Lecture Notes III

Waves & Wave Propagation

Sound waves propagate in a physical medium:

* gas, liquid or solid (and/or a *plasma* – the 4th state of matter @ very high temperature!)

* mass density of the medium ρ (= mass per unit volume) is important

* sound wave ("disturbance" = energy pulse) propagates in the medium with a characteristic speed of propagation *v* in that medium.

* propagation speed, *v* depends on density & elastic properties of the medium.

Simple model of (one-dimensional) medium:

Atom   *EM* forces between atoms

(microscopic picture of macroscopic medium)

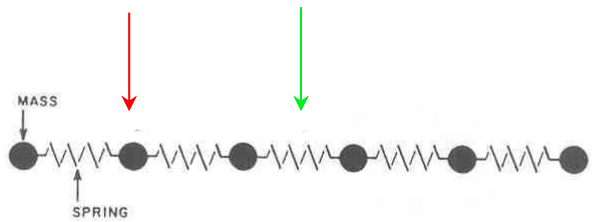
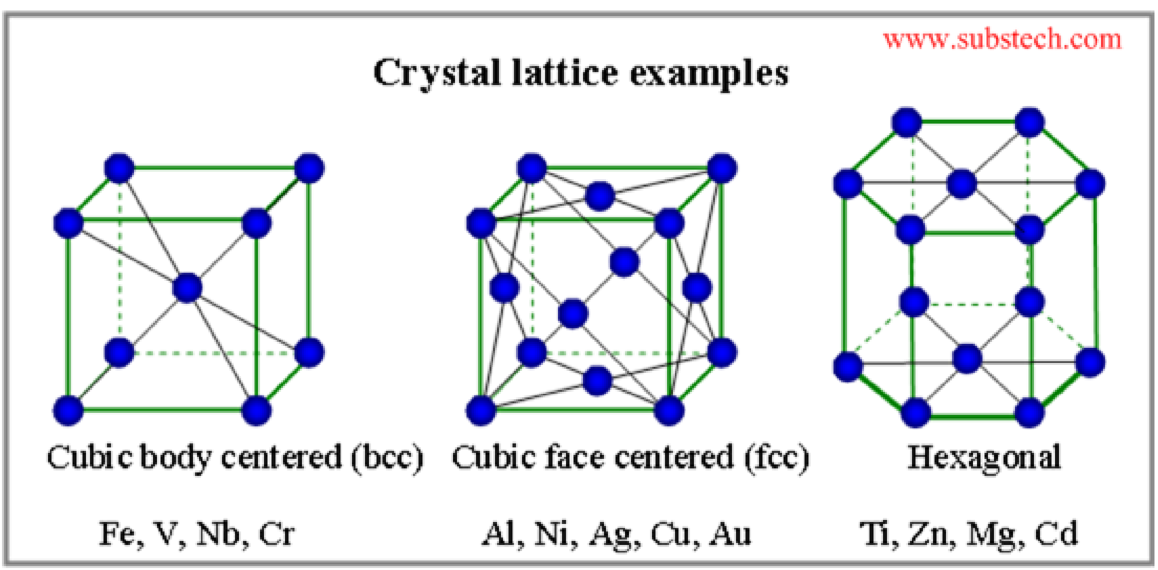
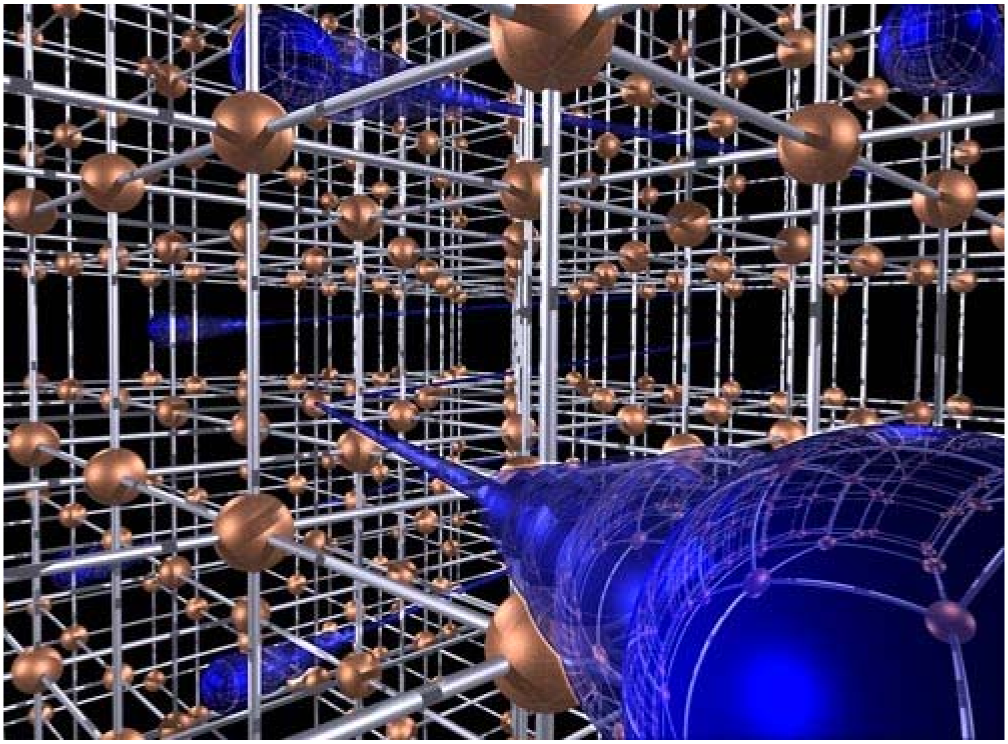


FIG. 1. A simple one-dimensional medium.

