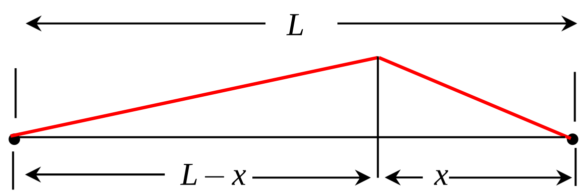
Complex Vibrations & Resonance

Simple vibrating systems have only ***one*** frequency (the fundamental).
Few such systems exist in real life (*n.b.* they are also musically less interesting/boring..).

Real vibrating systems are "complex" – rich structure of harmonics/overtones.
Overtone structure may also change/shift with time – not constant – more interesting!

Vibrating Strings - Standing Waves:

Consider a stretched string of length L, vibrating from fixed (*i.e.* rigid) end supports:



fixed endpoints (rigid)

Plucking the string at position x launches two ***counter-propagating traveling*** waves:

* One traveling wave moves to the ***right***, the other traveling wave moves to the ***left***.

* When the traveling wave(s) hit the rigid/fixed ends at $x = 0$ and $x = L$, they are reflected;
A polarity flip (= phase change of 180°) also occurs there.

Compare this situation to that for two counter-propagating traveling waves reflected from ***free*** ends - ***no*** polarity change (*i.e.* ***no*** phase shift) occurs!

The ***superposition*** {*i.e.* the ***linear*** addition $y_{tot}(x,t) = y_1(x,t) + y_2(x,t)$} of two ***counter-propagating*** traveling waves (one ***right***-moving, $y_1(x,t)$ and one ***left***-moving, $y_2(x,t)$) creates a ***standing*** wave on the string!

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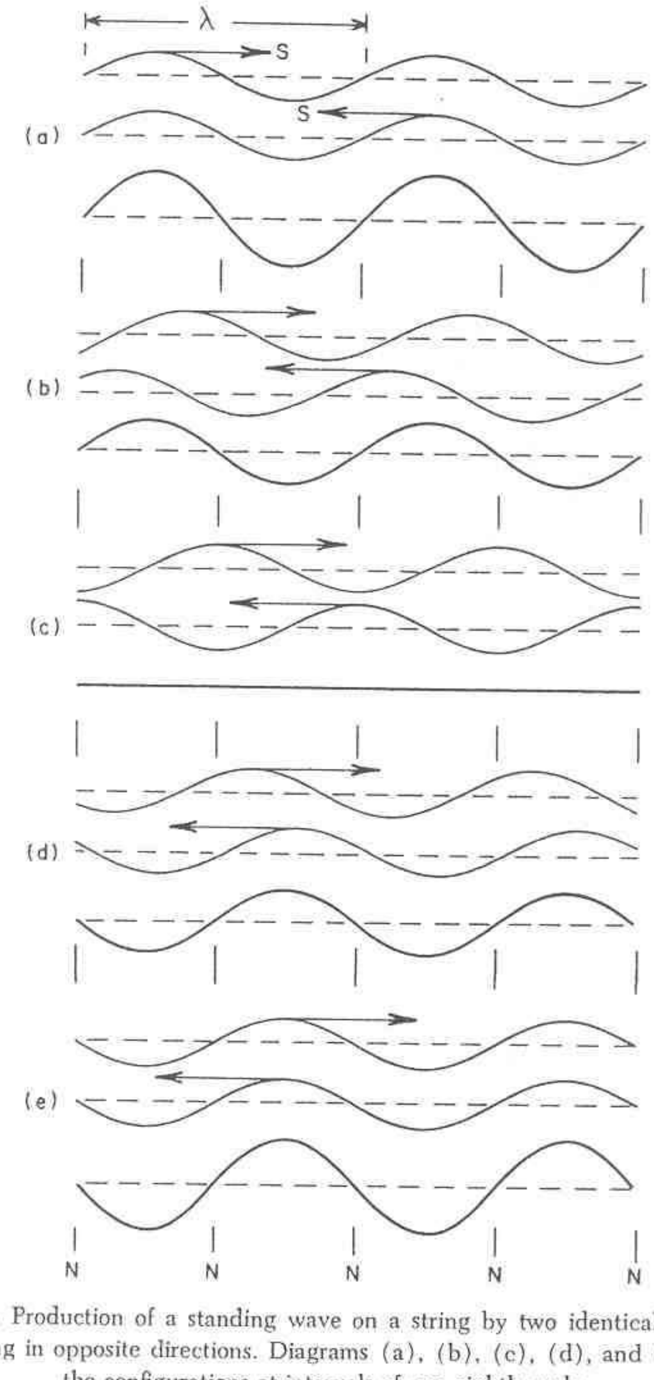

FIG. 1. Production of a standing wave on a string by two identical waves traveling in opposite directions. Diagrams (a), (b), (c), (d), and (e) are the configurations at intervals of one-eighth cycle.

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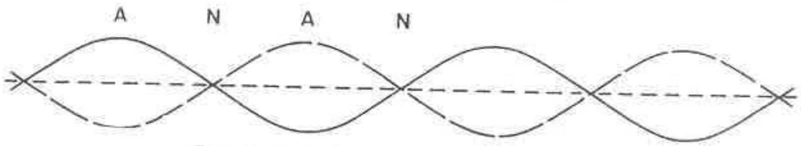
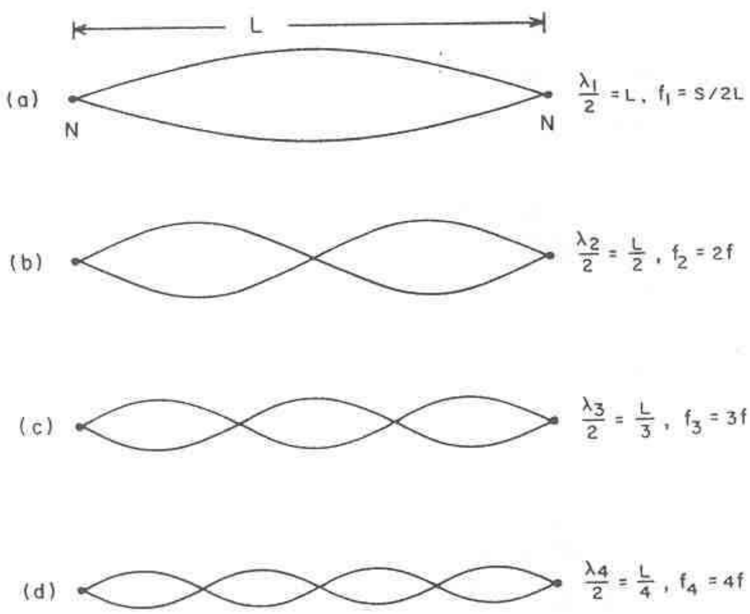


FIG. 2. Standing wave on a long string.



