Time Domain *vs*. Frequency Domain Sound Field Quantities

For "everyday" sound pressure levels $SPL(\vec{r}) \ll 134\,dB$ in air (at NTP), the ***purely real***, ***instantaneous*** sound field $S(\vec{r},t)$ at a listener point $\vec{r}$ in space is uniquely/fully specified ***iff*** (if and only if) ***two*** physical measurements are ***simultaneously*** made at the listener's point $\vec{r}$: the ***purely real***, ***instantaneous*** acoustic over-pressure $p(\vec{r},t)$ {*n.b.* a ***scalar*** quantity} and the ***purely real***, ***instantaneous*** 3-D particle velocity $\vec{u}(\vec{r},t)$ {*n.b.* a ***vector*** quantity}. Both of these ***instantaneous*** quantities are also known as ***time-domain*** quantities.

Note that simultaneous measurement of these two physical quantities {alone} at the listener point $\vec{r}$ is ***not*** sufficient to enable ***extrapolation*** of knowledge of the ***instantaneous*** sound field $S(\vec{r},t)$ to ***other*** space points $\vec{r}'$ and/or ***other*** times, t'. The ***gradient*** of scalar $p(\vec{r},t)$ – and the divergence and curl of 3-D vector $\vec{u}(\vec{r},t)$ – must ***also*** be known/measured…

Instantaneous *vs*. Complex Pressure and 3-D Particle Velocity:

For a ***monochromatic harmonic*** sound field – associated with a ***single*** frequency $\omega = 2\pi f$, the physical, ***purely real***, ***instantaneous*** acoustic over-pressure at the space point $\vec{r}$ is: $p(\vec{r},t) = p_o(\vec{r},\omega)\cos\left(\omega t + \varphi_p(\vec{r},\omega)\right)$ {SI units = *Pascals*}, the physical, ***purely real***, ***instantaneous*** 3-D particle velocity {SI units = *m/sec*} at the space point $\vec{r}$ is:

$$\begin{aligned}\vec{u}(\vec{r},t) &= u_x(\vec{r},t)\hat{x} + u_y(\vec{r},t)\hat{y} + u_z(\vec{r},t)\hat{z} \\ &= u_{o_x}(\vec{r},\omega)\cos\left(\omega t + \varphi_{u_x}(\vec{r},\omega)\right)\hat{x} \\ &+ u_{o_y}(\vec{r},\omega)\cos\left(\omega t + \varphi_{u_y}(\vec{r},\omega)\right)\hat{y} \\ &+ u_{o_z}(\vec{r},\omega)\cos\left(\omega t + \varphi_{u_z}(\vec{r},\omega)\right)\hat{x}\end{aligned}$$

Both the physical, ***purely real instantaneous*** acoustic over-pressure and ***instantaneous*** particle velocity are manifestly what we refer to as a ***time-domain*** quantities.

Using the trigonometric relation: $\cos(A+B) = \cos A\cos B - \sin A\sin B$ we can rewrite these two physical, ***purely real instantaneous time-domain*** quantities as:

$$p(\vec{r},t) = p_o(\vec{r},\omega)\cos\left(\omega t + \varphi_p(\vec{r},\omega)\right) = p_o(\vec{r},\omega)\left[\cos\omega t\cos\varphi_p(\vec{r},\omega) - \sin\omega t\sin\varphi_p(\vec{r},\omega)\right]$$

$$u_k(\vec{r},t) = u_{o_k}(\vec{r},\omega)\cos\left(\omega t + \varphi_{u_k}(\vec{r},\omega)\right) = u_{o_k}(\vec{r},\omega)\left[\cos\omega t\cos\varphi_{u_k}(\vec{r},\omega) - \sin\omega t\sin\varphi_{u_k}(\vec{r},\omega)\right]$$

where $k = x,y,z$. Note that a ***"generic"*** physical, ***purely real instantaneous monochromatic*** harmonically-varying/time-dependent signal $V(t)$ can similarly be written as:

$$V(t) = V_o\cos\left(\omega t + \varphi_V(\omega)\right) = V_o(\omega)\left[\cos\omega t\cos\varphi_V(\omega) - \sin\omega t\sin\varphi_V(\omega)\right]$$

A dual-channel lock-in amplifier {such as the Stanford Research Systems SRS-830 that we routinely use in the UIUC P406 POM lab} measures the ***in-phase*** and 90° ***out-of-phase*** (*aka* ***quadrature***) ***amplitude components*** $V_{\mathrm{x}}(\omega) \equiv V_o(\omega)\cos\varphi_V(\omega)$ and: $V_{\mathrm{y}}(\omega) \equiv V_o(\omega)\sin\varphi_V(\omega)$, respectively, of whatever "generic" purely real, ***instantaneous monochromatic harmonic*** signal $V(t) = V_o\cos(\omega t + \varphi_V(\omega))$ is input to the dual-channel lock-in amplifier.

However, order to obtain such measurements, the dual-channel lock-in amplifier ***must*** be ***phase-referenced relative*** to a periodic/harmonic ***reference*** signal {of the same frequency ω} – *e.g.* a sine-wave signal output from a driving function generator. Detailed information on how a dual-channel lock-in amplifier works is discussed in P406 POM Lect. Notes XIII – Part 2.

We can thus form the ***complex*** "generic" ***amplitude*** (*aka* ***phasor***): $\tilde{V}(\omega) \equiv V_{\mathrm{x}}(\omega) + iV_{\mathrm{y}}(\omega)$ where:

$$V_{\mathrm{x}}(\omega) \equiv \mathrm{Re}\{\tilde{V}(\omega)\} = V_o(\omega)\cos\varphi_V(\omega) \quad \text{and:} \quad V_{\mathrm{y}}(\omega) \equiv \mathrm{Im}\{\tilde{V}(\omega)\} = V_o(\omega)\sin\varphi_V(\omega).$$

Using the Euler relation: $e^{i\phi} = \cos\varphi + i\sin\varphi$, we see that we can also equivalently write the ***complex*** "generic" ***amplitude*** as:

$$\tilde{V}(\omega) \equiv V_{\mathrm{x}}(\omega) + iV_{\mathrm{y}}(\omega) = V_o(\omega)\left[\cos\varphi_V(\omega) + i\sin\varphi_V(\omega)\right] = V_o(\omega)e^{i\varphi_V(\omega)} = \left|\tilde{V}(\omega)\right|e^{i\varphi_V(\omega)}.$$

Note further that the ***complex*** "generic" ***amplitude*** $\tilde{V}(\omega) = V_o(\omega)e^{i\varphi_V(\omega)}$ is what we refer to as a ***frequency domain*** quantity – *i.e.* a dual-channel lock-in amplifier manifestly/intrinsically carries out measurements of ***complex*** "generic" ***amplitudes*** in the ***frequency domain***.

Thus, we can similarly construct ***complex time-domain*** acoustic over-pressure and ***complex time-domain*** acoustic 3-D particle velocity as follows:

$$\begin{aligned}\tilde{p}(\vec{r},t) &= p_o(\vec{r},\omega)\left[\cos\left(\omega t + \varphi_p(\vec{r},\omega)\right) + i\sin\left(\omega t + \varphi_p(\vec{r},\omega)\right)\right] \\ &= p_o(\vec{r},\omega)e^{i\left(\omega t + \varphi_p(\vec{r},\omega)\right)} = \underbrace{p_o(\vec{r},\omega)e^{i\varphi_p(\vec{r},\omega)}}_{\equiv \tilde{p}_o(\vec{r},\omega)} \cdot e^{i\omega t} = \tilde{p}_o(\vec{r},\omega)\cdot e^{i\omega t}\end{aligned}$$

and:

$$\begin{aligned}\vec{\tilde{u}}(\vec{r},t) &= \tilde{u}_x(\vec{r},t)\hat{x} + \tilde{u}_y(\vec{r},t)\hat{y} + \tilde{u}_z(\vec{r},t)\hat{z} \\ &= u_{o_x}(\vec{r},\omega)\left[\cos\left(\omega t + \varphi_{u_x}(\vec{r},\omega)\right) + i\sin\left(\omega t + \varphi_{u_x}(\vec{r},\omega)\right)\right]\hat{x} \\ &\quad + u_{o_y}(\vec{r},\omega)\left[\cos\left(\omega t + \varphi_{u_y}(\vec{r},\omega)\right) + i\sin\left(\omega t + \varphi_{u_y}(\vec{r},\omega)\right)\right]\hat{y} \\ &\quad + u_{o_z}(\vec{r},\omega)\left[\cos\left(\omega t + \varphi_{u_z}(\vec{r},\omega)\right) + i\sin\left(\omega t + \varphi_{u_z}(\vec{r},\omega)\right)\right]\hat{z} \\ &= u_{o_x}(\vec{r},\omega)e^{i\left(\omega t + \varphi_{u_x}(\vec{r},\omega)\right)}\hat{x} + u_{o_y}(\vec{r},\omega)e^{i\left(\omega t + \varphi_{u_y}(\vec{r},\omega)\right)}\hat{y} + u_{o_z}(\vec{r},\omega)e^{i\left(\omega t + \varphi_{u_z}(\vec{r},\omega)\right)}\hat{z} \\ &= \underbrace{u_{o_x}(\vec{r},\omega)e^{i\varphi_{u_x}(\vec{r},\omega)}}_{\equiv \tilde{u}_{o_x}(\vec{r},\omega)} \cdot e^{i\omega t}\hat{x} + \underbrace{u_{o_y}(\vec{r},\omega)e^{i\varphi_{u_y}(\vec{r},\omega)}}_{\equiv \tilde{u}_{o_y}(\vec{r},\omega)} \cdot e^{i\omega t}\hat{y} + \underbrace{u_{o_z}(\vec{r},\omega)e^{i\varphi_{u_z}(\vec{r},\omega)}}_{\equiv \tilde{u}_{o_z}(\vec{r},\omega)} \cdot e^{i\omega t}\hat{z}\end{aligned}$$

Thus:

$$\vec{\tilde{u}}(\vec{r},t) = \left[u_{o_x}(\vec{r},\omega) e^{i\varphi_{u_x}(\vec{r},\omega)} \hat{x} + u_{o_y}(\vec{r},\omega) e^{i\varphi_{u_y}(\vec{r},\omega)} \hat{y} + u_{o_z}(\vec{r},\omega) e^{i\varphi_{u_z}(\vec{r},\omega)} \hat{z} \right] \cdot e^{i\omega t}$$
$$= \left[\tilde{u}_{o_x}(\vec{r},\omega) \hat{x} + \tilde{u}_{o_y}(\vec{r},\omega) \hat{y} + \tilde{u}_{o_z}(\vec{r},\omega) \hat{z} \right] \cdot e^{i\omega t} \equiv \vec{\tilde{u}}_o(\vec{r},\omega) \cdot e^{i\omega t}$$

The ***complex*** scalar quantity:

$$\begin{aligned} \tilde{p}_o(\vec{r},\omega) &\equiv p_o(\vec{r},\omega) e^{i\varphi_p(\vec{r},\omega)} \\ &= p_o(\vec{r},\omega)\left[\cos\varphi_p(\vec{r},\omega) + i\sin\varphi_p(\vec{r},\omega) \right] \\ &\equiv \mathrm{Re}\left\{ \tilde{p}_o(\vec{r},\omega) \right\} + i\,\mathrm{Im}\left\{ \tilde{p}_o(\vec{r},\omega) \right\} \end{aligned}$$

is known as the ***frequency domain*** complex over-pressure ***amplitude***. We can experimentally measure complex $\tilde{p}_o(\vec{r},\omega)$ *e.g.* using a dual-channel lock-in amplifier.

Likewise, *e.g.* in Cartesian coordinates, the ***complex*** 3-D vector quantity:

$$\vec{\tilde{u}}_o(\vec{r},\omega) = \left[u_{o_x}(\vec{r},\omega) e^{i\varphi_{u_x}(\vec{r},\omega)} \hat{x} + u_{o_y}(\vec{r},\omega) e^{i\varphi_{u_y}(\vec{r},\omega)} \hat{y} + u_{o_z}(\vec{r},\omega) e^{i\varphi_{u_z}(\vec{r},\omega)} \hat{z} \right]$$
$$\begin{aligned} &= u_{o_x}(\vec{r},\omega)\left[\cos\varphi_{u_x}(\vec{r},\omega) + i\sin\varphi_{u_x}(\vec{r},\omega) \right]\hat{x} &&= \left[\mathrm{Re}\left\{ u_{o_x}(\vec{r},\omega) \right\} + i\,\mathrm{Im}\left\{ u_{o_x}(\vec{r},\omega) \right\} \right]\hat{x} \\ &+ u_{o_y}(\vec{r},\omega)\left[\cos\varphi_{u_y}(\vec{r},\omega) + i\sin\varphi_{u_y}(\vec{r},\omega) \right]\hat{y} &&+ \left[\mathrm{Re}\left\{ u_{o_y}(\vec{r},\omega) \right\} + i\,\mathrm{Im}\left\{ u_{o_y}(\vec{r},\omega) \right\} \right]\hat{y} \\ &+ u_{o_z}(\vec{r},\omega)\left[\cos\varphi_{u_z}(\vec{r},\omega) + i\sin\varphi_{u_z}(\vec{r},\omega) \right]\hat{z} &&+ \left[\mathrm{Re}\left\{ u_{o_z}(\vec{r},\omega) \right\} + i\,\mathrm{Im}\left\{ u_{o_z}(\vec{r},\omega) \right\} \right]\hat{z} \end{aligned}$$

is the ***frequency domain*** complex 3-D particle velocity ***amplitude***. We can experimentally measure complex 3-D vector $\vec{\tilde{u}}_o(\vec{r},\omega)$ *e.g.* using ***three*** dual-channel lock-in amplifiers.

The relationship between ***time-domain*** *vs*. ***frequency-domain*** complex acoustic quantities such as complex over-pressure and complex particle velocity, associated with a single-frequency/ monochromatic/harmonic sound field are:

$$\tilde{p}(\vec{r},t) = \tilde{p}_o(\vec{r},\omega) \cdot e^{i\omega t} \quad \text{and:} \quad \vec{\tilde{u}}(\vec{r},t) = \vec{\tilde{u}}_o(\vec{r},\omega) \cdot e^{i\omega t}$$

We thus see that the ***frequency-domain*** complex acoustic quantities $\tilde{p}_o(\vec{r},\omega)$ and $\vec{\tilde{u}}_o(\vec{r},\omega)$ physically represent the complex ***amplitudes*** of these acoustic quantities:

$$\tilde{p}_o(\vec{r},\omega) = \tilde{p}_{o_r}(\vec{r},\omega) + i\tilde{p}_{o_i}(\vec{r},\omega) = \left| \tilde{p}_o(\vec{r},\omega) \right| e^{i\varphi_p(\vec{r},\omega)}$$

and:

$$\vec{\tilde{u}}_o(\vec{r},\omega) = \vec{\tilde{u}}_{o_r}(\vec{r},\omega) + i\vec{\tilde{u}}_{o_i}(\vec{r},\omega) = \left| \tilde{u}_{o_x}(\vec{r},\omega) \right| e^{i\varphi_{u_x}(\vec{r},\omega)} \hat{x} + \left| \tilde{u}_{o_y}(\vec{r},\omega) \right| e^{i\varphi_{u_y}(\vec{r},\omega)} \hat{y} + \left| \tilde{u}_{o_z}(\vec{r},\omega) \right| e^{i\varphi_{u_z}(\vec{r},\omega)} \hat{z}$$

<u>Instantaneous Physical *vs*. Complex 3-D Vector Acoustic Intensity</u>:

The ***instantaneous physical*** 3-D acoustic intensity (*aka* sound intensity) {*n.b*. a ***purely <u>real</u>*** 3-D ***<u>vector</u>*** quantity} is a ***<u>time</u>-<u>domain</u>*** quantity, defined as the ***<u>product</u>*** of the ***instantaneous physical*** scalar over-pressure $p(\vec{r},t)$ and the ***instantaneous*** 3-D vector particle velocity $\vec{u}(\vec{r},t)$:

$$\begin{aligned}\vec{I}_a(\vec{r},t) &\equiv p(\vec{r},t)\cdot\vec{u}(\vec{r},t) = p(\vec{r},t)\cdot\left[u_x(\vec{r},t)\hat{x}+u_y(\vec{r},t)\hat{y}+u_z(\vec{r},t)\hat{z}\right] \\ &= \left[p(\vec{r},t)\cdot u_x(\vec{r},t)\right]\hat{x}+\left[p(\vec{r},t)\cdot u_y(\vec{r},t)\right]\hat{y}+\left[p(\vec{r},t)\cdot u_z(\vec{r},t)\right]\hat{z} \\ &= I_{a_x}(\vec{r},t)\hat{x}+I_{a_y}(\vec{r},t)\hat{y}+I_{a_z}(\vec{r},t)\hat{z}\end{aligned}$$

Since $p(\vec{r},t)$ has SI units of *Pascals* (= *Newtons*/*m*²) and $\vec{u}(\vec{r},t)$ has SI units of *m*/*s*, we see that $\vec{I}_a(\vec{r},t)$ has SI units of (*Newton*-*m*)/*m*²-*s* = (*Joules*)/*m*²-*s* = (*Joules*/*s*)/*m*² = *Watts*/*m*².