Crystalline metal – sound waves propagating in a 3-D lattice of atoms:





<u>Longitudinal Sound Waves Propagating in 1-Dimension</u>:

Displacement of atoms from equilibrium positions – *i.e*. compression/rarefaction is in/along/parallel to direction of propagation of soundwave

Longitudinal sound waves – in gases, liquids and solids (*i.e*. bulk materials)

<u>Propagation of Longitudinal Waves in One Dimension as a Function of Time</u>:

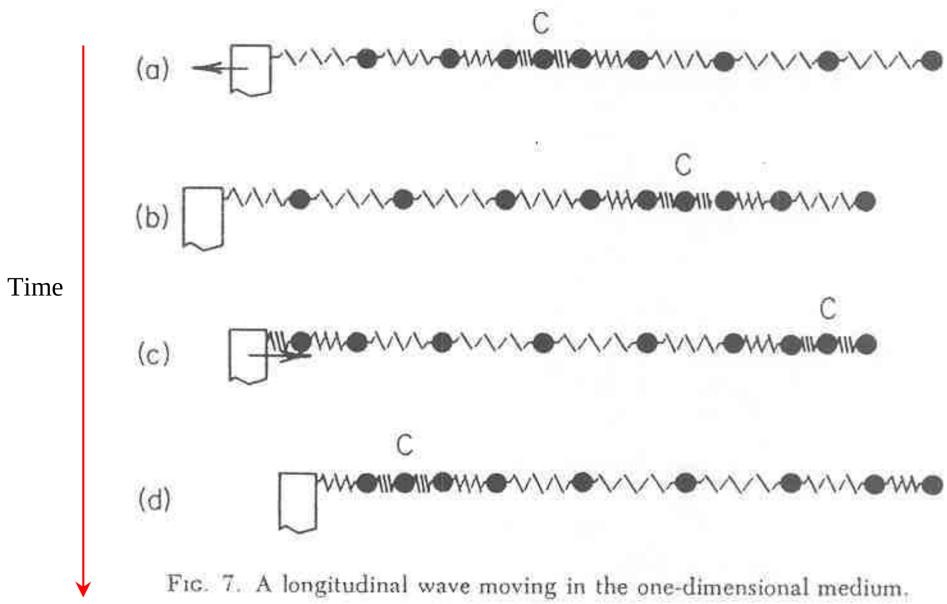


FIG. 7. A longitudinal wave moving in the one-dimensional medium.

Tuning Fork Used to Generate 1-Dimensional Longitudinal Waves:

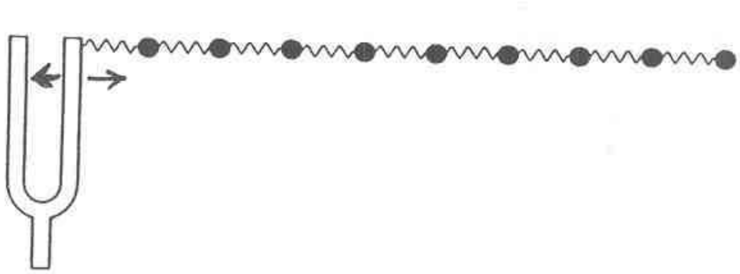


FIG. 6. Tuning fork attached to the one-dimensional medium to generate longitudinal waves.

Transverse Sound Waves Propagating in 1-Dimension:

Displacement of atoms from equilibrium positions is perpendicular (i.e. transverse) to direction of propagation of wave.

Waves in a solid – *e.g.* a vibrating string (1-D), or a vibrating rectangular, triangular or circular sheet/membrane (2-D), a vibrating hollow box, pyramid, cylinder or sphere (3-D)!

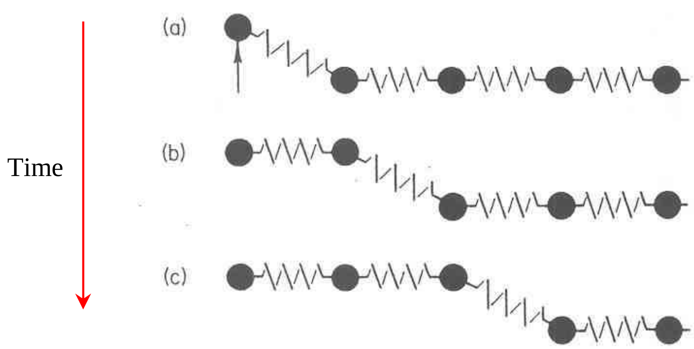


FIG. 3. (a), (b), (c). Successive positions of a transverse disturbance traveling in the one-dimensional medium.