$n = 1$
fundamental
(first harmonic)

$n = 2$
2nd harmonic
(1st overtone)

$n = 3$
3rd harmonic
(2nd overtone)

$n = 4$
4th harmonic
(3rd overtone)

FIG. 3. First four vibration modes of a string fastened at both ends.

Longitudinal wave speed, v

$v = f_n \bullet \lambda_n = f_1\lambda_1 = f_3\lambda_3 = \ldots f_n\lambda_n$

n = integer = 1, 2, 3, 4, ….

$f_n = nf_1$

$\lambda_n = \lambda_n = \lambda_1/n$

$v = \sqrt{\frac{T}{\mu}}$

T = string tension (*Newtons*)
μ = mass per unit length of string = M/L (kg/m)

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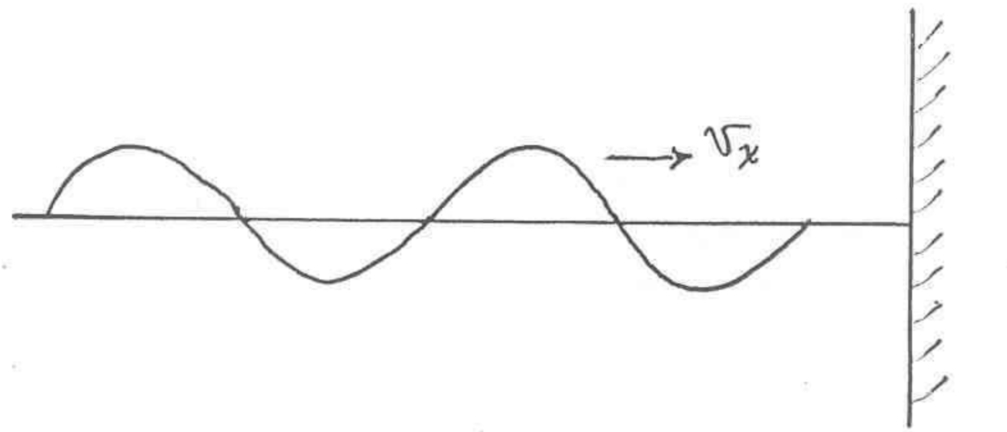
Standing Waves

Created when harmonic traveling wave reflects *e.g.* from a ***fixed*** (*i.e.* rigid, immovable) end:



x = L

- Resultant wave (in the $x < L$ region) is of the general form:

$$y(x,t) = f(x - vt) - f(-x - vt) \quad \text{where:} \quad f(x - vt) = A\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{\tau}\right)\right]$$

Analytic form for two counter-propagating traveling waves:

$$y(x,t) = A\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{\tau}\right)\right] - A\sin\left[2\pi\left(-\frac{x}{\lambda} - \frac{t}{\tau}\right)\right] \quad \Leftarrow$$

n.b. the – sign for the left-moving reflected wave is due to the polarity flip (*i.e.* phase change of 180° upon reflection) of the incident right-moving wave from the fixed/immovable endpoint.

$$= A\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{\tau}\right)\right] + A\sin\left[2\pi\left(\frac{x}{\lambda} + \frac{t}{\tau}\right)\right] \quad \Leftarrow n.b. \ \boxed{\sin(-u) = -\sin u} \ \ i.e. \ \textbf{\textit{odd}} \text{ fcn of } u.$$

Now use the trigonometric identity: $\sin(A \pm B) = \sin A\cos B \pm \sin B\cos A$

$$y(x,t) = A\sin\left(\frac{2\pi x}{\lambda}\right)\cos\left(\frac{2\pi t}{\tau}\right) - \cancel{A\cos\left(\frac{2\pi x}{\lambda}\right)\sin\left(\frac{2\pi t}{\tau}\right)}$$

$$+ A\sin\left(\frac{2\pi x}{\lambda}\right)\cos\left(\frac{2\pi t}{\tau}\right) + \cancel{A\cos\left(\frac{2\pi x}{\lambda}\right)\sin\left(\frac{2\pi t}{\tau}\right)}$$

Thus: $y(x,t) = 2A\sin\left(\frac{2\pi x}{\lambda}\right)\cos\left(\frac{2\pi t}{\tau}\right)$ for ***standing*** wave

= two ***counter-propagating traveling*** waves.

- Note: The analytic form describing the transverse displacement $y(x,t)$ associated with a **standing** wave is the ***product*** of two harmonic functions: fcn(space) × fcn(time).

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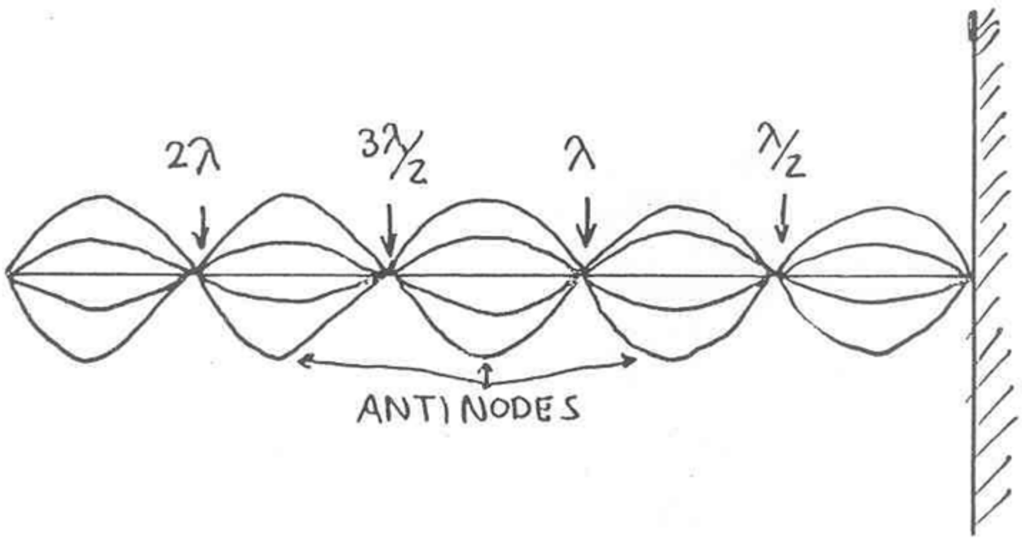
Nodes of transverse displacement occur at *x*-values along the string where $\sin(2\pi x/\lambda) = 0$
= *x*-positions along the string where the transverse displacement is ***minimum***: $y(x,t) = 0$

