On The Concord of Sweet Sounds

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Analysis of music for its consonant and dissonant properties has been performed by philosophers, musicians, physicists, and mathematicians for centuries. Analysis of powerful models developed by these researchers can be simplified and strengthened using basic physical concepts. By observing standard musical waveforms on an oscilloscope and comparing them to standard mathematical models a new understanding involving the stability of the waveform can be compiled.

I. INTRODUCTION

Harmony is the property of music described as how well sounds coincide with one another. Dissonance is the aspect of harmony wherein sounds fight against one another. Consonance is the opposite; this occurs when sounds blend.

Music is an assortment of sounds in varying degrees of harmony. These sounds have been studied for centuries to try to determine why the some sounds of music hold power over people. In many realms people have joined this pursuit: musicians create the music, philosophers muse at its influence on people, and mathematicians and physicists study its physical properties.

Musicians study music on the basis of chords and harmonic structure. When looking at chords, flow of sound is a major consideration, as well as the small-scale movement of one tone to the next. These main topics of study are important contributions to composing new music and understanding musical literature form the past.

Philosophers consider the general effect of music on people. They are concerned with the strength of chords and the meaning or feeling of specific chords to individual people. The research musical philosophers undertake is based primarily on subjective chord quality, not scientific principles.

Mathematicians and physicists have tried to find a basic model for the consonance and dissonance of chord qualities. This has been accomplished through scientific study and mathematical extrapolation. These theories, however, have become difficult to understand and even more difficult to implement.

II. BACKGROUND

Study has primarily involved the ratio of one note to another. These interval relationships are governed by two different historically developed tuning schemes. The first, the Pythagorean intervals, are based off of whole number ratios between notes in a scale. For example, a fifth is a ratio of 3 to 2 from to the fundamental or base tone. These values are shown in table 1 below.

The second method, also in table 1 below, is based on the twelfth root of two. For example, the third scale degree is the three-twelfth root of 2.

Degree	Pythagorean	Root
Root	1:1	1
Minor Second	16:15	1.059
Major Second	9:8	1.122
Minor Third	6:5	1.189
Major Third	5:4	1.260
Perfect Fourth	7:5	1.335
Tri-tone	13:9	1.414
Perfect Fifth	3:2	1.498
Minor Sixth	8:5	1.587
Major Sixth	5:3	1.682
Minor Seventh	9:5	1.782
Major Seventh	15:8	1.888
Octave	2:1	2

Table 1: Frequency Ratios

When talking about musical structures, the ratios between the various scale degrees and the tonic is important. For a major fifth, the 3:2:1 ratio defines the relationship. For something like a major chord (the first, major third, and perfect fifth) the relationship is a 6:5:4:1 ratio. (3:2 is converted to 6:4)

The major research addressed in this study was Leonhard Euler's study of consonance. He categorized chords of two or more sounds into what he deemed "degrees of sweetness". His categorization is based on the lowest common multiples of the scale degrees of the notes within the chord and putting them into specific categories. These categories are organized from the smoothest sounds to the most dissonant, as shown in Table 2.

The first degree consists of only the unison interval as these two waves align completely. The second degree consists of the octave, where the two waves are aligned such that one is exactly twice the frequency of the other.

Then as the degrees of sweetness then increase the lowest common multiples of their ratios are organized according to an unstable naming scheme. The scheme that was used by Euler is generalized to a scheme of the multiples of the prime factors of the ratios as shown:

$$\text{For a ratio of } 1{:}p_1^{k_1}p_2^{k_2}\ldots p_m^{k_m}$$

$$\text{Degree} = \sum_{1}^{m}(k_n p_n - k_n) + 1$$

For example, the fifth will be (1*3-1)+(1*2-1)+1=4. The fifth is found at Euler's degree of sweetness four.