Since the ***purely <u>real</u> instantaneous <u>time</u>-<u>domain</u>*** over-pressure and 3-D particle velocity are:

$$\begin{aligned}p(\vec{r},t) &= p_o(\vec{r},\omega)\cos\left(\omega t+\varphi_p(\vec{r},\omega)\right) = p_o(\vec{r},\omega)\left[\cos\omega t\cos\varphi_p(\vec{r},\omega)-\sin\omega t\sin\varphi_p(\vec{r},\omega)\right] \\ u_x(\vec{r},t) &= u_{o_x}(\vec{r},\omega)\cos\left(\omega t+\varphi_{u_x}(\vec{r},\omega)\right) = u_{o_x}(\vec{r},\omega)\left[\cos\omega t\cos\varphi_{u_x}(\vec{r},\omega)-\sin\omega t\sin\varphi_{u_x}(\vec{r},\omega)\right] \\ u_y(\vec{r},t) &= u_{o_y}(\vec{r},\omega)\cos\left(\omega t+\varphi_{u_y}(\vec{r},\omega)\right) = u_{o_y}(\vec{r},\omega)\left[\cos\omega t\cos\varphi_{u_y}(\vec{r},\omega)-\sin\omega t\sin\varphi_{u_y}(\vec{r},\omega)\right] \\ u_z(\vec{r},t) &= u_{o_z}(\vec{r},\omega)\cos\left(\omega t+\varphi_{u_z}(\vec{r},\omega)\right) = u_{o_z}(\vec{r},\omega)\left[\cos\omega t\cos\varphi_{u_z}(\vec{r},\omega)-\sin\omega t\sin\varphi_{u_z}(\vec{r},\omega)\right]\end{aligned}$$

Then {temporarily suppressing the arguments $(\vec{r},t)$ and $(\vec{r},\omega)$ for notational clarity}, the $k = x, y, z$ component of the ***purely <u>real</u> instantaneous <u>time</u>-<u>domain</u>*** 3-D acoustic intensity is:

$$\begin{aligned}I_{a_k} &= p_o\cos\left(\omega t+\varphi_p\right)\cdot u_{o_k}\cos\left(\omega t+\varphi_{u_\beta}\right) \\ &= p_o\cdot u_k\cos\left(\omega t+\varphi_p\right)\cos\left(\omega t+\varphi_{\vec{u}}\right) \\ &= p_o\cdot u_{o_k}\left[\cos\omega t\cos\varphi_p-\sin\omega t\sin\varphi_p\right]\left[\cos\omega t\cos\varphi_{u_k}-\sin\omega t\sin\varphi_{u_k}\right] \\ &= p_o\cdot u_{o_k}\left[\cos^2\omega t\cos\varphi_p\cos\varphi_{u_k}+\sin^2\omega t\sin\varphi_p\sin\varphi_{u_k}-\sin\omega t\cos\omega t\left(\sin\varphi_p\cos\varphi_{u_k}+\cos\varphi_p\sin\varphi_{u_k}\right)\right]\end{aligned}$$

Now: $\sin^2\omega t = 1-\cos^2\omega t$ and thus for the first two terms on the RHS:

$$\begin{aligned}&\cos^2\omega t\cos\varphi_p\cos\varphi_{u_k}+\sin^2\omega t\sin\varphi_p\sin\varphi_{u_k} = \cos^2\omega t\cos\varphi_p\cos\varphi_{u_k}+\left(1-\cos^2\omega t\right)\sin\varphi_p\sin\varphi_{u_k} \\ &= \cos^2\omega t\left(\cos\varphi_p\cos\varphi_{u_k}-\sin\varphi_p\sin\varphi_{u_k}\right)+\sin\varphi_p\sin\varphi_{u_k} = \cos^2\omega t\cos\left(\varphi_p+\varphi_{u_k}\right)+\sin\varphi_p\sin\varphi_{u_k}\end{aligned}$$

But: $\sin\varphi_p\cos\varphi_{u_k}+\cos\varphi_p\sin\varphi_{u_k} = \sin\left(\varphi_p+\varphi_{u_k}\right)$, and: $\sin\omega t\cos\omega t = \tfrac{1}{2}\sin 2\omega t$ and then, using: $\sin A\sin B = \tfrac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$ then: $\sin\varphi_p\sin\varphi_{u_k} = \tfrac{1}{2}\left[\cos\left(\varphi_p-\varphi_{u_k}\right)-\cos\left(\varphi_p+\varphi_{u_k}\right)\right]$
Additionally: $\cos^2\omega t = \tfrac{1}{2}+\tfrac{1}{2}\cos 2\omega t = \tfrac{1}{2}\left(1+\cos 2\omega t\right)$, thus we have:

$$
\begin{aligned}
I_{a_k} &= \tfrac{1}{2} p_o \cdot u_{o_k} \left[\left(\cancel{1} + \cos 2\omega t \right) \cos\left(\varphi_p + \varphi_{u_k}\right) + \left[\cos\left(\varphi_p - \varphi_{u_k}\right) - \cancel{\cos\left(\varphi_p + \varphi_{u_k}\right)} \right] - \sin 2\omega t \sin\left(\varphi_p + \varphi_{u_k}\right) \right] \\
&= \tfrac{1}{2} p_o \cdot u_{o_k} \left[\cos 2\omega t \cos\left(\varphi_p + \varphi_{u_k}\right) + \cos\left(\varphi_p - \varphi_{u_k}\right) - \sin 2\omega t \sin\left(\varphi_p + \varphi_{u_k}\right) \right] \\
&= \tfrac{1}{2} p_o \cdot u_{o_k} \left[\left\{ \cos 2\omega t \cdot \cos\left(\varphi_p + \varphi_{u_k}\right) - \sin 2\omega t \cdot \sin\left(\varphi_p + \varphi_{u_k}\right) \right\} + \cos\left(\varphi_p - \varphi_{u_k}\right) \right]
\end{aligned}
$$

Again using: $\cos(A+B) = \cos A \cos B - \sin A \sin B$, the above term in the curly brackets can be rewritten as: $\cos 2\omega t \cdot \cos\left(\varphi_p + \varphi_{u_k}\right) - \sin 2\omega t \cdot \sin\left(\varphi_p + \varphi_{u_k}\right) = \cos\left[2\omega t + \left(\varphi_p + \varphi_{u_k}\right)\right]$

Thus:

$$
I_{a_k} = \tfrac{1}{2} p_o \cdot u_{o_k} \left[\cos\left[2\omega t + \left(\varphi_p + \varphi_{u_k}\right)\right] + \cos\left(\varphi_p - \varphi_{u_k}\right) \right]
$$

Next, we define: $\Delta\varphi_{p-u_k} \equiv \left(\varphi_p - \varphi_{u_k}\right)$ and then: $\left(\varphi_p + \varphi_{u_k}\right) = \left[2\varphi_p + \left(\varphi_{u_k} - \varphi_p\right)\right] = \left(2\varphi_p - \Delta\varphi_{p-u_k}\right)$.