$\sin(2\pi x/\lambda) = 0$ when: $(2\pi x/\lambda) = 0\pi, 1\pi, 2\pi, 3\pi .... = n\pi, \quad n = 0,1,2,3\ldots$

Thus, we see that **<u>nodes</u>** occur at: $x = \frac{n}{2}\lambda = \frac{0}{2}\lambda, \frac{1}{2}\lambda, \frac{2}{2}\lambda, \frac{3}{2}\lambda.... \quad n = 0,1,2,3\ldots$



Anti-Nodes of transverse displacement occur at *x*-values along the string where $\sin(2\pi x/\lambda) = 1$
= *x*-positions along the string where transverse displacement is ***maximum***: $y(x,t) = A$

$\sin(2\pi x/\lambda) = 1$ when $(2\pi x/\lambda) = \frac{1\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} .... = m\frac{\pi}{2}, \quad m = 1,3,5,\ldots$

Thus, we see that ***<u>anti</u>*-*<u>nodes</u>*** occur at: $x = \frac{m}{4}\lambda = \frac{1}{4}\lambda, \frac{3}{4}\lambda, \frac{5}{4}\lambda.... \quad m = 1,3,5,\ldots$

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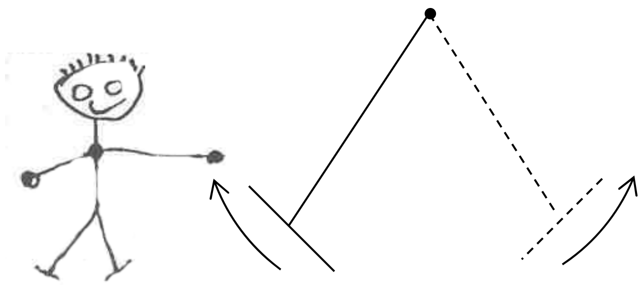
STANDING WAVES

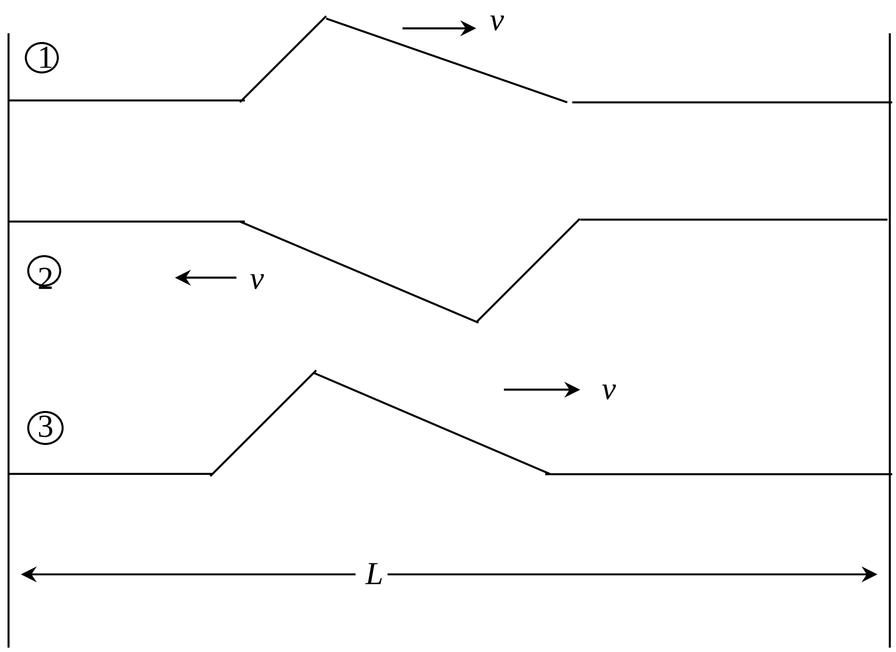
= Resonance Phenomenon

Input energy to create a "stable" configuration:

e.g. A person swinging on a swing:



e.g. A traveling wave on a string:



Traveling wave "gets in phase" after it travels a distance $2L$ in time $\tau = 2L/v$

$\therefore$ "PUSH" with frequency $f \equiv 1/\tau = v/2L$ excites the fundamental!

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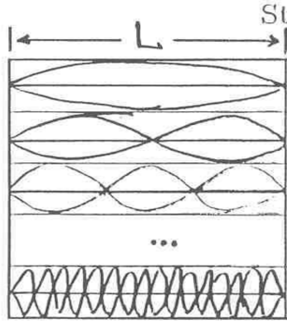
Resonant Frequencies for Standing Waves on a String of Length, L: $f_n = v/\lambda_n$

Transverse displacement nodes $\sin(2\pi x/\lambda) = 0$ at $x = 0$ and $x = L$ (endpoints of string).