Transverse Sinusoidal Traveling Wave in 1-Dimension:

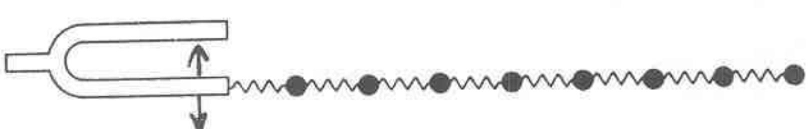


FIG. 4. Tuning fork attached to the one-dimensional medium to generate transverse waves.

Propagation of Transverse Waves on a Stretched String as a Function of Time:

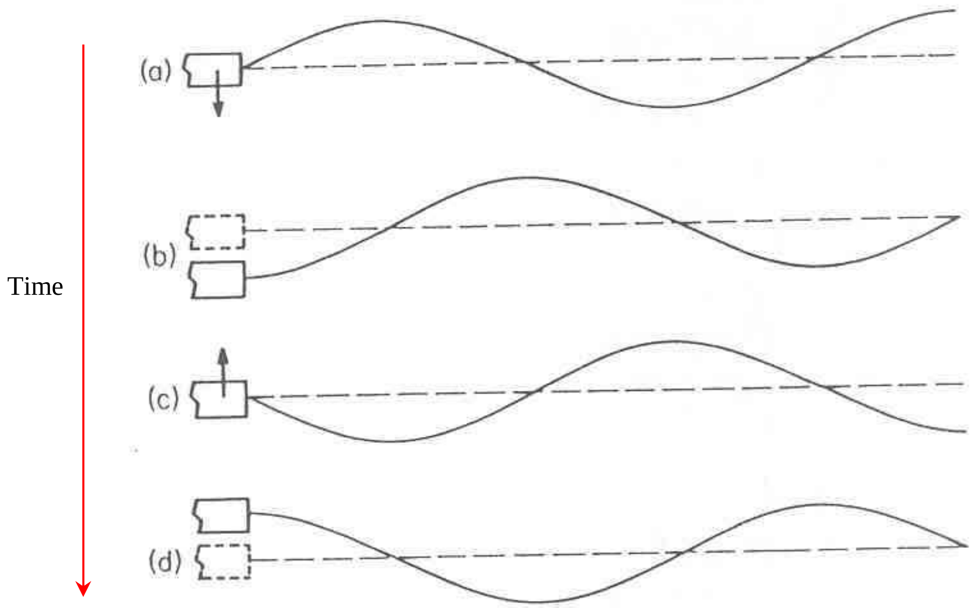


FIG. 8. A transverse wave in a string.

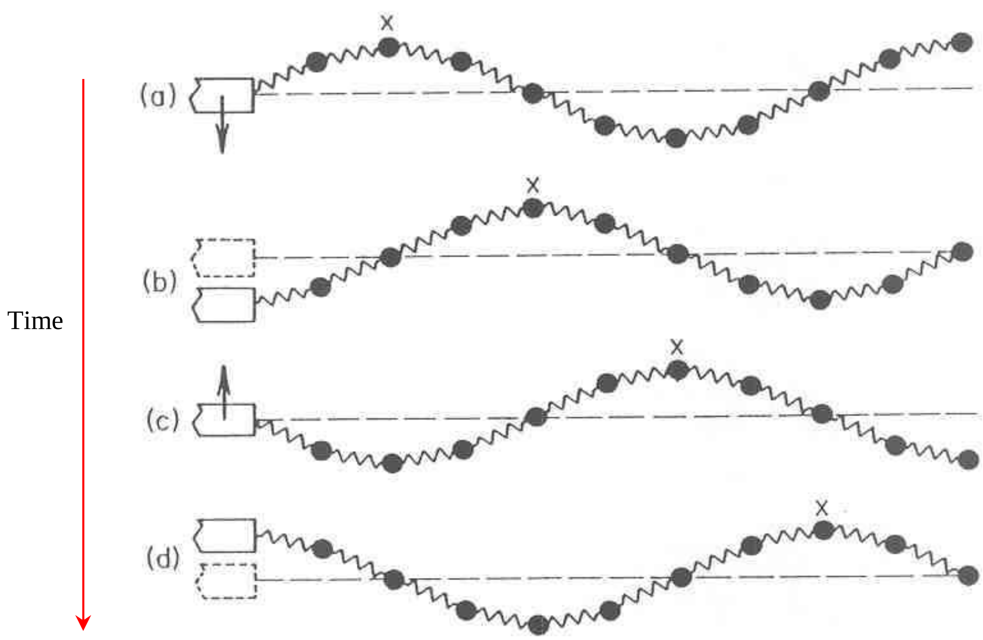


FIG. 5. A transverse wave moving in the one-dimensional medium.

Propagation of Longitudinal Waves in an Air Column:

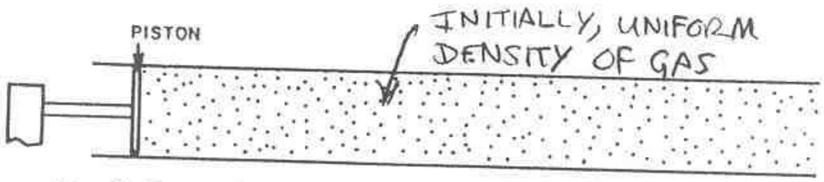


FIG. 9. Tuning fork attached to piston at the end of an air column.

