\omega t)$, $K.E.(t) = E_o\sin^2(\omega t)$

We define the *total* energy, $E_{tot}(t)$ as the sum of instantaneous potential + kinetic energies:

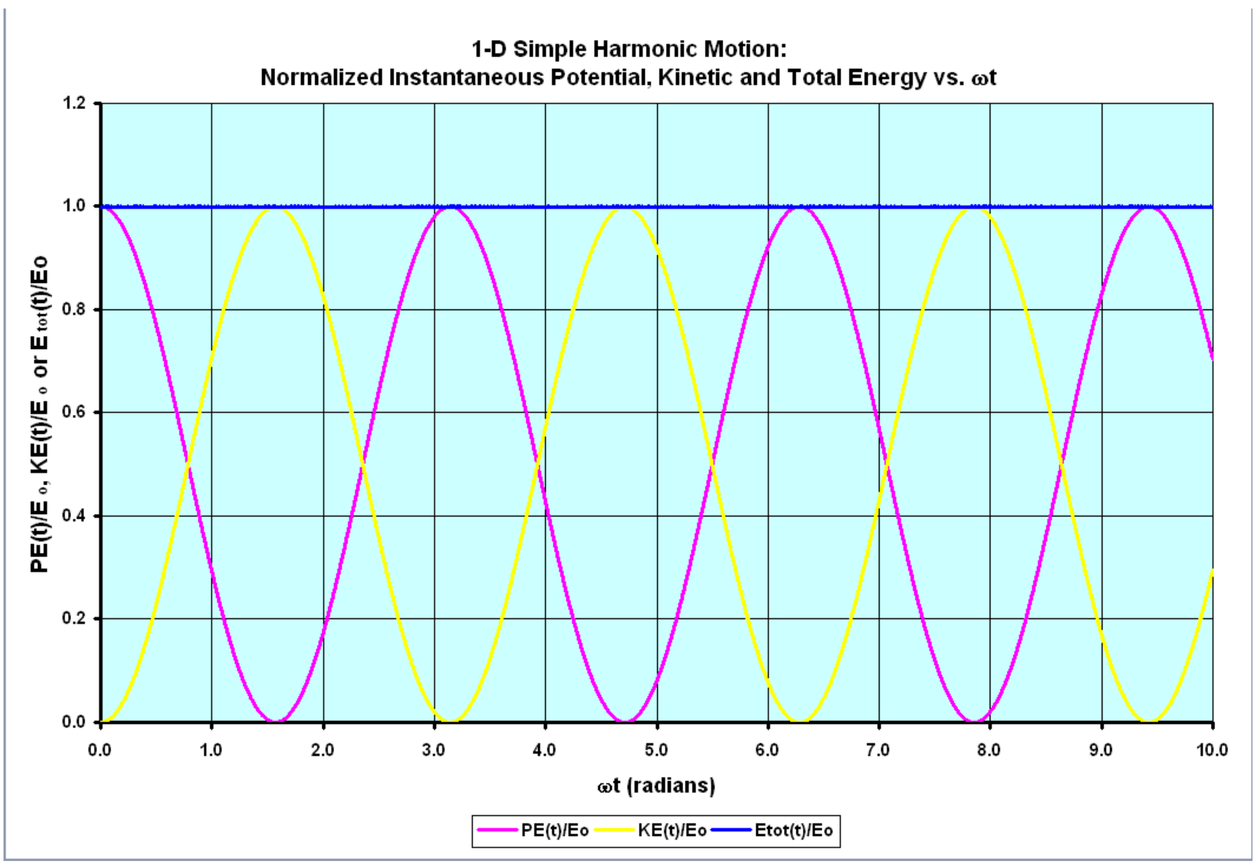
$$E_{tot}(t) = P.E.(t) + K.E.(t) = E_o\cos^2(\omega t) + E_o\sin^2(\omega t) = E_o\left\{\cos^2(\omega t) + \sin^2(\omega t)\right\}$$

Using the trigonometric identity $1 = \cos^2 x + \sin^2 x$ we see that:

$$E_{Tot}(t) = E_o = \frac{1}{2}kx_o^2 = \frac{1}{2}M\omega^2x_o^2 = \text{constant } (>0)\text{, independent of time!}$$

Thus, the total energy in (spring + mass) system *is* constant – due to conservation of energy!!