Standing wave patterns – "normal modes"



	1st harm.	$L = \frac{1\lambda_1}{2}$	$\lambda_1 = \frac{2}{1}\ L$	$f_1 = \frac{1V_x}{2L} = 1\ f_1$
	2nd harm.	$L = \frac{2\lambda_2}{2}$	$\lambda_2 = \frac{2}{2}\ L$	$f_2 = \frac{2V_x}{2L} = 2\ f_1$
	3rd harm.	$L = \frac{3\lambda_3}{2}$	$\lambda_3 = \frac{2}{3}\ L$	$f_3 = \frac{3V_x}{2L} = 3\ f_1$
...	...	...	...	...
	nth harm.	$L = \frac{n\lambda_n}{2}$	$\lambda_n = \frac{2}{n}\ L$	$f_n = \frac{nV_x}{2L} = n\ f_1$

Note: 1st harmonic (n = 1) also known as the Fundamental
2nd harmonic (n = 2) also known as the 1st Overtone
3rd harmonic (n = 3) also known as the 2nd Overtone

etc.

$$f_n = n\frac{v}{2L} = nf_1\ ; \qquad f_1 = \frac{v}{2L}$$

$$\lambda_n = \frac{2L}{n} = \frac{\lambda_1}{n}; \qquad n = 1, 2, 3, \ldots$$

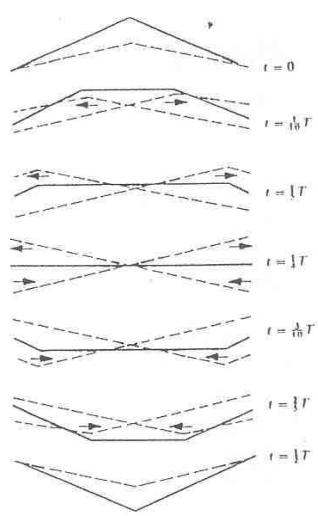


FIGURE 2.6. Time analysis of the motion of a string plucked at its midpoint through one half cycle. Motion can be thought of as due to two pulses traveling in opposite directions.

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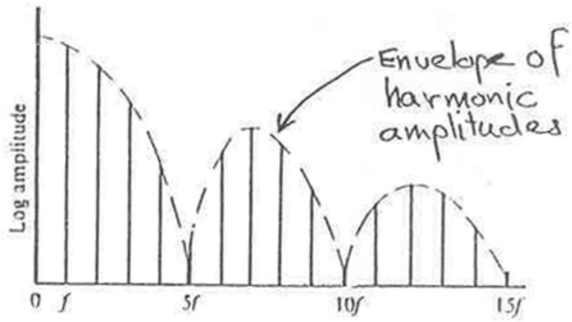

FIGURE 2.7. Spectrum of a string plucked one-fifth of the distance from one end.

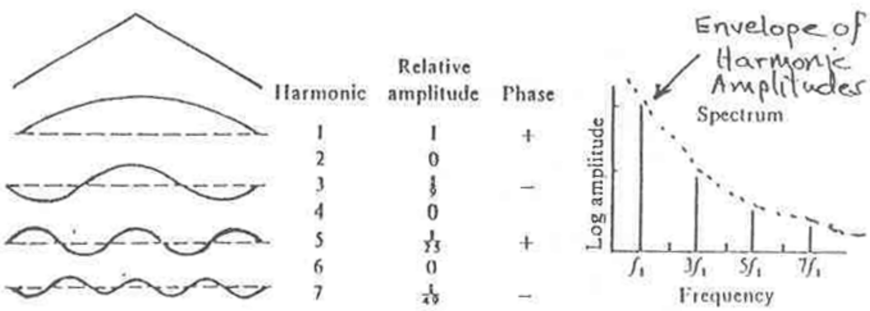

FIGURE 2.5. Frequency analysis of a string plucked at its center. Odd-numbered modes of vibration add up in appropriate amplitude and phase to give the shape of the string.

Please see/hear/touch UIUC Physics 406POM **Guitar.exe** demo – shows/demos the Fourier harmonic amplitudes associated with a guitar string plucked at arbitrary point along its length…. Reconstructs the geometrical shape of the plucked string (@ $t = 0$) from Fourier components…

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PICKING/PLUCKING A GUITAR STRING

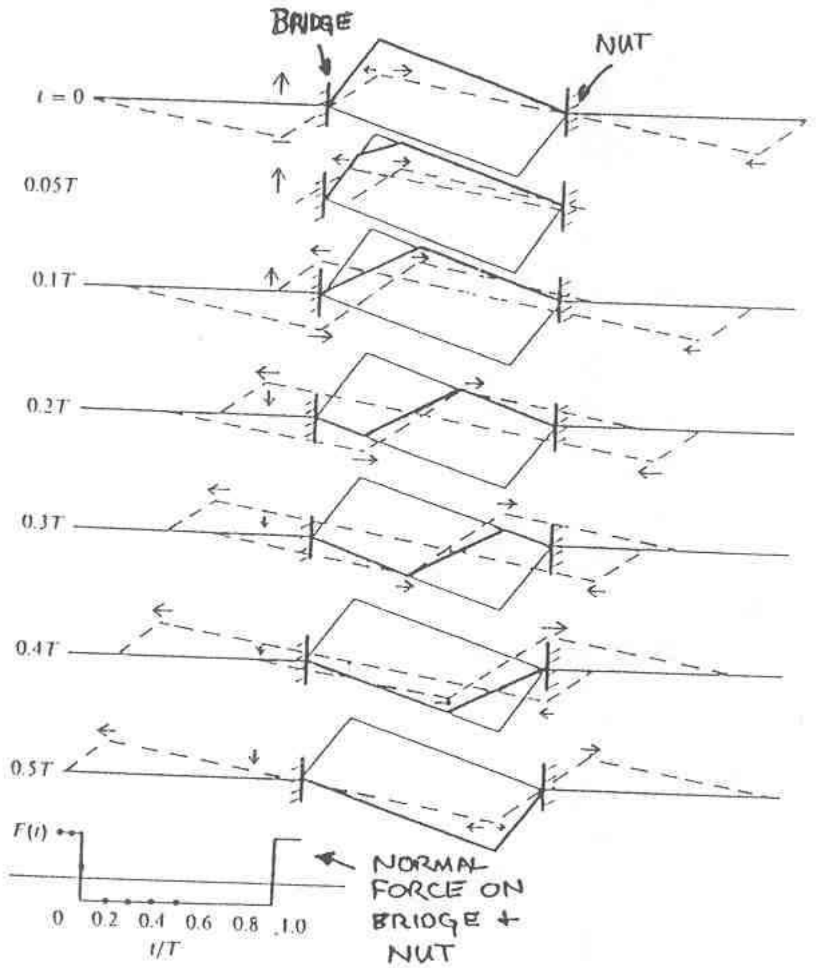

FIGURE 2.8. Time analysis through one half cycle of the motion of a string plucked one-fifth of the distance from one end. The motion can be thought of as due to two pulses [representing the two terms in Eq. (2.5)] moving in opposite directions (dashed curves). The resultant motion consists of two bends, one moving clockwise and the other counterclockwise around a parallelogram. The normal force on the end support, as a function of time, is shown at the bottom.