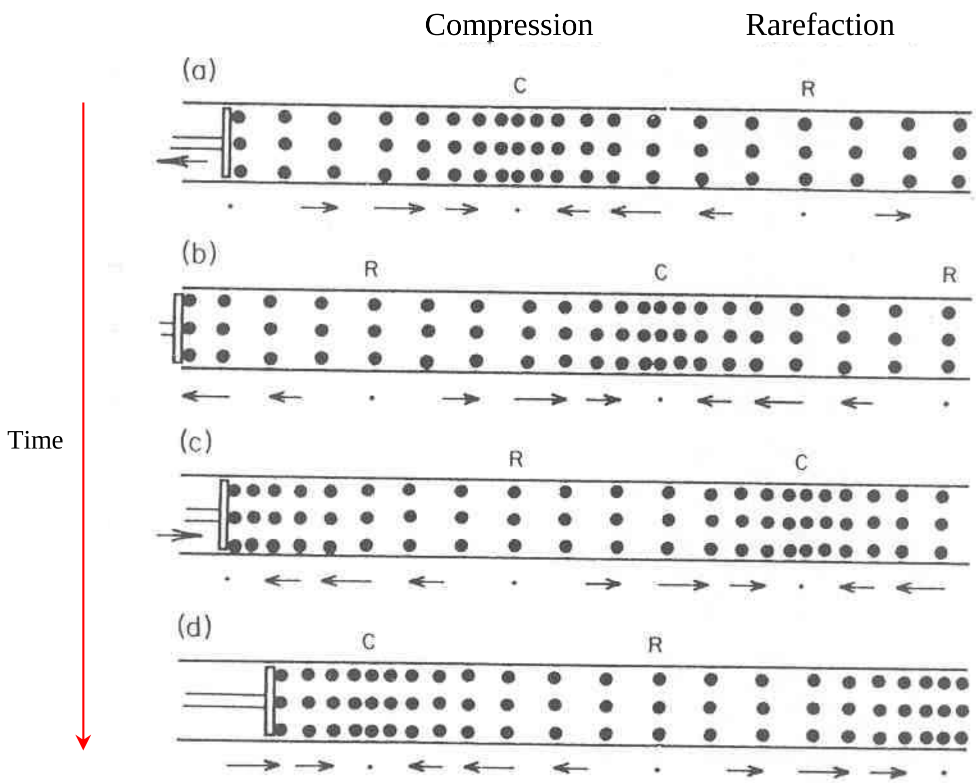


FIG. 10. A longitudinal wave in an air column.

(displacement of air molecules from their equilibrium positions) & (over-pressure "pressure displacement" from equilibrium pressure – 1 *Atm.*)

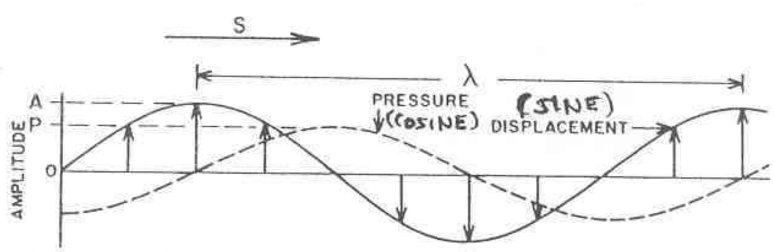


FIG. 11. Graph of the displacements of the air molecules and the pressure variation, plotted against distance along the column.

n.b. Over-pressure and displacement of air molecules are 90° out of phase with each other!

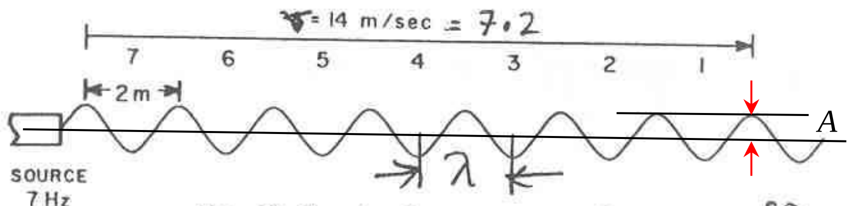


FIG. 12. Speed = frequency × wavelength.

A = displacement amplitude of wave = max ***displacement*** from equilibrium position (meters, *m*)

f = frequency of wave = number of complete cycles per second of wave (*cycles per second* = *cps* = *Hertz*, or *Hz*).

λ = "lambda" = spatial wavelength of wave – distance to complete one oscillation cycle (meters, *m*)

τ = "tau" = period of wave = time to complete one oscillation cycle = "temporal wavelength" (*seconds* (*secs*, or *s*)):

$$\boxed{\tau = 1/f}$$

v = speed of propagation of wave (*meters*/*second* = *m*/*s*): $\boxed{v = f\lambda}$

ω = "omega" = angular frequency (*radians*/*second* = *rads*/*sec*, or *rads*/s):

$$\boxed{\omega = 2\pi f, \quad f = \omega/2\pi}$$

k = spatial wave number (<u>inverse</u> *meters*, i.e. 1/*meters* = 1/*m*):

$$\boxed{k = 2\pi/\lambda}$$

Hence, we also see that: $\boxed{v = f\lambda = 2\pi f \cdot \lambda/2\pi = \omega/k}$