The problem with this method lies in higher complexity chords. The generalized scheme is difficult to perform for all chords. Euler has supplied a table of the various elements as shown, but his calculation scheme is not easily computable. Given any higher degree chord, for example a seventh chord, the lowest common multiple fits into Euler's table properly, but the scheme for deriving that place in the table breaks down.

Major seven chords consist of a root, major third, fifth, and major

I	1;
II	2;
III	3, 4;
IY	6, 8;
Y	5, 9, 12, 16;
YI	10, 18, 24, 32.
YII	7, 15, 20, 27, 36, 48, 64;
YIII	14, 30, 40, 54, 72, 96, 128;
IX	21, 25, 28, 45, 60, 80, 81, 108, 144, 192, 256;
X	42, 50, 56, 90, 120, 160, 162, 216, 288, 384, 512;
XI	11, 35, 63, 75, 84, 100, 112, 135, 180, 240, 243, 320, 324, 432, 576, 768, 1024;
XII	22, 70, 126, 150, 168, 200, 224, 270, 360, 480, 486, 640, 648, 864, 1152, 1536, 2048;
XIII	13, 33, 44, 49, 105, 125, 140, 189, 225, 252, 300, 336, 400, 405, 448, 540, 720, 729, 960, 972, 1280, 1296, 1728, 2304, 3072, 4096;
XIY	26, 66, 88, 98, 210, 250, 280, 378, 450, 504, 600, 672, 800, 810, 896, 1080, 1440, 1458, 1920, 1944, 2560, 2592, 3456, 4608, 6144, 8192;
XY	39, 52, 55, 99, 132, 147, 175, 176, 196, 315, 375, 450, 500, 560, 567, 675, 756, 900, 1008, 1200, 1215, 1344, 1600, 1620, 1792, 2160, 2187, 2880, 2916, 3840, 3888, 5120, 5184, 6912, 9216, 12288, 16384;
XYI	78, 104, 110, 198, 264, 294, 350, 352, 392, 630, 750, 840, 1000, 1120, 1134, 1350, 1512, 1800, 2016, 2400, 2430, 2688, 3200, 3240, 3584, 4320, 4374, 5760, 5832, 7680, 7776, 10240, 10368, 13824, 18432, 24576, 32768.

Table 2: Euler's Chart of Lowest Common Multiples and Degrees of Sweetness.

seventh. The math according to Euler is as follows:

$LCM(1:8:10:12:15)=120$

Factorization: $1, 2^3, 2*5, 3*2^2, 3*5$

Simplifies to: $1, 2^6, 3^2, 5^2$

$Degree=(1*1-1)+(6*2-6)+(2*3-2)+(2*5-2)+1$

$=0+6+4+8+1=19$

This chord is supposed to fit into degree 10, but seems to fit into degree 19 when calculated.

III. METHODS

Two main methods were employed to look at possible physical and mathematical relationships between tones. The first method employs a unity gain op-amp connected to function generators and an oscilloscope while the second is a purely mathematical exploration of Euler's basic concepts and ideas.

The basic construction of the op-amp is illustrated in figure 1.

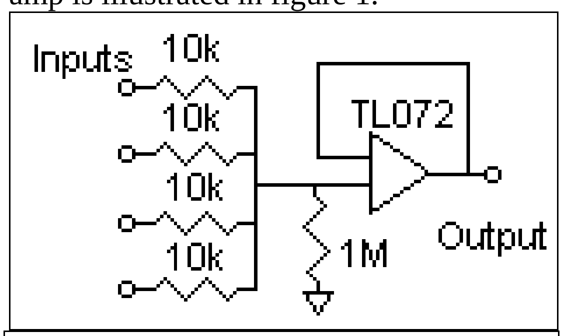


Figure 1: Op Amp Diagram

Overall, the connections are based on the setup in figure 2. The op-amp is housed in the aluminum box adjacent to four input function generators. The output of the op-amp is then connected to the oscilloscope to view the summed wave.