Thus:

$$
\begin{aligned}
I_{a_k} &= \tfrac{1}{2} p_o \cdot u_{o_k} \left[\cos\left[2\omega t + \left(2\varphi_p - \Delta\varphi_{p-u_k}\right)\right] + \cos \Delta\varphi_{p-u_k} \right] \\
&= \tfrac{1}{2} p_o \cdot u_{o_k} \left[\cos\left[\left(2\omega t + 2\varphi_p\right) - \Delta\varphi_{p-u_k}\right] + \cos \Delta\varphi_{p-u_k} \right] \\
&= \tfrac{1}{2} p_o \cdot u_{o_k} \left[\cos\left[2\left(\omega t + \varphi_p\right) - \Delta\varphi_{p-u_k}\right] + \cos \Delta\varphi_{p-u_k} \right]
\end{aligned}
$$

Then using: $\cos(A-B) = \cos A \cos B + \sin A \sin B$:

$$
\cos\left[2\left(\omega t + \varphi_p\right) - \Delta\varphi_{p-u_k}\right] = \cos 2\left(\omega t + \varphi_p\right) \cos \Delta\varphi_{p-u_k} + \sin 2\left(\omega t + \varphi_p\right) \sin \Delta\varphi_{p-u_k}
$$

Hence:

$$
I_{a_k} = \tfrac{1}{2} p_o \cdot u_{o_k} \left[\cos 2\left(\omega t + \varphi_p\right) \cos \Delta\varphi_{p-u_k} + \sin 2\left(\omega t + \varphi_p\right) \sin \Delta\varphi_{p-u_k} + \cos \Delta\varphi_{p-u_k} \right]
$$

Now, note that:

$$
\mathrm{Re}\left\{ e^{i\Delta\varphi_{p-u_k}} \left[1 + e^{-2i\left(\omega t + \varphi_p\right)} \right] \right\} = \cos 2\left(\omega t + \varphi_p\right) \cos \Delta\varphi_{p-u_k} + \sin 2\left(\omega t + \varphi_p\right) \sin \Delta\varphi_{p-u_k} + \cos \Delta\varphi_{p-u_k}
$$

<u>Proof</u>:

$$e^{i\Delta\varphi_{p-u_k}}\left[1+e^{-2i(\omega t+\varphi_p)}\right]=\mathrm{Re}\left\{e^{i\Delta\varphi_{p-u_k}}\left[1+e^{-2i(\omega t+\varphi_p)}\right]\right\}+i\,\mathrm{Im}\left\{e^{i\Delta\varphi_{p-u_k}}\left[1+e^{-2i(\omega t+\varphi_p)}\right]\right\}$$
$$=\left[\cos\Delta\varphi_{p-u_k}+i\sin\Delta\varphi_{p-u_\beta}\right]\left[1+\cos 2\left(\omega t+\varphi_p\right)-i\sin 2\left(\omega t+\varphi_p\right)\right]$$
$$=\left\{\cos\Delta\varphi_{p-u_k}+\cos 2\left(\omega t+\varphi_p\right)\cos\Delta\varphi_{p-u_k}+\sin 2\left(\omega t+\varphi_p\right)\sin\Delta\varphi_{p-u_k}\right\}$$
$$+i\left\{\sin\Delta\varphi_{p-u_k}+\cos 2\left(\omega t+\varphi_p\right)\sin\Delta\varphi_{p-u_k}-\sin 2\left(\omega t+\varphi_p\right)\cos\Delta\varphi_{p-u_k}\right\}$$

i.e.

$$\mathrm{Re}\left\{e^{i\Delta\varphi_{p-u_k}}\left[1+e^{-2i(\omega t+\varphi_p)}\right]\right\}=\left\{\cos\Delta\varphi_{p-u_k}+\cos 2\left(\omega t+\varphi_p\right)\cos\Delta\varphi_{p-u_k}+\sin 2\left(\omega t+\varphi_p\right)\sin\Delta\varphi_{p-u_k}\right\}$$

and:

$$\mathrm{Im}\left\{e^{i\Delta\varphi_{p-u_k}}\left[1+e^{-2i(\omega t+\varphi_p)}\right]\right\}=\left\{\sin\Delta\varphi_{p-u_k}+\cos 2\left(\omega t+\varphi_p\right)\sin\Delta\varphi_{p-u_k}-\sin 2\left(\omega t+\varphi_p\right)\cos\Delta\varphi_{p-u_k}\right\}$$

Thus, we can equivalently write the ***purely <u>real</u>***, ***instantaneous <u>time</u>-<u>domain</u>*** 3-D vector sound intensity as:

$$I_{a_k}=\tfrac{1}{2}\,p_o\cdot u_{o_k}\,\mathrm{Re}\left\{e^{i\Delta\varphi_{p-u_k}}\left[1+e^{-2i(\omega t+\varphi_p)}\right]\right\}$$

Explicitly reinstating the arguments $(\vec{r},t)$ and $(\vec{r},\omega)$, this is:

$$I_{a_k}(\vec{r},t)=\tfrac{1}{2}\,p_o(\vec{r},\omega)\cdot u_{o_k}(\vec{r},\omega)\,\mathrm{Re}\left\{e^{i\Delta\varphi_{p-u_k}(\vec{r},\omega)}\left[1+e^{-2i(\omega t+\varphi_p(\vec{r},\omega))}\right]\right\}$$

Since: $\Delta\varphi_{p-u_k}(\vec{r},\omega)\equiv\left(\varphi_p(\vec{r},\omega)-\varphi_{u_k}(\vec{r},\omega)\right)$, then: $e^{i\Delta\varphi_{p-u_k}(\vec{r},\omega)}=e^{i\left(\varphi_p(\vec{r},\omega)-\varphi_{u_k}(\vec{r},\omega)\right)}=e^{i\varphi_p(\vec{r},\omega)}\cdot e^{-i\varphi_{u_k}(\vec{r},\omega)}$, and thus we see that since:

$$\tilde{p}(\vec{r},t)\equiv p_o(\vec{r},\omega)e^{i\varphi_p(\vec{r},\omega)}\cdot e^{i\omega t}=\tilde{p}(\vec{r},\omega)e^{i\omega t}\qquad\{\text{where: }\tilde{p}(\vec{r},\omega)\equiv p_o(\vec{r},\omega)e^{i\varphi_p(\vec{r},\omega)}\}$$

and:

$$\tilde{u}_k(\vec{r},t)\equiv u_{o_k}(\vec{r},\omega)e^{i\varphi_{u_k}(\vec{r},\omega)}\cdot e^{i\omega t}=\tilde{u}_k(\vec{r},\omega)e^{i\omega t}\quad\{\text{where: }\tilde{u}_k(\vec{r},\omega)\equiv u_{o_k}(\vec{r},\omega)e^{i\varphi_{u_k}(\vec{r},\omega)}\}$$

hence:

$$\tilde{u}_k^*(\vec{r},t)\equiv u_{o_k}(\vec{r},\omega)e^{-i\varphi_{u_k}(\vec{r},\omega)}\cdot e^{-i\omega t}=\tilde{u}_k^*(\vec{r},\omega)e^{-i\omega t}\,\{\text{where: }\tilde{u}_k^*(\vec{r},\omega)\equiv u_{o_k}(\vec{r},\omega)e^{-i\varphi_{u_k}(\vec{r},\omega)}\}$$

the ***purely <u>real</u>***, ***instantaneous <u>time</u>-<u>domain</u>*** 3-D vector sound intensity can ***<u>also</u>*** equivalently be written as:

$$I_{a_k}(\vec{r},t)=\mathrm{Re}\left\{\tfrac{1}{2}\,\tilde{p}(\vec{r},\omega)\cdot\tilde{u}_k^*(\vec{r},\omega)\left[1+e^{-2i(\omega t+\varphi_p(\vec{r},\omega))}\right]\right\}$$

Thus, we can now define the ***frequency-domain complex "amplitude"*** of the 3-D vector sound intensity as:

$$\begin{aligned}\tilde{\vec{I}}_a(\vec{r},\omega) &\equiv \tfrac{1}{2}\tilde{p}(\vec{r},\omega)\cdot\tilde{\vec{u}}^*(\vec{r},\omega)\\ &= \tfrac{1}{2}\tilde{p}(\vec{r},\omega)\cdot\tilde{u}_x^*(\vec{r},\omega)\hat{x}+\tfrac{1}{2}\tilde{p}(\vec{r},\omega)\cdot\tilde{u}_y^*(\vec{r},\omega)\hat{y}+\tfrac{1}{2}\tilde{p}(\vec{r},\omega)\cdot\tilde{u}_z^*(\vec{r},\omega)\hat{z}\\ &= \tilde{I}_{a_x}(\vec{r},\omega)\hat{x}+\tilde{I}_{a_y}(\vec{r},\omega)\hat{y}+\tilde{I}_{a_z}(\vec{r},\omega)\hat{z}\end{aligned}$$

Finally, we can thus explicitly show that the ***purely real, instantaneous time-domain*** 3-D vector sound intensity $\vec{I}_a(\vec{r},t)$ is related to the ***frequency-domain complex*** 3-D vector sound intensity ***"amplitude"*** $\tilde{\vec{I}}_a(\vec{r},\omega)$ by:

$$\vec{I}_a(\vec{r},t) = \mathrm{Re}\left\{\tilde{\vec{I}}_a(\vec{r},\omega)\left[1+e^{-2i(\omega t+\varphi_p(\vec{r},\omega))}\right]\right\}$$