Graphs of $P.E.(t)$, $K.E.(t)$, and $E_{tot}(t)$ vs. time (all normalized to E_o) are shown in the figure below:



Note that $P.E.(t)$, $K.E.(t)$ and $E_{tot}(t)$ are always ≥ 0 (*i.e.* never negative)!!!

Note further that energy/energies are ***additive***, ***scalar*** quantities.

A ***real*** vibrating spring – mass system suffers from various energy loss mechanisms:
* friction – the mass M slides on surface, mass M also slides through viscous air
* spring also dissipates energy internally each time it is flexed (another type of friction)

* Thus, the motion of a *real* mass on a *real* spring is damped by frictional processes.
* The original/initial energy, $E_{tot}(t) = E_o$ = constant is dissipated by frictional processes.
* The initial energy E_o ultimately winds up as heat (another form of energy) - thus the mass, spring, horizontal surface and the air all heat up with time…

Mathematically, we can represent the effect(s) of frictional damping associated with a 1-D simple harmonic oscillator as a velocity-dependent (and hence time-dependent) force $F_d(t)$ acting horizontally on the mass M, which opposes the motion, which, for the initial conditions of our problem, this damping force is given by: $\boxed{F_d(t) = +bv(t)}$ where b is a positive constant, known as the ***viscous damping coefficient***, with *SI* units of *kg*/*sec*.

Then since $\boxed{v(t) = \dfrac{dx(t)}{dt} \equiv \dot{x}(t)}$, the equation of motion for the damped 1-D simple harmonic oscillator becomes: $\boxed{M\dfrac{d^2x(t)}{dt^2} + b\dfrac{dx(t)}{dt} + kx(t) = 0}$ or: $\boxed{M\ddot{x}(t) + b\dot{x}(t) + kx(t) = 0}$

We can rewrite this differential equation as: $\boxed{\ddot{x}(t) + (b/M)\dot{x}(t) + (k/M)x(t) = 0}$ and defining: the ***damping constant*** $\boxed{\gamma \equiv (b/2M) > 0}$ and $\boxed{\omega^2 \equiv (k/M) = (2\pi)^2 f^2}$, then our linear, homogeneous 2nd-order differential equation can also be written as: $\boxed{\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega^2 x(t) = 0}$. The general solution to this differential equation is of the form: $\boxed{x(t) = x_o e^{\zeta t}}$.

Explicitly carrying out the time-differentiation we obtain: $\boxed{\zeta^2 x(t) + 2\gamma\zeta x(t) + \omega^2 x(t) = 0}$ or: $\boxed{\zeta^2 + 2\gamma\zeta + \omega^2 = 0}$ which in turn is a quadratic equation in ζ, the solution for which has two roots: $\boxed{\zeta = \dfrac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega^2}}$.

When ***no*** damping is present ($\gamma = 0$), then: $\boxed{\gamma \equiv (b/2M) = 0}$, and thus: $\boxed{\zeta = \pm\sqrt{-\omega^2}}$.

Defining $i \equiv \sqrt{-1}$, then for $\gamma = 0$ we see that $\boxed{\zeta = \pm i\omega}$.
Next, we use Euler's complex relations for cosine and sine functions: $\boxed{\cos\omega t = \tfrac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)}$ and $\boxed{\sin\omega t = \tfrac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)}$. For the no-damping situation, we already know (from above) what the solution ***must*** be for the 1-D harmonic motion, with our initial conditions:
$\boxed{x(t) = x_o\cos(\omega t) = \tfrac{1}{2}x_o\left(e^{i\omega t} + e^{-i\omega t}\right)}$.

Now suppose that a ***small*** amount of damping is present in the system. Mathematically this is represented by $\gamma \equiv (b/2M) < \omega$, hence $\sqrt{\gamma^2 - \omega^2} = i\sqrt{\omega^2 - \gamma^2}$, with purely real $\sqrt{\omega^2 - \gamma^2} > 0$. We define $\omega'^2 \equiv \omega^2 - \gamma^2$ or equivalently $\omega' \equiv \sqrt{\omega^2 - \gamma^2} > 0$. Thus, for ***under-damped*** 1-D harmonic motion, with $0 < \gamma \equiv (b/2M) < \omega$, we see that $\zeta = -\gamma \pm \sqrt{\gamma^2 - \omega^2} = -\gamma \pm i\sqrt{\omega^2 - \gamma^2} = -\gamma \pm i\omega'$, and thus the *physical* solution for ***under-damped*** 1-D harmonic motion, for our ***initial*** conditions is given by: $\boxed{x(t) = x_o e^{\zeta t} = x_o e^{-\gamma t}\cos(\omega' t) = \tfrac{1}{2} x_o e^{-\gamma t}\left(e^{i\omega' t} + e^{-i\omega' t}\right)}$ where the ***damping constant*** $0 < \gamma \equiv (b/2M) < \omega$ and $\boxed{0 < \omega' \equiv \sqrt{\omega^2 - \gamma^2} < \omega}$. The motion is ***exponentially*** damped as time increases, with ***damping time constant*** $\boxed{\tau_d \equiv 1/\gamma = (2M/b)}$ (*seconds*), where the ***envelope*** of the 1-D oscillation falls to $1/e = e^{-1} \simeq 0.3679$ of its initial value at time $\boxed{t = \tau_d \equiv 1/\gamma = (2M/b)}$ (*seconds*), as shown in the figure below:

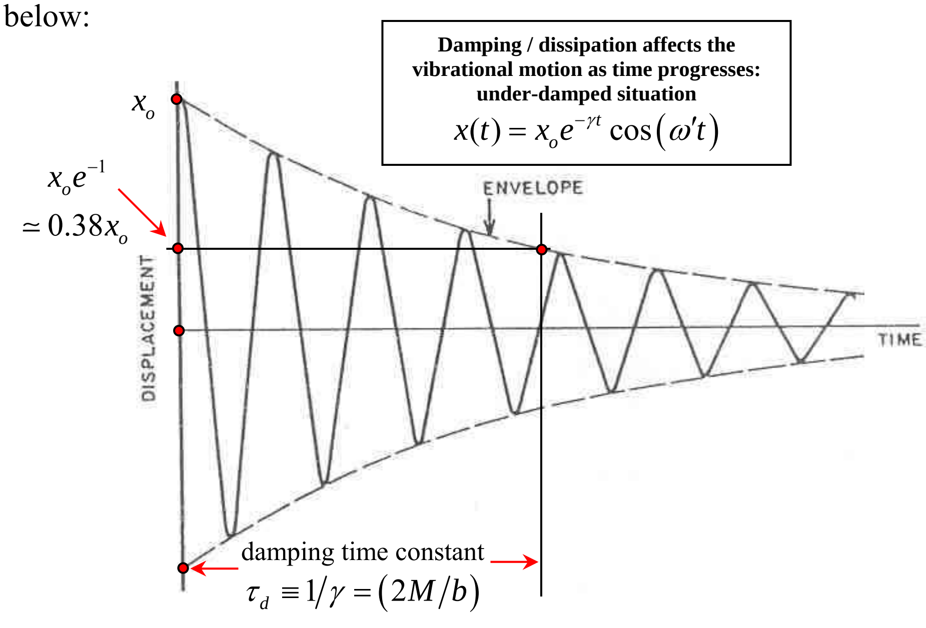


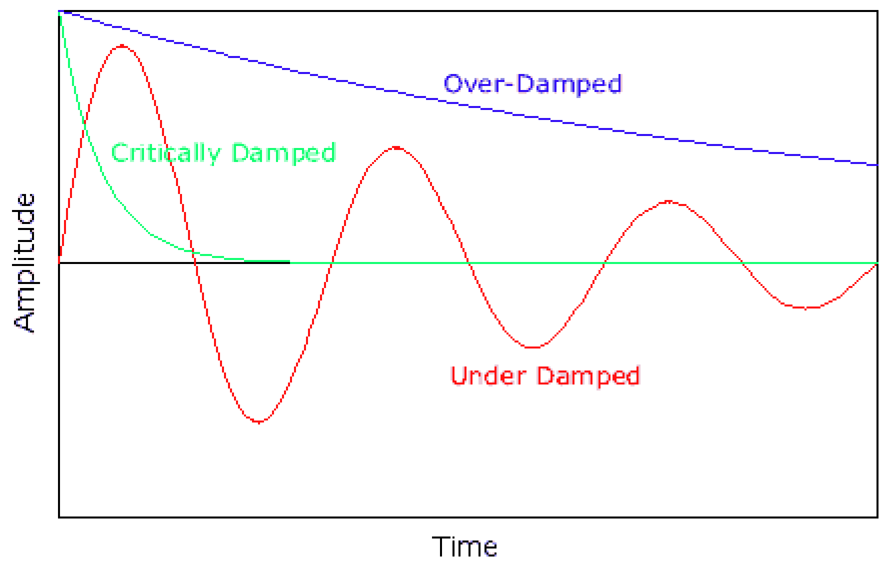
FIG. 3. Graph of displacement versus time for a damped vibration.