$$y_{string}(t,x) = \underbrace{y_R(vt-x)}_{\text{right-moving traveling wave}} + \underbrace{y_L(vt+x)}_{\text{left-moving traveling wave}} = \text{"standing" wave}$$

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Vibrating Air Columns

(Longitudinal) Standing Waves in a Pipe:

= superposition of two counter-propagating traveling waves (one right moving, one left moving)

Rarefaction and compression of air molecules = displacement of air molecules from their equilibrium positions

See UIUC Physics 406 animation of longitudinal displacement of air molecules in a pipe…

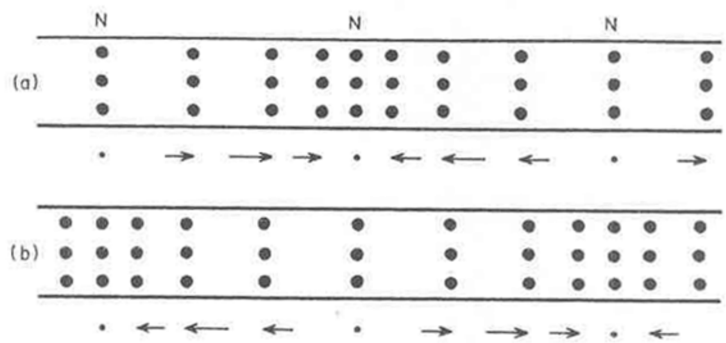

FIG. 5. Longitudinal standing wave in an air column. (a) At an instant of maximum displacement of the air molecules. (b) One-half cycle later.

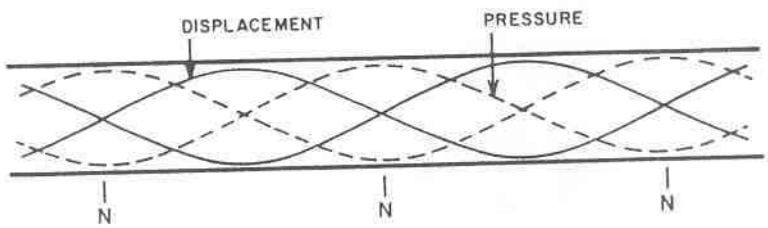

FIG. 6. Graphic representation of a longitudinal standing wave.

Three basic kinds of "organ pipes":

a.) Both ends ***closed*** (analogous to "fixed" ends on a vibrating string)
b.) Both ends ***open*** (analogous to "free" ends on a vibrating string)
c.) One end ***open***, one end ***closed*** (analogous to one end fixed, one end free on string)

⇒ **Boundary Conditions** on mathematical allowed solutions to the wave equation that describes the longitudinal waves propagating in an organ pipe

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a.) **Both Ends Closed:**

$$v = f_n \lambda_n$$
$$f_n = nf_1 = n\frac{v}{2L}$$
$$\lambda_n = \frac{\lambda_1}{n} = \frac{2L}{n}$$
$$n = 1, 2, 3, 4\ldots$$

$$f_1 = \frac{c}{2L}, \qquad (8)$$

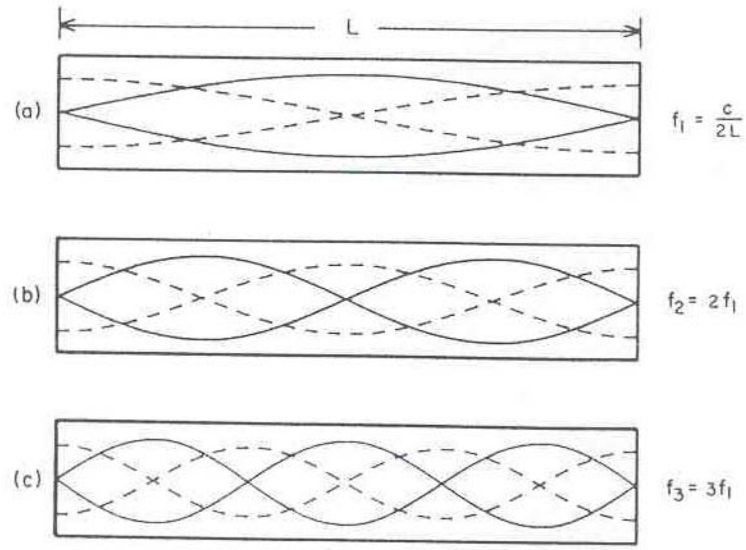

FIG. 7. First three vibration modes of an air column closed at both ends. Solid lines give displacement amplitudes; dashed lines, pressure amplitudes.

Closed Ends: ⇒ **Pressure** ***anti-nodes*** and **displacement** ***nodes*** at $x = 0$ and $x = L$.

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b.) **Both Ends Open:**

$$v = f_n \lambda_n$$
$$f_n = n f_1 = n\frac{v}{2L}$$
$$\lambda_n = \frac{\lambda_1}{n} = \frac{2L}{n}$$
$$n = 1, 2, 3, 4\ldots$$

Open Ends: ⇒ **Pressure *nodes*** and **displacement *anti-nodes*** at $x = 0$ and $x = L$.

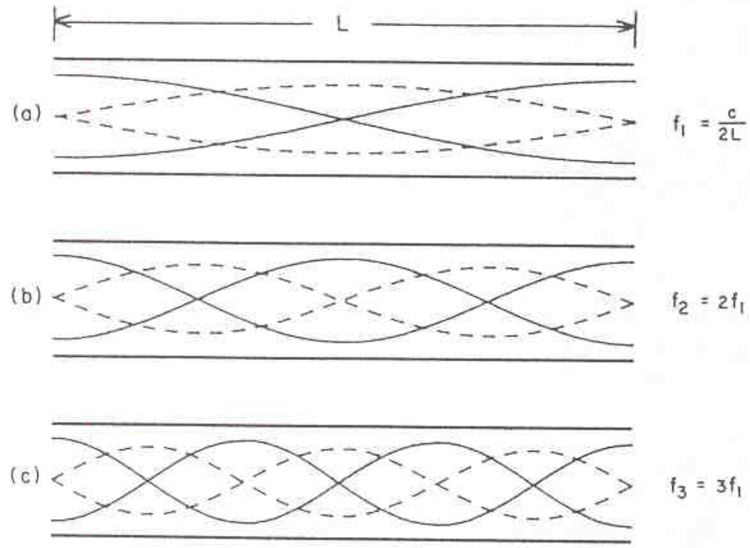

FIG. 8. First three vibration modes of an air column open at both ends. Solid lines give displacement amplitudes; dashed lines, pressure amplitudes.