Sinusoidal ***<u>longitudinal</u>*** wave propagation {in the +*ve* *z*-direction} (relevant *e.g*. for sound propagation in air/water) is mathematically described by:

$$\boxed{Z(z,t) = Z_o \sin(\omega t - kz)}$$

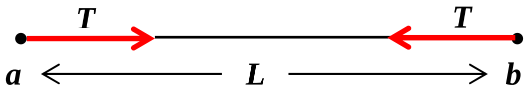
where *z* = listener's position, Z_o = longitudinal displacement amplitude (i.e. along the direction of propagation) and *Z*(*z*,*t*) is the instantaneous ***longitudinal*** <u>displacement</u> (*i.e*. longitudinal <u>deviation</u>) from the wave's <u>equilibrium</u> position at the point *z* at time *t*.

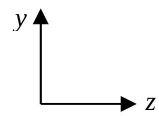
Note that the ***argument*** of the sine function $(\omega t - kz)$ = **<u>constant</u>**. Thus, as time *t* increases, the position *z* must also increase – hence the name **<u>traveling</u>** wave. Hence, we also see that an argument of the sine function of the form $(\omega t + kz)$ = <u>constant</u> mathematically represents a traveling longitudinal displacement plane wave propagating in the –*z* direction: $\boxed{Z(z,t) = Z_o \sin(\omega t + kz)}$.

The longitudinal speed of propagation *v* of ***transverse*** displacement traveling waves, where the displacement (from equilibrium position) is *e.g.* in the *y*-direction, perpendicular (*i.e.* transverse) to the direction of propagation, *e.g.* in the ± *z*-direction is mathematically described by:

$\boxed{Y(z,t) = Y_o \sin(\omega t \mp kz)}$ for a ***transverse*** traveling plane wave propagating in the ± *z*-direction.

For ***transverse*** waves propagating on a ***stretched string*** having tension, *T* (*n.b.* the *SI* / metric units of tension *T* (= force) is Newtons, *N* = *kg-m/s*^{2}), the string has mass *M* and length *L* , the longitudinal speed of propagation of transverse traveling waves on a string is given by:

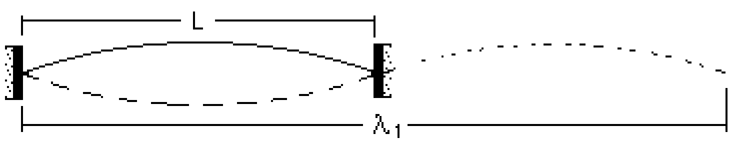




$\boxed{v = \sqrt{T/\mu}}$ where: $\boxed{\mu = M/L}$ = mass per unit length of string (*SI* units: *kg*/*m*)

Example: Tension and Transverse {Standing} Waves on the High-E String of a Guitar:

n.b. A standing wave = superposition of two traveling waves propagating in opposite directions!



$f_{Hi\text{-}E}$ = 332 *Hz* for open Hi-E string.

The high-*E* string on a guitar has diameter, *D* = 0.009" (~230μm) $(0.001" = 1/1000" = 25.4\mu m)$

L = string length = 63.5 *cm* = 0.635*m* (= 25.0" = scale length *e.g.* of a Fender electric guitar)

ρ = density of string = 7.9 gms/cm^3 = 7900 *kg*/*m*^{3} for ***steel***.

$A_\perp$ = cross-sectional area of string = $\pi R^2 = \pi(D^2/4) = 4.104\times10^{-3}\ cm^2 = 4.104\times10^{-9}\ m^2$.

V = volume of string $= A_\perp \cdot L = \pi R^2 \cdot L = \pi(D/2)^2 \cdot L = \pi D^2 L/4 = 0.026\ cm^3 = 0.026\times10^{-6}\ m^3$

Mass of string: $M = \rho V = \rho(A_\perp \cdot L) = \rho(\pi D^2 L/4) = 0.206\ gms = 0.206\times10^{-3}\ kg$

$$\mu = \frac{M}{L} = \frac{\rho V}{L} = \frac{\rho(A_\perp \cdot \cancel{L})}{\cancel{L}} = \rho \cdot A_\perp = \frac{0.206\times10^{-3}\ kg}{0.635 m} = 3.242\times10^{-4}\ kg/m$$

Now $f_{Hi\text{-}E}$ = 332 Hz, and $\lambda_{Hi\text{-}E}$ = 2*L* = 2×0.635 *m* = 1.27 *m* (see above pix, for fundamental)

Thus, the longitudinal speed of transverse traveling waves on the Hi-*E* string of a guitar is:

$$v_{Hi\text{-}E} = f_{Hi\text{-}E} * \lambda_{Hi\text{-}E} = 332\ Hz * 1.27\ m = 421.6\ m/s$$

Since $v = \sqrt{T/\mu}$, the string tension T on the Hi-E string of the guitar is:

$T = \mu v^2 = (3.242\times10^{-4}\ kg/m)*(421.6\ m/s)^2 = 57.6\ kg\text{-}m/s^2 = 57.6\ N \approx 12.95\ lbs$ of force.

1 N = 0.2248 *lbs* of force, 1 *lb* of force = 4.448 N

The typical string tension on a steel-stringed acoustic and/or electric guitar is $T \sim 50 - 60\ N$.