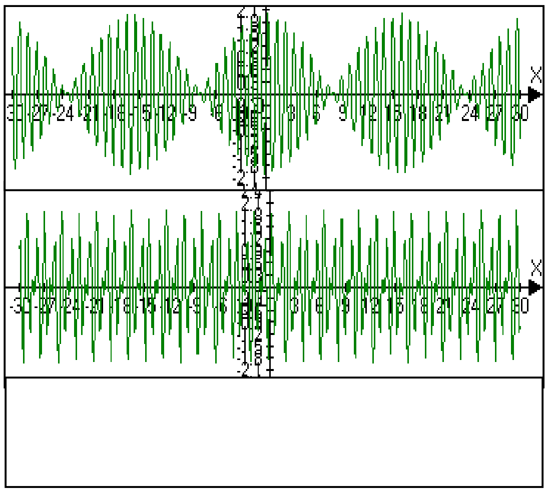
This setup was to observe the waveform of summed waves such as intervals and chords. The general relationships of tones, such as a root and fifth (1 and 1.5 times a fundamental) are input from two of the signal generators to the op-amp, and, in turn to the oscilloscope. Once the superposed wave is apparent on the scope, a physical property comparable to Euler's mathematics may be observed.

The other method involved finding a mathematical explanation for Euler's formulas that correlates with a physical idea. This only requires pen, paper, and a calculator. In general, this method looked at the Lowest Common Multiples of the component frequencies and looked for ways of fitting them to Euler's scheme.

IV. RESULTS AND DISCUSSION

In general, the sweetness of a sound is its quality of smoothness. Euler used this as the name for his ideas, but his mathematics is in trouble. Using physics, however, a simpler solution was found. The stability of a wave is in direct relation to its smoothness in time. Therefore, physically speaking, the smoothness of the sound is determined by the stability the wave exhibits.

The best way of looking at the stability of waves is to compare physics to what Euler was trying to do. The method of finding the lowest common multiple was just another way of finding out how many repetitions of the fundamental frequency were required until the entire pattern repeated itself. For example, a simple octave will repeat every two cycles of the fundamental whereas a fifth will repeat every three cycles. This is a direct correlation to Euler's work.



The difference between the physical and mathematical observations is the end classification. As a set of sounds requires many more repeats to coincide or has a very irregular cycle, it is a higher complexity then something that repeats readily and has a simple symmetric waveform. This comparison does not provide for an absolute scale, but provides for a way to meaningfully compare sounds without getting bogged down in classification. This differs from the mathematical calculations in this respect.

Problems:

Unfortunately, the oscilloscope method is very difficult to use. The signal generators do not hold steady at the desired frequency, so the summed signal does not hold steady. Therefore, only the general idea of the wave can be understood from looking at the signal this way. This means the mathematical approach is absolutely necessary.

V. CONCLUSIONS

The stability of a wave is the determination of sweetness. Complex sounds shift and grind to people's ears sounding unpleasant, whereas simpler sounds align and sound soothing, pleasing, and even. Eulers' method takes this simple system but tries to make it too complex for meaningful analysis. In this situation no absolute scale exists, but a relative scale of wave repetition can serve as a basis for comparison of soothing and rampantly grinding sounds. Euler's chart of multiples is a good basis for comparison, but not realistic to derive for an arbitrary complex chord.

Overall, Euler's methods make a definite method of classification for musical chords. Unfortunately, this method is difficult to understand and reproduce. Therefore, a better method using physics was found. Euler's methods aligned with easily understood physics, and thusly it can be explained.

The longer it takes a wave to repeat, the harsher it sounds to the ear. The method of Euler's classifications fits in with this method, and thusly it is a simpler way of classifying them than looking at prime factorizations of each component wave. This discovery is a simpler method of understanding how a consonant sound can be pleasing and settled to the ear and a dissonant tone as grating.

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VII. REFERENCES

- Bailhache, Patrice. “Music Translated into Mathematics: Leonhard Euler”. http://www.ixpress.com/interval/monzo/euler/euler-en.htm. Jan 17, 1997.