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c.) **<u>One End Open, One End Closed</u>:**

$$v = f_m \lambda_m$$
$$f_m = m f_1 = m \frac{v}{4L}$$
$$\lambda_m = \frac{\lambda_1}{m} = \frac{4L}{m}$$
$$m = 1, 3, 5, 7 \ldots$$

n.b. Only ***<u>odd</u>***-*m* integers allowed!

<u>Closed End</u>: $\Rightarrow$ **Displacement *<u>node</u>*** & **pressure *<u>anti</u>*-*<u>node</u>*** at $x = 0$.
<u>Open End</u>: $\Rightarrow$ **Displacement *<u>anti</u>*-*<u>node</u>*** & **pressure *<u>node</u>*** at $x = L$.

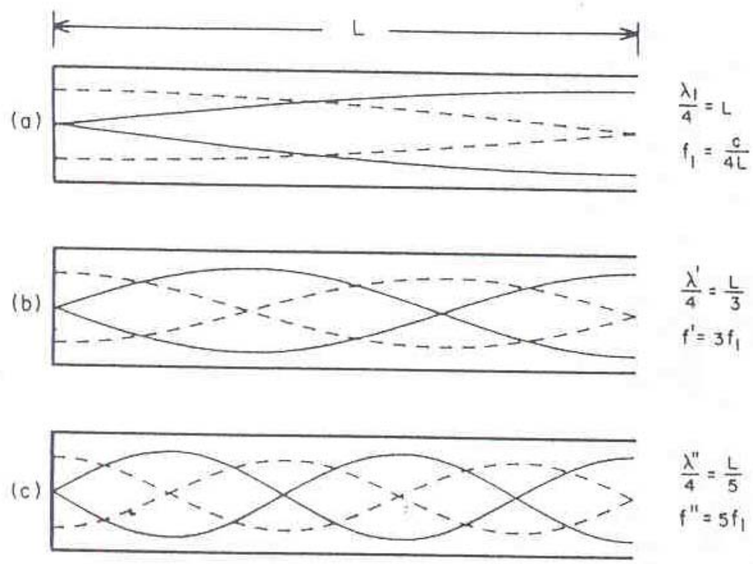

FIG. 9. First three vibration modes of an air column closed at one end and open at the other. Solid lines give displacement amplitudes; dashed lines, pressure amplitudes.

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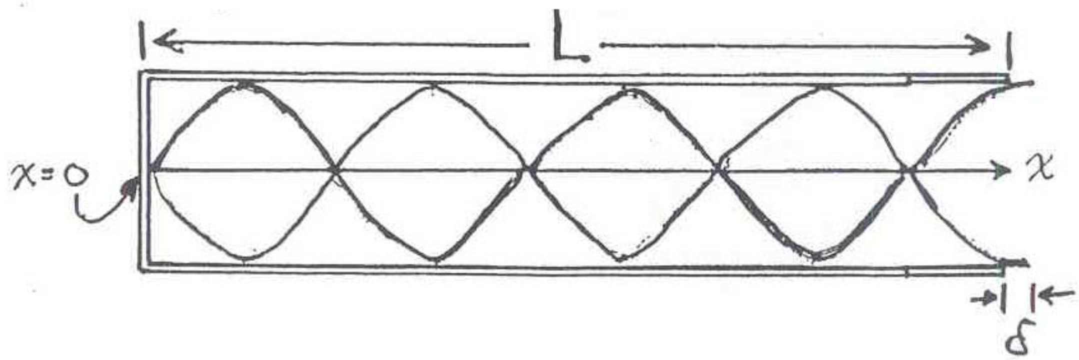
Normal Modes & Standing Waves

1.) **<u>Standing Sound Waves in an Organ Pipe</u>**:

(a) Standing displacement wave:

$$y = A\sin\left(\frac{2\pi x}{\lambda}\right)\cos\left(\frac{2\pi t}{\tau}\right) \qquad \text{(Standing Wave)}$$

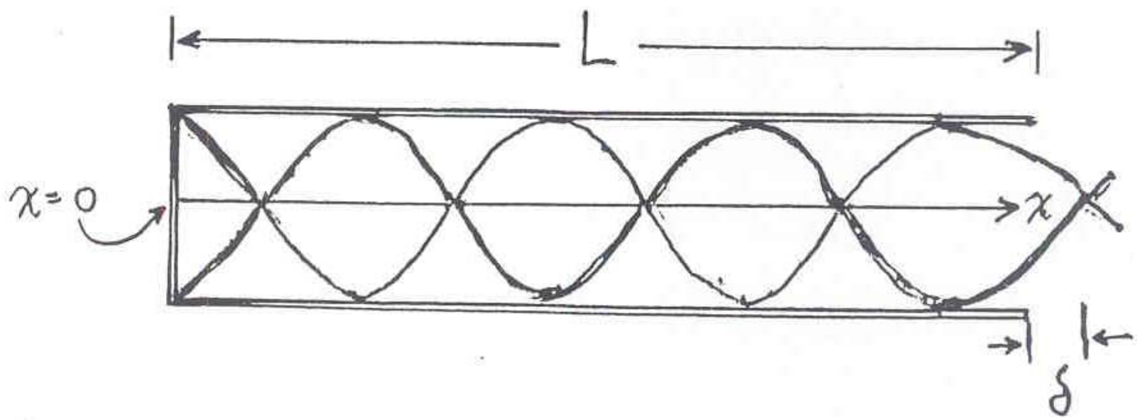
- Displacement ***node*** at $x = 0$



(b) Standing pressure wave:

$$\triangle P = -B\frac{\partial y}{\partial x} = -B\,A\left\{\frac{\partial}{\partial x}\sin\left(2\pi x/\lambda\right)\right\}\cos\left(\frac{2\pi t}{\tau}\right) = -\frac{2\pi\ B\ A}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right)\cos\left(\frac{2\pi t}{\tau}\right)$$

- Explains why **displacement *<u>nodes</u>*** are **pressure *<u>anti</u>-<u>nodes</u>***!