⇒ For steel 6 (12)-string guitar, *total* string tension is ~ 300 – 360 (600 – 720) N !!!

PROPAGATION OF SOUND WAVES IN 2 & 3 DIMENSIONS
SOUNDWAVES EMANATING FROM A POINT SOURCE

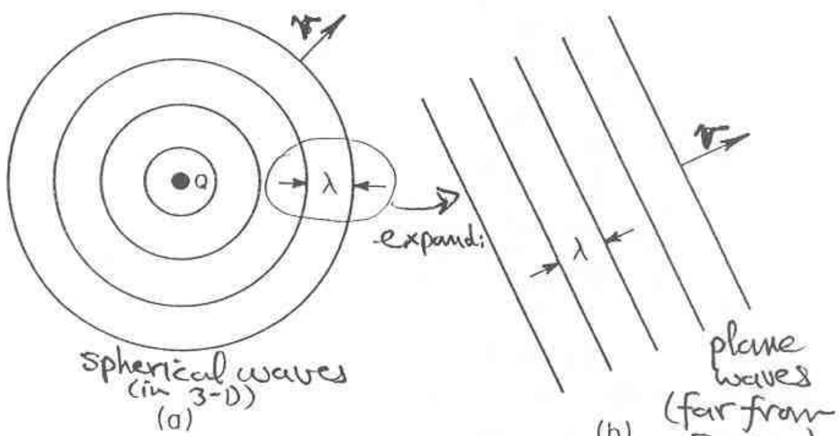


FIG. 13. Wave fronts. (a) Waves spreading out from a small source. (b) Plane waves.

PROPAGATION OF SOUND WAVES IN A 3-D GAS (*e.g.* AIR)

A sound wave freely propagating in a gas = *longitudinal* compression/rarefaction of the gas.

Gives rise to a *longitudinal* displacement wave. If *harmonic/sinusoidal* in nature, far away from the sound source we have traveling plane waves, *e.g.* propagating in the + z-direction:

$\boxed{\xi_z(z,t) = \xi_o \cos(\omega t - kz)}$ = mean (or avg.) *longitudinal* (*i.e.* z-) displacement of air molecules

and a corresponding *over-pressure* wave of the form: $p(z,t) = P(z,t) - P_{atm}$, where $P(z,t)$ is the instantaneous absolute pressure, and P_{atm} = atmospheric pressure = **constant**, typically ~ 14.7 *psi* ≈ 1.03×10^5 *Pascals* (*Pa*) at NTP (*i.e.* sea level and temperature $T = 20\ ^oC$).

For a harmonic traveling plane wave, the instantaneous over-pressure is: $\boxed{p(z,t) = p_o \sin(\omega t - kz)}$

Sound waves can propagate through elastic, compressible media as *longitudinal* waves.

The over-pressure $p = P - P_{atm} = \Delta P$ required to compress a gas of initial volume V to $V - \Delta V$ is:

$$p = P - P_{atm} = \Delta P = -B\left(\frac{\Delta V}{V}\right)$$

Change in the pressure for a fractional change in volume (adiabatic conditions)

Fractional change in volume

(Adiabatic) bulk modulus, B of fluid (here, a gas)

B is the so-called adiabatic bulk modulus, $B = 1/\kappa$ where κ = compressibility of the fluid (liquid or gas) – *n.b.* B has same *SI* units as pressure, p (from dimensional analysis of above formula)!

Thus, we see that the adiabatic bulk modulus B of a fluid (liquid or gas) is the (negative) of the change in the {over-pressure} divided by the fractional change in the volume of the fluid due to the change in the over-pressure:

$$B = \frac{-\Delta P}{(\Delta V/V)} \quad (Pascals)$$

Now, for so-called adiabatic (*i.e.* slow) compression of a gas due to *e.g.* propagation of sound waves in the gas:

γ = "gamma" ≡ C_V/C_P = Ratio of: $\dfrac{\text{specific heat of gas @ constant volume}}{\text{specific heat of gas @ constant pressure}}$

γ = 5/3 for monatomic gases (*e.g.* helium, neon, argon & xenon)

γ = 7/5 for diatomic gases (*e.g.* oxygen & nitrogen molecules – O_2 & N_2)

The Ideal Gas Law:

$$PV = NRT$$

Absolute temperature (*degrees Kelvin*)

Pressure (N/m^2) Volume (m^3) # Moles of gas

R = universal gas constant = 8.3145 (*Joules/mole/deg.K*)

e.g. Carbon atom has 12 atomic mass units (amu's), and thus 1 *mole* (*mol*) of carbon {having Avogadro's number, $N_A = 6.022 \times 10^{23}$ *atoms/mole*} weighs 12 *grams*.

Now air @ NTP (a mixture of oxygen & nitrogen molecules, traces of argon, *etc.*) is NOT a perfect ideal gas – but is close to an ideal gas.

For the so-called adiabatic condition: PV^{γ} = constant = K, thus: $P = KV^{-\gamma}$

Then for ***small*** pressure variations $\left(p = P - P_{atm} = \Delta P \ll P_{atm}\right)$: $\Rightarrow \left(\Delta P/P\right) = -\gamma\left(\Delta V/V\right)$

However, from above: $B \equiv -\dfrac{\Delta P}{\left(\Delta V/V\right)}$. Thus, we see that: $B = \gamma P \simeq \gamma P_{atm}$ for $p = P - P_{atm} \ll P_{atm}$.