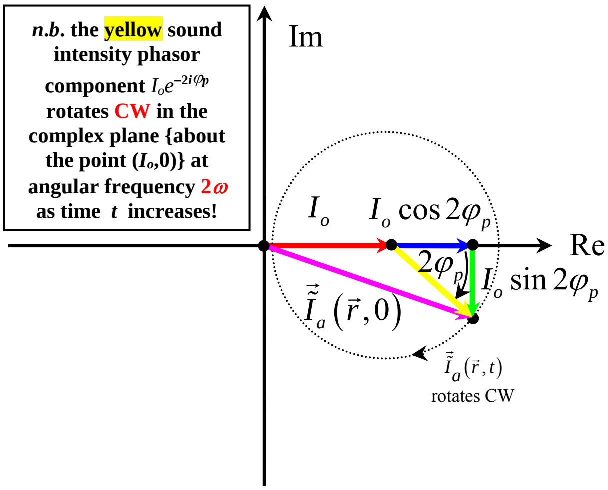
Hence, the ***complex time-domain*** instantaneous 3-D vector sound intensity is thus:

$$\tilde{\vec{I}}_a(\vec{r},t) = \tilde{\vec{I}}_a(\vec{r},\omega)\left[1+e^{-2i(\omega t+\varphi_p(\vec{r},\omega))}\right]$$

A phasor plot of the behavior of the ***complex time-domain*** 3-D vector sound intensity $\tilde{\vec{I}}_a(\vec{r},t)$ is shown below for a snapshot in time at $t = 0$, for the simple 1-D case of a ***purely real frequency-domain*** $\tilde{\vec{I}}_a(\vec{r},\omega) = \tilde{I}_a(\vec{r},\omega)\hat{z} = I_o\hat{z} = \tfrac{1}{2}p_o u_{o_z}\hat{z}$:

$$\tilde{I}_a(\vec{r},t) = I_o\left[1+e^{-2i(\omega t+\varphi_p(\vec{r},\omega))}\right] = I_o\left[1+\cos 2(\omega t+\varphi_p(\vec{r},\omega)) - i\sin 2(\omega t+\varphi_p(\vec{r},\omega))\right]$$

At $t = 0$: $\tilde{I}_a(\vec{r},0) = I_o\left[1+e^{-2i\varphi_p}\right] = I_o + I_o e^{-2i\varphi_p} = I_o + I_o\cos 2\varphi_p - iI_o\sin 2\varphi_p$



For completeness' sake, note that each of the $k = x, y, z$ components of the ***purely real***, ***instantaneous <u>time</u>-<u>domain</u>*** 3-D vector acoustic intensity $\vec{I}_a(\vec{r},t)$, using the fact that $\mathrm{Re}\{e^{\pm i\theta}\} = \mathrm{Re}\{\cos\theta \pm i\sin\theta\} = \cos\theta$ can be written as:

$$\begin{aligned} I_{a_k}(\vec{r},t) &\equiv p(\vec{r},t)\cdot u_k(\vec{r},t) = p_o(\vec{r},\omega)\cos(\omega t+\varphi_p)\cdot u_{o_k}(\vec{r},\omega)\cos(\omega t+\varphi_{u_k}) \\ &= \mathrm{Re}\{\tilde{p}(\vec{r},t)\}\cdot\mathrm{Re}\{\tilde{u}_k(\vec{r},t)\} = \mathrm{Re}\{\tilde{p}(\vec{r},t)\}\cdot\mathrm{Re}\{\tilde{u}_k^*(\vec{r},t)\} \\ &= \mathrm{Re}\{\tilde{p}(\vec{r},\omega)e^{i\omega t}\}\cdot\mathrm{Re}\{\tilde{u}_k^*(\vec{r},\omega)e^{-i\omega t}\} \\ &= \mathrm{Re}\{p_o(\vec{r},\omega)e^{i\varphi_p}\cdot e^{i\omega t}\}\cdot\mathrm{Re}\{u_{o_k}(\vec{r},\omega)e^{-i\varphi_{u_k}}\cdot e^{-i\omega t}\} \end{aligned}$$

Note also that each of the $k = x, y, z$ components of the ***<u>frequency</u>-<u>domain</u> complex*** 3-D vector acoustic intensity ***"amplitude"*** $\tilde{\vec{I}}_a(\vec{r},\omega)$ can be written as:

$$\begin{aligned} \tilde{I}_{a_k}(\vec{r},\omega) &\equiv \tfrac{1}{2}\tilde{p}(\vec{r},\omega)\cdot\tilde{u}_k^*(\vec{r},\omega) \\ &= \mathrm{Re}\{\tilde{I}_{a_k}(\vec{r},\omega)\} + i\,\mathrm{Im}\{\tilde{I}_{a_k}(\vec{r},\omega)\} \\ &= I_{a_k}^{\mathrm{r}}(\vec{r},\omega) + iI_{a_k}^{\mathrm{i}}(\vec{r},\omega) \end{aligned}$$

Since complex $\tilde{p} = p_{\mathrm{r}} + ip_{\mathrm{i}} = |\tilde{p}|e^{i\varphi_p}$ and complex {conjugated} $\tilde{u}_k^* = u_{\mathrm{r}_k} - iu_{\mathrm{i}_k} = |\tilde{u}_k^*|e^{-i\varphi_{u_k}} = |\tilde{u}_k|e^{-i\varphi_{u_k}}$, then for each of the $k = x, y, z$ components of $\tilde{\vec{I}}_a(\vec{r},\omega)$:

$$\begin{aligned} \tilde{I}_{a_k} &\equiv \tfrac{1}{2}\tilde{p}\cdot\tilde{u}_k^* = \tfrac{1}{2}(p_{\mathrm{r}}+ip_{\mathrm{i}})\cdot(u_{\mathrm{r}_k}-iu_{\mathrm{i}_k}) = \tfrac{1}{2}\left[(p_{\mathrm{r}}u_{\mathrm{r}_k}+p_{\mathrm{i}}u_{\mathrm{i}_k}) + i(p_{\mathrm{i}}u_{\mathrm{r}_k}-p_{\mathrm{r}}u_{\mathrm{i}_k})\right] \\ &= \tfrac{1}{2}|\tilde{p}|e^{i\varphi_p}\cdot|\tilde{u}_k^*|e^{-i\varphi_{u_k}} = \tfrac{1}{2}|\tilde{p}||\tilde{u}_k^*|e^{i\varphi_p}\cdot e^{-i\varphi_{u_k}} = \tfrac{1}{2}|\tilde{p}||\tilde{u}_k^*|e^{i(\varphi_p-\varphi_{u_k})} = \tfrac{1}{2}|\tilde{p}\cdot\tilde{u}_k^*|e^{i\Delta\varphi_{p-u_k}} \\ &= |\tilde{I}_{a_k}|e^{i\varphi_{I_k}} \end{aligned}$$

Thus, we have learned several things:

a.) The phase of complex $\tilde{I}_{a_k}(\vec{r},\omega)$ is equal to the phase ***difference*** between complex $\tilde{p}$ and $\tilde{u}_k$, *i.e.* $\varphi_{I_k} = \Delta\varphi_{p-u_k} \equiv (\varphi_p - \varphi_{u_k})$, and:

$$\begin{aligned} I_{a_k}^{\mathrm{r}} &\equiv \mathrm{Re}\{\tilde{I}_{a_k}\} = \mathrm{Re}\{|\tilde{I}_{a_k}|e^{i\varphi_{I_k}}\} = \tfrac{1}{2}\mathrm{Re}\{|\tilde{p}\cdot\tilde{u}_k^*|e^{i\Delta\varphi_{p-u_k}}\} = \tfrac{1}{2}\mathrm{Re}\{|\tilde{p}\cdot\tilde{u}_k^*|e^{i(\varphi_p-\varphi_{u_k})}\} \\ &= |\tilde{I}_{a_k}|\cos\varphi_{I_{a_k}} = \tfrac{1}{2}|\tilde{p}\cdot\tilde{u}_k^*|\cos\Delta\varphi_{p-u_k} = \tfrac{1}{2}(p_{\mathrm{r}}u_{\mathrm{r}_k} + p_{\mathrm{i}}u_{\mathrm{i}_k}) \end{aligned}$$

and:

$$\begin{aligned} I_{a_k}^{\mathrm{i}} &\equiv \mathrm{Im}\{\tilde{I}_{a_k}\} = \mathrm{Im}\{|\tilde{I}_{a_k}|e^{\varphi_{I_k}}\} = \tfrac{1}{2}\mathrm{Im}\{|\tilde{p}\cdot\tilde{u}_k^*|e^{i\Delta\varphi_{p-u_k}}\} = \tfrac{1}{2}\mathrm{Im}\{|\tilde{p}\cdot\tilde{u}_k^*|e^{i(\varphi_p-\varphi_{u_k})}\} \\ &= |\tilde{I}_{a_k}|\sin\varphi_{I_{a_k}} = \tfrac{1}{2}|\tilde{p}\cdot\tilde{u}_k^*|\sin\Delta\varphi_{p-u_k} = \tfrac{1}{2}(p_{\mathrm{i}}u_{\mathrm{r}_k} - p_{\mathrm{r}}u_{\mathrm{i}_k}) \end{aligned}$$

b.) The ***time average*** of the ***time-domain***, ***purely real***, ***instantaneous*** 3-D vector sound intensity is:

$$\left\langle \vec{I}_a(\vec{r},t)\right\rangle_t = \left\langle \mathrm{Re}\left\{\tilde{\vec{I}}_a(\vec{r},\omega)\left[1+e^{-2i(\omega t+\varphi_p(\vec{r},\omega))}\right]\right\}\right\rangle_t \equiv \frac{1}{\tau}\int_{\substack{one\\cycle}} \mathrm{Re}\left\{\tilde{\vec{I}}_a(\vec{r},\omega)\left[1+e^{-2i(\omega t+\varphi_p(\vec{r},\omega))}\right]\right\}dt$$

$$= \frac{1}{\tau}\mathrm{Re}\left\{\tilde{\vec{I}}_a(\vec{r},\omega)\int_{\substack{one\\cycle}}\left[1+e^{-2i(\omega t+\varphi_p(\vec{r},\omega))}\right]dt\right\}$$

$$= \frac{1}{\tau}\mathrm{Re}\left\{\tilde{\vec{I}}_a(\vec{r},\omega)\left[\int_0^\tau 1\cdot dt+\int_0^\tau \cos 2\left(\omega t+\varphi_p(\vec{r},\omega)\right)dt - i\int_0^\tau \sin 2\left(\omega t+\varphi_p(\vec{r},\omega)\right)dt\right]\right\}$$

$$= \frac{1}{\tau}\mathrm{Re}\left\{\tilde{\vec{I}}_a(\vec{r},\omega)\left[\tau + \underbrace{\int_0^\tau \cos 2\left(\omega t+\varphi_p(\vec{r},\omega)\right)dt}_{=0 \text{ over one cycle}} - i\underbrace{\int_0^\tau \sin 2\left(\omega t+\varphi_p(\vec{r},\omega)\right)dt}_{=0 \text{ over one cycle}}\right]\right\}$$

$$= \mathrm{Re}\left\{\tilde{\vec{I}}_a(\vec{r},\omega)\right\} = \mathrm{Re}\left\{\tfrac{1}{2}\tilde{p}(\vec{r},\omega)\cdot\tilde{\vec{u}}^*(\vec{r},\omega)\right\}$$

<u>In words</u>: The ***time average*** of the ***purely real***, ***instantaneous time-domain*** 3-D vector sound intensity $\left\langle \vec{I}_a(\vec{r},t)\right\rangle_t$ is equal to the ***real part*** of the ***frequency-domain complex*** 3-D vector sound intensity ***"amplitude"*** $\mathrm{Re}\left\{\tilde{\vec{I}}_a(\vec{r},\omega)\right\} = \mathrm{Re}\left\{\tfrac{1}{2}\tilde{p}(\vec{r},\omega)\cdot\tilde{\vec{u}}^*(\vec{r},\omega)\right\}$.

<u>For a monochromatic/single-frequency harmonic sound field</u>:

a.) The ***real part*** of the ***frequency-domain*** complex 3-D vector sound intensity ***"amplitude"*** $\vec{I}_a^{\,r}(\vec{r},\omega) = \mathrm{Re}\left\{\tilde{\vec{I}}_a(\vec{r},\omega)\right\}$ is associated with ***propagating*** ("***active***") sound – *i.e.* sound ***radiation***.

b.) The ***imaginary part*** of the ***frequency-domain*** complex 3-D vector sound intensity ***"amplitude"*** $\vec{I}_a^{\,i}(\vec{r},\omega) = \mathrm{Im}\left\{\tilde{\vec{I}}_a(\vec{r},\omega)\right\}$ is associated with ***non-propagating sound energy*** – *i.e.* ("***reactive***") sound energy - that only "sloshes" back and forth {locally} during each cycle of oscillation.

Note that for a {lossless} ***standing*** acoustic wave consisting of ***two equal-amplitude counter-propagating traveling*** waves, the ***time average*** of the ***purely real***, ***instantaneous time-domain*** 3-D vector sound intensity $\left\langle \vec{I}_a(\vec{r},t)\right\rangle_t = 0$ because the ***real part*** of the ***frequency-domain complex*** 3-D vector sound intensity ***"amplitude"*** $\mathrm{Re}\left\{\tilde{\vec{I}}_a(\vec{r},\omega)\right\} = \mathrm{Re}\left\{\tfrac{1}{2}\tilde{p}(\vec{r},\omega)\cdot\tilde{\vec{u}}^*(\vec{r},\omega)\right\} \equiv 0$.

There is ***no net*** sound ***propagation*** in this situation. {*n.b.* These are analogous to the ***real*** and ***imaginary*** components of complex scalar electrical power $\tilde{P}_e(\omega)$ associated *e.g.* with an *LCR* circuit – the ***real*** component is the electrical power ***dissipated*** in the resistance *R*; the imaginary component is power transitorily ***stored***/***circulating*** in the inductance *L* and/or the capacitance *C* of the *LCR* circuit!}

Note also that in this situation, the ***imaginary part*** of the ***frequency-domain complex*** 3-D vector acoustic intensity ***"amplitude"*** $\operatorname{Im}\left\{\tilde{\vec{I}}_a(\vec{r},\omega)\right\}=\operatorname{Im}\left\{\tfrac{1}{2}\tilde{p}(\vec{r},\omega)\cdot\tilde{\vec{u}}^*(\vec{r},\omega)\right\}\neq 0$, because the acoustic energy associated with a ***standing*** acoustic wave {consisting of ***two equal-amplitude counter-propagating traveling*** waves} simply "sloshes" back and forth {locally} during each cycle of oscillation. A {lossless} standing acoustic wave has ***no net*** sound ***propagation***.

Thus, we see that the ***complex frequency-domain*** 3-D vector sound intensity ***"amplitude"*** $\tilde{\vec{I}}_a(\vec{r},\omega)=\tfrac{1}{2}\tilde{p}(\vec{r},\omega)\cdot\tilde{\vec{u}}^*(\vec{r},\omega)$ actually gives us ***more*** physical insight into what is going on than just the ***time average*** of the ***purely real***, ***instantaneous time-domain*** 3-D vector sound intensity $\left\langle\vec{I}_a(\vec{r},t)\right\rangle_t$, since $\left\langle\vec{I}_a(\vec{r},t)\right\rangle_t$ ***is*** equal to the ***real part*** of the ***frequency-domain complex*** 3-D vector sound intensity ***"amplitude"***: $\left\langle\vec{I}_a(\vec{r},t)\right\rangle_t=\operatorname{Re}\left\{\tilde{\vec{I}}_a(\vec{r},\omega)\right\}=\operatorname{Re}\left\{\tfrac{1}{2}\tilde{p}(\vec{r},\omega)\cdot\tilde{\vec{u}}^*(\vec{r},\omega)\right\}$.