(c) Pressure ***<u>node</u>*** $\left(p = p_{\text{ambient}}\right)$ just ***beyond*** open end $x = L + \delta \Leftarrow$ not precisely at $x = L$!

- so-called "end correction" $\delta \approx 0.6D$, where D = diameter of pipe.

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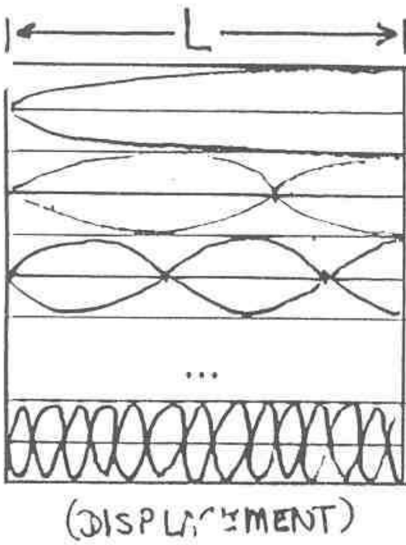
2.) **<u>Standing Sound Waves in Closed-Open Organ Pipes</u>:**

<u>Closed End</u>: $\Rightarrow$ **Displacement** ***<u>node</u>*** & **pressure** ***<u>anti</u>*-*<u>node</u>*** at $x = 0$.
<u>Open End</u>: $\Rightarrow$ **Displacement** ***<u>anti</u>*-*<u>node</u>*** & **pressure** ***<u>node</u>*** at $x = L$.



n'			
1st harm.	$L = \frac{1}{4}\lambda_1$	$\lambda_1 = \frac{4}{1}L$	$f_1 = \frac{1}{4}\frac{V}{L} = 1\ f_1$
3rd harm.	$L = \frac{3}{4}\lambda_3$	$\lambda_3 = \frac{4}{3}L$	$f_3 = \frac{3}{4}\frac{V}{L} = 3\ f_1$
5th harm.	$L = \frac{5}{4}\lambda_5$	$\lambda_5 = \frac{4}{5}L$	$f_5 = \frac{5}{4}\frac{V}{L} = 5\ f_1$
...	...	...	....
nth harm.	$L = \frac{n'}{4}\lambda_{n'}$	$\lambda_{n'} = \frac{4}{n'}L$	$f_{n'} = \frac{n'}{4}\frac{V}{L} = n'\ f_1$

$$f_{n'} = \frac{v}{\lambda_{n'}} = n' f_1; \quad f_1 = \frac{v}{4L}$$

where: $n' = 2n - 1, \quad n = 1, 2, 3, \ldots$ so $n' = 1, 3, 5, \ldots$ (*i.e.* the ***<u>odd</u>*** integers)

- First harmonic also known as the fundamental.
- Second harmonic also known as the first overtone, *etc*.
- Replace L by $L + \delta$ for "exact" answer

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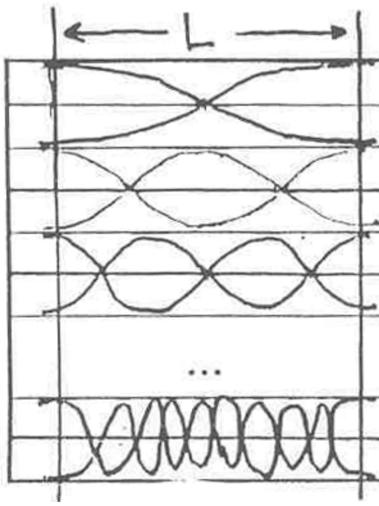
3.) <u>Standing Waves in Open-Open (and Closed-Closed) Organ Pipes</u>:

<u>Open Ends</u>: $\Rightarrow$ **Pressure** ***<u>nodes</u>*** and **displacement** ***<u>anti-nodes</u>*** at $x = 0$ and $x = L$.
<u>Closed Ends</u>: $\Rightarrow$ **<u>Pressure</u>** ***<u>anti-nodes</u>*** and **<u>displacement</u>** ***<u>nodes</u>*** at $x = 0$ and $x = L$.



n

1st harm.	$L = \frac{1}{2}\lambda_1$	$\lambda_1 = \frac{2}{1}L$	$f_1 = \frac{1}{2}\frac{V}{L} = 1\ f_1$
2nd harm.	$L = \frac{2}{2}\lambda_2$	$\lambda_2 = \frac{2}{2}L$	$f_2 = \frac{2}{2}\frac{V}{L} = 2\ f_1$
3rd harm.	$L = \frac{3}{2}\lambda_3$	$\lambda_3 = \frac{2}{3}L$	$f_3 = \frac{3}{2}\frac{V}{L} = 3\ f_1$
...	...	...	....
nth harm.	$L = \frac{n}{2}\lambda_n$	$\lambda_n = \frac{2}{n}L$	$f_n = \frac{n}{2}\frac{V}{L} = n\ f_1$

(*n.b.* open-open standing wave modes drawn)

$$f_n = \frac{v}{\lambda_n} = n\, f_1; \quad f_1 = \frac{v}{2L}; \quad n = 1, 2, 3, \ldots$$

- First harmonic also known as the fundamental
- Second harmonic also known as the first overtone, *etc*.
- Replace L by $L + 2\,\delta$ for "exact" answer.
- Note: Since $v_{\text{helium}} \gg v_{air}$, $f_1(\text{helium}) > f_1(\text{air})$

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Conical-Shaped Air Columns

Some wind instruments - *e.g.* whistles, recorders, flutes, oboe, bagpipes (chanter) have ***conical-shaped*** air columns: ≈ more complicated organ pipes – one end open; one end closed…