For 1-D traveling plane waves: $p = \Delta P = -B\left(\Delta V/V\right) = -B\dfrac{\partial \xi_z(z,t)}{\partial z}$.

Thus, we see that there is a relation between overpressure $p(z,t)$ and the local <u>*slope*</u> (*i.e.* the spatial gradient) of the longitudinal displacement $\partial \xi(z,t)/\partial z$:

If longitudinal displacement: $\boxed{\xi_z(z,t) = \xi_o \cos(\omega t - kz)}$ then: $p(z,t) = -B\,\partial \xi_z(z,t)/\partial z$ gives:

$$\boxed{p(z,t) = p(z,t) - P_{atm} = \Delta P(z,t) = -B\frac{\partial \xi_z(z,t)}{\partial z} = -Bk\xi_o \sin(\omega t - kz) = p_o \sin(\omega t - kz)}$$

Hence, we see that the over-pressure $p(z,t)$ and the longitudinal displacement of air molecules from their equilibrium positions $Z(z,t)$ are 90° out-of-phase with each other.

The (mean, or avg.) longitudinal <u>speed</u> of air molecules $u_z(z,t)$ is the time rate of change of the (mean, or avg.) longitudinal displacement of air molecules, *i.e*. the time derivative $\partial \xi_z(z,t)/\partial t$.

Thus, for a harmonic/sinusoidal sound wave in air, the instantaneous particle velocity is:

$$\boxed{u_z(z,t) = \partial \xi_z(z,t)/\partial t = -\omega \xi_o \sin(\omega t - kz) = u_o \sin(\omega t - kz)}$$

Thus, we also see that the longitudinal speed of air molecules $u_z(z,t)$ and the overpressure $p(z,t)$ are <u>in</u>-<u>phase</u> with each other for sound waves propagating in "free" air:

$$\boxed{p(z,t) = p_o \sin(\omega t - kz) = -B\frac{\partial \xi(z,t)}{\partial z} = -Bk\xi_0 \sin(kz - \omega t) = -\frac{Bk}{\omega}\omega\xi_0 \sin(kz - \omega t) = \frac{Bk}{\omega}u_z(z,t)}$$

A medium has a so-called <u>***characteristic specific acoustic impedance***</u> $z_a(\vec{r})$ associated with it at the listener point $\vec{r}$ – which physically is a measure of how easy (or difficult) it is for (acoustic) energy to <u>***flow***</u> from one point to another in the medium. For <u>***longitudinal***</u> traveling acoustic plane waves propagating ***freely*** in a gas, the ***characteristic longitudinal specific acoustic impedance*** $z_a^{\parallel}(\vec{r})$ is defined as (using $v = \omega/k = f\lambda$, and $B = \rho v^2$ {see below…}):

$$\boxed{z_a^{\parallel}(\vec{r}) \equiv p(\vec{r})/u_z(\vec{r}) = Bk/\omega = B/v = \rho v}$$ (*SI* units of $z_a(\vec{r})$: *Pa-s/m* = acoustic Ohms).

aka <u>***Rayls***</u>, in honor of Lord Rayleigh (John William Strutt)

The ***characteristic longitudinal specific acoustic impedance*** for ***air*** is:

$$\boxed{z_a^{\parallel,air}(\vec{r}) = \rho_o^{air} v_o^{air} = 1.204\,kg/m^3 \cdot 345\,m/s \simeq 415\,Pa\text{-}s/m = 415\,\Omega_{ac} = 415\,Rayls}$$ @ NTP.

Because $z_a^{\parallel,air}(\vec{r}) = \rho^{air}v^{air}$, since both the mass density of air ρ^{air} and the longitudinal speed of propagation v^{air} have a slight temperature (and pressure-) dependences (as well as slight humidity dependences), $z_a^{\parallel,air}(\vec{r})$ also has a slight temperature (and pressure-) dependence (as well as a slight humidity dependence) – see below….

The wave equation for describing propagation of longitudinal sound waves in a gas relates the speed of longitudinal sound propagation in the gas v (m/s) to the adiabatic bulk modulus of the gas B (in N/m^2 = *Pascals*) and the gas density, $\rho = M/V$ (in kg/m^3) by the following formula:

$$\boxed{v = \sqrt{B/\rho}}$$

<u>From dimensional analysis</u>:

<u>SI units</u>: $B/\rho = \dfrac{N/m^2}{kg/m^3} = \dfrac{\left(kg\text{-}m/s^2\right)/m^2}{kg/m^3} = \dfrac{m^2}{s^2}$ <u>note</u>: $1\,N = 1\,kg\text{-}m/s^2$

Thus: $v = \sqrt{B/\rho} = \sqrt{m^2/s^2} = m/s =$ meters/second = dimensions of speed

Now, since $B = \gamma P$ then: $v = \sqrt{B/\rho} = \sqrt{\gamma P/\rho}$

The gas density $\rho = \dfrac{M}{V} = \dfrac{\mathrm{M}_{molar} n_{mole}}{V} = \dfrac{\mathrm{M}_{molar} P}{RT}$ {for an ideal gas, and using the ideal gas law}

where M_{molar} = <u>molar</u> mass (kg/mol), and n_{mole} = # of moles of gas.