Dissipative processes/friction tends to ***lower*** the frequency of oscillation of a vibrating system, as can be seen from the relation $\boxed{0 < \omega' = \sqrt{\omega^2 - \gamma^2} < \omega}$. Small damping corresponds to a slight decrease in the oscillation frequency from its "natural" un-damped value of $\omega = \sqrt{k/M}$.

If we now imagine slowly increasing the damping to "heavy" damping – eventually there will be no oscillation(s) at all! When $\gamma = \omega$, the system is said to be ***critically damped***, and $\zeta = -\gamma$, and the corresponding ***critically-damped*** motion is a purely-decaying exponential with time: $\boxed{x(t) = x_o e^{-\gamma t}}$. When $\gamma > \omega$, the system is said to be ***over-damped***, and $\sqrt{\gamma^2 - \omega^2} > 0$. Here, $\zeta = -\gamma \pm \sqrt{\gamma^2 - \omega^2}$, but the ***physical*** solution is: $\zeta = -\gamma + \sqrt{\gamma^2 - \omega^2} = -\gamma\left(1 - \sqrt{1 - (\omega/\gamma)^2}\right)$.

The ***over-damped*** motion is again a decaying exponential with time: $\boxed{x(t) = x_o e^{\zeta t} = x_o e^{-\gamma\left(1 - \sqrt{1 - (\omega/\gamma)^2}\right)t}}$.

The following figure shows the effect of under-, critical and over-damping on the motion of a 1-D harmonic oscillator system:



Note that damping processes that are operative in ***all*** musical instruments are in the ***under-damped*** regime (since by definition, to be musical, they ***must*** vibrate at frequencies > 0), typically with small amounts of damping, *i.e.* $\gamma \ll \omega$, such that: $\boxed{0 < \omega' = \sqrt{\omega^2 - \gamma^2} = \omega\sqrt{1-(\gamma/\omega)^2} \lesssim \omega}$.

A more realistic motion of a vibrating mass on spring is that associated with *e.g.* driving it with a periodic force (corresponding to a linear, ***inhomogeneous*** 2^{nd}-order differential equation):

- Have to get the mass moving first (initially at rest), takes a while for oscillations to build up
- Takes a finite time to reach a steady state displacement amplitude x_o
- When switch off the driving force, displacement amplitude decays away, as shown below:

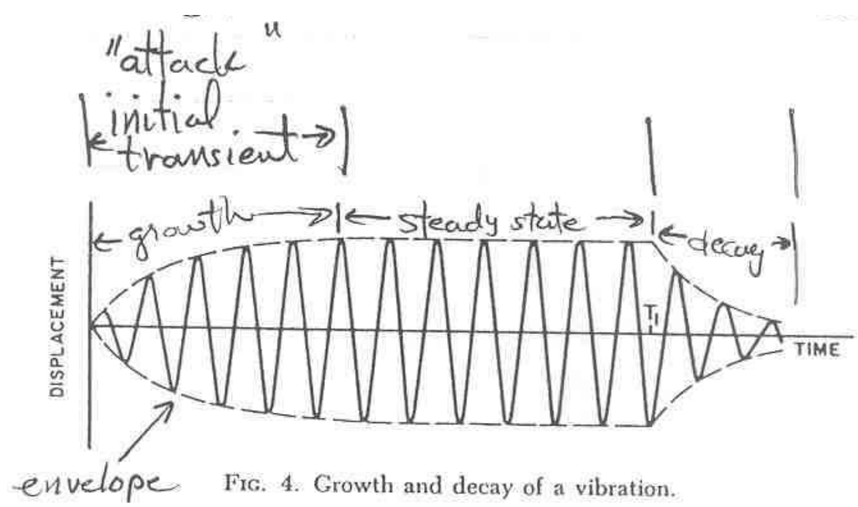


FIG. 4. Growth and decay of a vibration.

Slow attack – *e.g.* flute-like sound. Fast attack – *e.g.* more like trumpet/sax/*etc*.… type sounds
Slow decay → large sustain (*e.g.* solid-body electric guitar). Fast decay → little sustain (*e.g.* acoustic and/or hollow-body, archtop-type jazz guitar). Fast *vs*. slow attack & decay times are important aspects/attributes of the overall sound(s) produced by musical instruments!

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