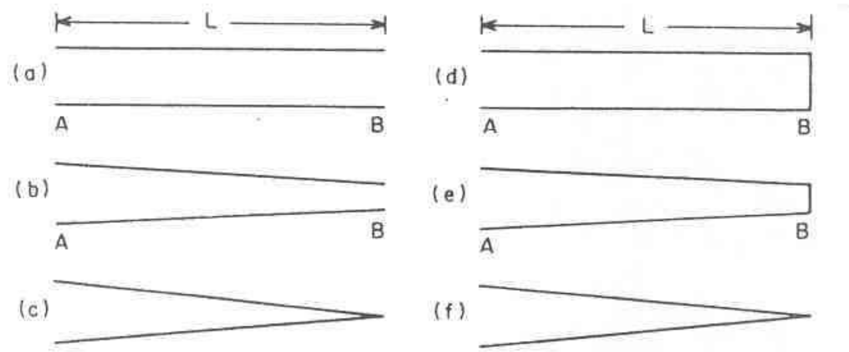

FIG. 10. (a), (b), (c) Going from an open tube to a cone. (d), (e), (f) Going from a closed tube to the same cone.

$$v = f_n \lambda_n$$
$$f_n = n f_1$$
$$\lambda_n = \frac{\lambda_1}{n} = \frac{2L}{n}$$
$$n = 1, 2, 3....$$

A **complete** cone has the ***same*** mode vibration frequencies as that for an **open-open** tube of the **same** length – the ***tip*** of the cone reflects like the **open** end of a tube!!

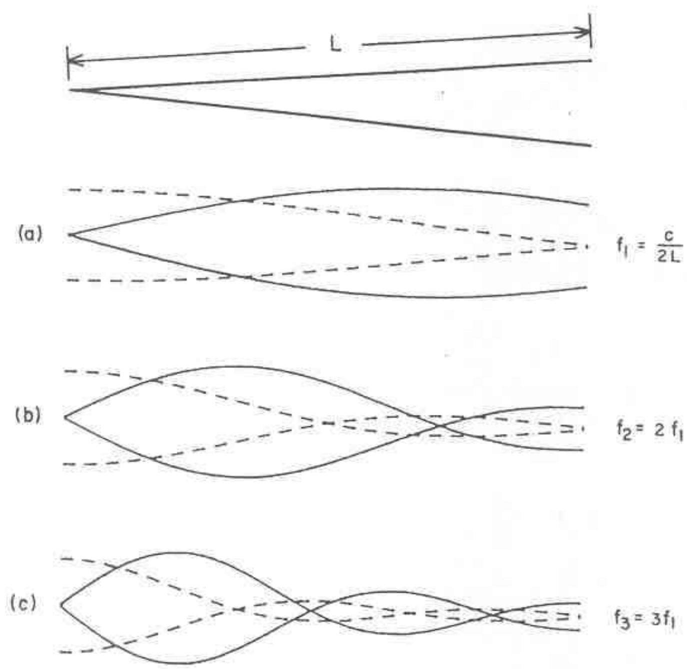

FIG. 11. First three vibration modes of a complete cone.

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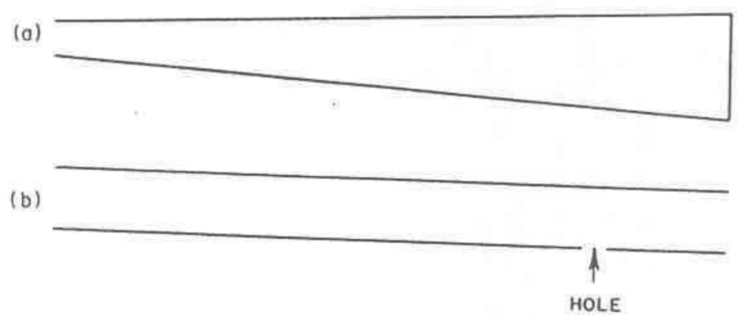

FIG. 12. Distorted air columns.

<u>The Tuning Fork</u>: Vibrations of a metal bar clamped @ one end (math not simple..):

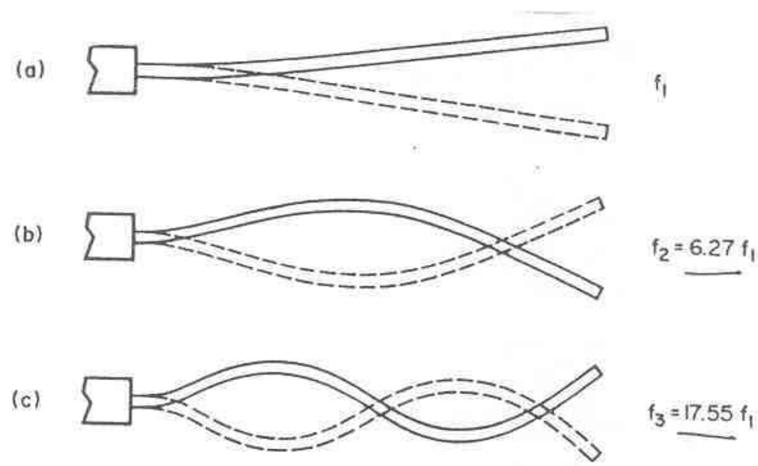

FIG. 13. First three transverse vibration modes of a metal bar clamped at one end.

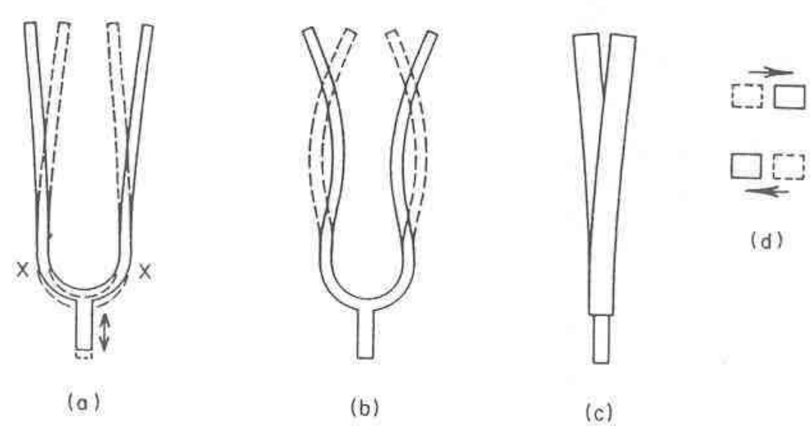

FIG. 14. Vibrations of a tuning fork. (a) Normal vibration. (b) Clang tone. (c) Another mode of vibration. (d) End view of this mode.

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