Then for an *<u>ideal</u>* gas: $\boxed{v = \sqrt{B/\rho} = \sqrt{\gamma P/\rho} = \sqrt{\gamma RT/\mathrm{M}_{molar}}}$

Hence, *e.g*. the speed of sound in air has a temperature-dependence!
Normalizing to the so-called standard temperature (300K) and pressure (1 *atm*):

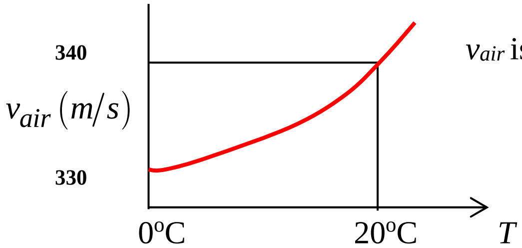
$$\boxed{v_{air} \cong 330\sqrt{\left[T(^{o}C) + 273^{o}\right]/273^{o}} \simeq 330 + 0.6T(^{o}C)} \quad (m/s)$$

The latter relation was obtained using the Taylor-series expansion for $\sqrt{1+\varepsilon} \simeq 1 + \tfrac{1}{2}\varepsilon$ for $\varepsilon << 1$ and keeping only the first non-trivial term in the Taylor-series expansion, which is linear in $\varepsilon = T(^{o}C)/273^{o}K << 1$.

$v_{air} \approx 330\ m/s$ @ $T =$ 0 ºC (@ p = 1 *atm* (*i.e*. at sea level))

$v_{air} \approx 340$ m/s @ T = 20 ºC (@ p = 1 *atm* (*i.e*. at sea level))

Thus, we see that that speed of sound in air, v_{air} *increases* with increasing temperature:



v_{air} is proportional to/grows with $\sqrt{T}$.

Note also that: $v_{helium} \sim 3v_{air}$ since: {M_{molar}(Helium) = 4 *gms*} << {M_{molar}(Air) = 28 *gms*}.
{monatomic} {diatomic}

Longitudinal speed of sound propagation (in bulk)
and
Longitudinal characteristic specific acoustic impedance for water:

$$v_{H_2O} = \sqrt{\frac{B_{H_2O}}{\rho_{H_2O}}} \approx 1480\ m/s >> v_{air} \approx 330\ m/s$$

$$z_{\parallel}^{H_2O} = \rho_o^{H_2O} v_o^{H_2O} \simeq 1000\, kg/m^3 \cdot 1480\, m/s \simeq 1.48\times10^6\ Pa\text{-}s/m = 1.48\times10^6\ \Omega_{ac}$$

Compare: $z_{\parallel}^{H_2O} \simeq 1.5\times10^6\ \Omega_{ac} \gg z_{\parallel}^{air} \simeq 415\,\Omega_{ac}$

Longitudinal speed of sound propagation (in bulk)
and
Longitudinal characteristic specific acoustic impedance for an elastic solid:

$v_{Solid} = \sqrt{\frac{Y_{Solid}}{\rho_{Solid}}}$ where: Y_{Solid} = Young's modulus (force/unit area, *i.e.* N/m^2) – *i.e. Pascals*!)

$Y \equiv \frac{S}{\Delta L/L}\left(N/m^2\right)$ = Ratio of compressive stress/compressive strain

$$z_{\parallel}^{solid} = \rho_o^{solid} v_o^{solid}$$

Now, *e.g.* for steel: $Y_{steel} = 2\times10^{11}$ *Pascals* and: $\rho_{steel} = 7.9\ gm/cm^3 = 7900\ kg/m^3$.

Thus: $v_{Steel} = \sqrt{\frac{Y_{Steel}}{\rho_{Steel}}} = \sqrt{\frac{2\times10^{11}}{7800}} \cong 5000\, m/s$

$$z_{\parallel}^{Steel} = \rho_o^{Steel} v_o^{Steel} \simeq 7900\, kg/m^3 \cdot 5000\, m/s \simeq 3.95\times10^7\ Pa\text{-}s/m = 3.95\times10^7\ \Omega_{ac}$$

Compare: $z_{\parallel}^{Steel} \simeq 4.0\times10^7\ \Omega_{ac} \gg z_{\parallel}^{H_2O} \simeq 1.5\times10^6\ \Omega_{ac} \gg z_{\parallel}^{air} \simeq 415\,\Omega_{ac}$

Longitudinal Speed of Sound Propagation in Various Bulk Media

Gases	
Material	*v* (*m*/*s*)
Hydrogen (0°C)	1286
Helium (0°C)	972
Air (20°C)	343
Air (0°C)	331
Liquids at 25°C	
Material	*v* (*m*/*s*)
Glycerol	1904
Sea water	1533
Water	1493
Mercury	1450
Kerosene	1324
Methyl alcohol	1143
Carbon tetrachloride	926
Solids	
Material	*v* (*m*/*s*)
Diamond	12000
Pyrex glass	5640
Iron	5130
Aluminum	5100
Brass	4700
Copper	3560
Gold	3240
Lucite	2680
Lead	1322
Rubber	1600