Complex 3-D Vector *Specific* Acoustic Immittances:

As we have previously discussed in considerable detail the manifestly ***frequency-domain*** complex 3-D vector ***specific*** acoustic immittances in Physics 406 Lect. Notes XI, Part 2 (*p*. 16-27), for completeness' sake, we summarize them again, here:

Complex 3-D Vector *Specific* Acoustic Admittance:

$$\begin{aligned}
\tilde{\vec{y}}_a(\vec{r},\omega)&=\tilde{y}_{a_x}(\vec{r},\omega)\hat{x}+\tilde{y}_{a_y}(\vec{r},\omega)\hat{y}+\tilde{y}_{a_z}(\vec{r},\omega)\hat{z}\\
&=\frac{\tilde{u}_x(\vec{r},\omega)}{\tilde{p}(\vec{r},\omega)}\hat{x}+\frac{\tilde{u}_y(\vec{r},\omega)}{\tilde{p}(\vec{r},\omega)}\hat{y}+\frac{\tilde{u}_z(\vec{r},\omega)}{\tilde{p}(\vec{r},\omega)}\hat{z}=\frac{\tilde{\vec{u}}(\vec{r},\omega)}{\tilde{p}(\vec{r},\omega)}=\frac{1}{\tilde{\vec{z}}_a(\vec{r},\omega)}\\
&=\vec{y}_a^{\,\mathrm{r}}(\vec{r},\omega)+i\vec{y}_a^{\,\mathrm{i}}(\vec{r},\omega)=\operatorname{Re}\left\{\tilde{\vec{y}}_a(\vec{r},\omega)\right\}+\operatorname{Im}\left\{\tilde{\vec{y}}_a(\vec{r},\omega)\right\}\\
&=\left[y_{a_x}^{\mathrm{r}}(\vec{r},\omega)+iy_{a_x}^{\mathrm{i}}(\vec{r},\omega)\right]\hat{x}+\left[y_{a_y}^{\mathrm{r}}(\vec{r},\omega)+iy_{a_y}^{\mathrm{i}}(\vec{r},\omega)\right]\hat{y}+\left[y_{a_z}^{\mathrm{r}}(\vec{r},\omega)+iy_{a_z}^{\mathrm{i}}(\vec{r},\omega)\right]\hat{z}\\
&=\left|\tilde{y}_{a_x}(\vec{r},\omega)\right|e^{i\varphi_{y_x}(\vec{r},\omega)}\hat{x}+\left|\tilde{y}_{a_y}(\vec{r},\omega)\right|e^{i\varphi_{y_{a_y}}(\vec{r},\omega)}\hat{y}+\left|\tilde{y}_{a_z}(\vec{r},\omega)\right|e^{i\varphi_{y_{a_z}}(\vec{r},\omega)}\hat{z}
\end{aligned}$$

Complex 3-D Vector *Specific* Acoustic Impedance:

$$\begin{aligned}
\tilde{\vec{z}}_a(\vec{r},\omega)&=\tilde{z}_{a_x}(\vec{r},\omega)\hat{x}+\tilde{z}_{a_y}(\vec{r},\omega)\hat{y}+\tilde{z}_{a_z}(\vec{r},\omega)\hat{z}=\frac{\tilde{p}(\vec{r},\omega)}{\tilde{\vec{u}}(\vec{r},\omega)}=\frac{1}{\tilde{\vec{y}}_a(\vec{r},\omega)}\\
&=\frac{\tilde{p}(\vec{r},\omega)\tilde{u}_x^*(\vec{r},\omega)}{\left|\tilde{\vec{u}}(\vec{r},\omega)\right|^2}\hat{x}+\frac{\tilde{p}(\vec{r},\omega)\tilde{u}_y^*(\vec{r},\omega)}{\left|\tilde{\vec{u}}(\vec{r},\omega)\right|^2}\hat{y}+\frac{\tilde{p}(\vec{r},\omega)\tilde{u}_z^*(\vec{r},\omega)}{\left|\tilde{\vec{u}}(\vec{r},\omega)\right|^2}\hat{z}=\frac{\tilde{p}(\vec{r},\omega)\tilde{\vec{u}}^*(\vec{r},\omega)}{\left|\tilde{\vec{u}}(\vec{r},\omega)\right|^2}\\
&=\vec{z}_a^{\,\mathrm{r}}(\vec{r},\omega)+i\vec{z}_a^{\,\mathrm{i}}(\vec{r},\omega)=\operatorname{Re}\left\{\tilde{\vec{z}}_a(\vec{r},\omega)\right\}+\operatorname{Im}\left\{\tilde{\vec{z}}_a(\vec{r},\omega)\right\}\\
&=\left[z_{a_x}^{\mathrm{r}}(\vec{r},\omega)+iz_{a_x}^{\mathrm{i}}(\vec{r},\omega)\right]\hat{x}+\left[z_{a_y}^{\mathrm{r}}(\vec{r},\omega)+iz_{a_y}^{\mathrm{i}}(\vec{r},\omega)\right]\hat{y}+\left[z_{a_z}^{\mathrm{r}}(\vec{r},\omega)+iz_{a_z}^{\mathrm{i}}(\vec{r},\omega)\right]\hat{z}\\
&=\left|\tilde{z}_{a_x}(\vec{r},\omega)\right|e^{i\varphi_{z_{a_x}}(\vec{r},\omega)}\hat{x}+\left|\tilde{z}_{a_y}(\vec{r},\omega)\right|e^{i\varphi_{z_{a_y}}(\vec{r},\omega)}\hat{y}+\left|\tilde{z}_{a_z}(\vec{r},\omega)\right|e^{i\varphi_{z_{a_z}}(\vec{r},\omega)}\hat{z}
\end{aligned}$$

where, for the $k = x, y,$ or z components {temporarily suppressing the $(\vec{r},\omega)$ arguments}:

$$\tilde{y}_{a_k} = y^{\mathrm{r}}_{a_k} + i y^{\mathrm{i}}_{a_k} = \frac{\tilde{u}_k}{\tilde{p}} = \frac{u_{\mathrm{r}_k} + i u_{\mathrm{i}_k}}{p_{\mathrm{r}} + i p_{\mathrm{i}}} = \left(\frac{u_{\mathrm{r}_k} + i_{\mathrm{i}_k}}{p_{\mathrm{r}} + i p_{\mathrm{i}}}\right)\cdot\left(\frac{p_{\mathrm{r}} - i p_{\mathrm{i}}}{p_{\mathrm{r}} - i p_{\mathrm{i}}}\right) = \left(\frac{p_{\mathrm{r}} u_{\mathrm{r}_k} + p_{\mathrm{i}} u_{\mathrm{i}_k}}{|\tilde{p}|^2}\right) + i\left(\frac{p_{\mathrm{i}} u_{\mathrm{r}_k} - p_{\mathrm{r}} u_{\mathrm{i}_k}}{|\tilde{p}|^2}\right)$$

and:

$$\tilde{z}_{a_k} = z^{\mathrm{r}}_{a_k} + i z^{\mathrm{i}}_{a_k} = \frac{\tilde{p}\cdot\tilde{u}^*_k}{|\tilde{\vec{u}}|^2} = \frac{(p_{\mathrm{r}} + i p_{\mathrm{i}})(u_{\mathrm{r}_k} + i u_{\mathrm{i}_k})^*}{|\tilde{\vec{u}}|^2} = \frac{(p_{\mathrm{r}} + i p_{\mathrm{i}})(u_{\mathrm{r}_k} - i u_{\mathrm{i}_k})}{|\tilde{\vec{u}}|^2} = \left(\frac{p_{\mathrm{r}} u_{\mathrm{r}_k} + p_{\mathrm{i}} u_{\mathrm{i}_k}}{|\tilde{\vec{u}}|^2}\right) + i\left(\frac{p_{\mathrm{i}} u_{\mathrm{r}_k} - p_{\mathrm{r}} u_{\mathrm{i}_k}}{|\tilde{\vec{u}}|^2}\right)$$

i.e. for the $k = x, y,$ or z components of the immittances $\tilde{\vec{y}}_a(\vec{r},\omega)$ and $\tilde{\vec{z}}_a(\vec{r},\omega)$:

$$y^{\mathrm{r}}_{a_k} = \mathrm{Re}\{\tilde{y}_{a_k}\} = \frac{p_{\mathrm{r}} u_{\mathrm{r}_k} + p_{\mathrm{i}} u_{\mathrm{i}_k}}{|\tilde{p}|^2} \text{ and: } y^{\mathrm{i}}_{a_k} = \mathrm{Im}\{\tilde{y}_{a_k}\} = \frac{p_{\mathrm{r}} u_{\mathrm{r}_k} - p_{\mathrm{i}} u_{\mathrm{i}_k}}{|\tilde{p}|^2} = -\frac{p_{\mathrm{i}} u_{\mathrm{r}_k} - p_{\mathrm{r}} u_{\mathrm{i}_k}}{|\tilde{p}|^2}$$

and:

$$z^{\mathrm{r}}_{a_k} = \mathrm{Re}\{\tilde{z}_{a_k}\} = \frac{p_{\mathrm{r}} u_{\mathrm{r}_k} + p_{\mathrm{i}} u_{\mathrm{i}_k}}{|\tilde{\vec{u}}|^2} \text{ and: } z^{\mathrm{i}}_{a_k} = \mathrm{Im}\{\tilde{z}_{a_k}\} = \frac{p_{\mathrm{i}} u_{\mathrm{r}_k} - p_{\mathrm{r}} u_{\mathrm{i}_k}}{|\tilde{\vec{u}}|^2}$$

However, in the ***frequency*** ***domain***, for the $k = x, y, z$ components of $\tilde{\vec{I}}_a(\vec{r},\omega)$ we have:

$$\begin{aligned}
\tilde{I}_{a_k} &\equiv \tfrac{1}{2}\tilde{p}\cdot\tilde{u}^*_k = \mathrm{Re}\{\tilde{I}_{a_k}\} + i\,\mathrm{Im}\{\tilde{I}_{a_k}\} = I^{\mathrm{r}}_{a_k} + i I^{\mathrm{i}}_{a_k} \\
&= \tfrac{1}{2}(p_{\mathrm{r}} + i p_{\mathrm{i}})\cdot(u_{\mathrm{r}_k} - i u_{\mathrm{i}_k}) = \tfrac{1}{2}\left[(p_{\mathrm{r}} u_{\mathrm{r}_k} + p_{\mathrm{i}} u_{\mathrm{i}_k}) + i(p_{\mathrm{i}} u_{\mathrm{r}_k} - p_{\mathrm{r}} u_{\mathrm{i}_k})\right] \\
&= \tfrac{1}{2}|\tilde{p}|e^{i\varphi_p}\cdot|\tilde{u}^*_k|e^{-i\varphi_{u_k}} = \tfrac{1}{2}|\tilde{p}||\tilde{u}^*_k|e^{i\varphi_p}\cdot e^{-i\varphi_{u_k}} = \tfrac{1}{2}|\tilde{p}||\tilde{u}^*_k|e^{i(\varphi_p - \varphi_{u_k})} = \tfrac{1}{2}|\tilde{p}\cdot\tilde{u}^*_k|e^{i\Delta\varphi_{p-u_k}} \\
&= |\tilde{I}_{a_k}|e^{i\varphi_{I_{a_k}}}
\end{aligned}$$

i.e. for the $k = x, y,$ or z components of the ***frequency*** ***domain*** complex 3-D vector acoustic intensity, $\tilde{\vec{I}}_a(\vec{r},\omega)$, we have:

$$I^{\mathrm{r}}_{a_k} = \mathrm{Re}\{\tilde{I}_{a_k}\} = \tfrac{1}{2}(p_{\mathrm{r}} u_{\mathrm{r}_k} + p_{\mathrm{i}} u_{\mathrm{i}_k}) \text{ and: } I^{\mathrm{i}}_{a_k} = \mathrm{Im}\{\tilde{I}_{a_k}\} = \tfrac{1}{2}(p_{\mathrm{i}} u_{\mathrm{r}_k} - p_{\mathrm{r}} u_{\mathrm{i}_k})$$

Hence, we see that ***frequency*** ***domain*** relations exist between the complex 3-D vector ***specific*** acoustic immittances $\tilde{\vec{z}}_a(\vec{r},\omega), \tilde{\vec{y}}_a(\vec{r},\omega)$ and the complex 3-D vector acoustic intensity $\tilde{\vec{I}}_a(\vec{r},\omega)$:

$$I^{\mathrm{r}}_{a_k} = \mathrm{Re}\{\tilde{I}_{a_k}\} = \tfrac{1}{2}(p_{\mathrm{r}} u_{\mathrm{r}_k} + p_{\mathrm{i}} u_{\mathrm{i}_k}) = \tfrac{1}{2}|\tilde{\vec{u}}|^2 z^{\mathrm{r}}_{a_k} \quad \text{and/or:} \quad I^{\mathrm{r}}_{a_k} = \mathrm{Re}\{\tilde{I}_{a_k}\} = \tfrac{1}{2}(p_{\mathrm{r}} u_{\mathrm{r}_k} + p_{\mathrm{i}} u_{\mathrm{i}_k}) = \tfrac{1}{2}|\tilde{p}|^2 y^{\mathrm{r}}_{a_k}$$

$$I^{\mathrm{i}}_{a_k} = \mathrm{Im}\{\tilde{I}_{a_k}\} = \tfrac{1}{2}(p_{\mathrm{i}} u_{\mathrm{r}_k} - p_{\mathrm{r}} u_{\mathrm{i}_k}) = \tfrac{1}{2}|\tilde{\vec{u}}|^2 z^{\mathrm{i}}_{a_k} \quad \text{and/or:} \quad I^{\mathrm{i}}_{a_k} = \mathrm{Im}\{\tilde{I}_{a_k}\} = \tfrac{1}{2}(p_{\mathrm{i}} u_{\mathrm{r}_k} - p_{\mathrm{r}} u_{\mathrm{i}_k}) = -\tfrac{1}{2}|\tilde{p}|^2 y^{\mathrm{i}}_{a_k}$$

$$\tilde{I}_{a_k} \equiv \tfrac{1}{2}\tilde{p}\cdot\tilde{u}^*_k = \tfrac{1}{2}|\tilde{\vec{u}}|^2 \tilde{z}_{a_k} \quad \text{and/or:} \quad \tilde{I}_{a_k} \equiv \tfrac{1}{2}\tilde{p}\cdot\tilde{u}^*_k = \tfrac{1}{2}|\tilde{p}|^2 \tilde{y}^*_{a_k}$$

$$\boxed{\tilde{\vec{I}}_a \equiv \tfrac{1}{2}\tilde{p}\cdot\tilde{\vec{u}}^* = \tfrac{1}{2}|\tilde{\vec{u}}|^2 \tilde{\vec{z}}_a} \quad \text{and/or:} \quad \boxed{\tilde{\vec{I}}_a \equiv \tfrac{1}{2}\tilde{p}\cdot\tilde{\vec{u}}^* = \tfrac{1}{2}|\tilde{p}|^2 \tilde{\vec{y}}^*_a}$$

Note also that the real and imaginary ***frequency domain*** components of complex 3-D vector acoustic intensity $\tilde{\vec{I}}_a(\vec{r},\omega)$, the real and imaginary ***frequency domain*** components of complex 3-D vector ***specific*** acoustic immittances $\tilde{\vec{z}}_a(\vec{r},\omega)$, $\tilde{\vec{y}}_a(\vec{r},\omega)$ are proportional to linear combinations of the real and imaginary ***frequency domain*** components of complex over-pressure $\tilde{p}(\vec{r},\omega)$ and complex 3-D vector particle velocity $\tilde{\vec{u}}(\vec{r},\omega)$, *i.e.* that:

$$I_{a_k}^{\mathrm{r}}, z_{a_k}^{\mathrm{r}}, y_{a_k}^{\mathrm{r}} \propto \left(p_{\mathrm{r}} u_{\mathrm{r}_k} + p_{\mathrm{i}} u_{\mathrm{i}_k}\right) \quad \text{and:} \quad I_{a_k}^{\mathrm{i}}, z_{a_k}^{\mathrm{i}}, y_{a_k}^{\mathrm{i}} \propto \left(p_{\mathrm{i}} u_{\mathrm{r}_k} - p_{\mathrm{r}} u_{\mathrm{i}_k}\right).$$

For a monochromatic/single-frequency harmonic sound field:

a.) The ***real part*** of the ***frequency-domain*** complex 3-D vector sound intensity ***"amplitude"*** $\vec{I}_a^{\mathrm{r}}(\vec{r},\omega) = \mathrm{Re}\left\{\tilde{\vec{I}}_a(\vec{r},\omega)\right\}$, and the ***real part*** of the ***frequency-domain*** complex 3-D vector ***specific*** acoustic immittances $\vec{z}_a^{\mathrm{r}}(\vec{r},\omega) = \mathrm{Re}\left\{\tilde{\vec{z}}_a(\vec{r},\omega)\right\}$ and $\vec{y}_a^{\mathrm{r}}(\vec{r},\omega) = \mathrm{Re}\left\{\tilde{\vec{y}}_a(\vec{r},\omega)\right\}$, are associated with ***propagating*** sound – *i.e.* sound ***radiation***.

b.) The ***imaginary part*** of the ***frequency-domain*** complex 3-D vector sound intensity ***"amplitude"*** $\vec{I}_a^{\mathrm{i}}(\vec{r},\omega) = \mathrm{Im}\left\{\tilde{\vec{I}}_a(\vec{r},\omega)\right\}$, and the ***imaginary part*** of the ***frequency-domain*** complex 3-D vector ***specific*** acoustic immittances $\vec{z}_a^{\mathrm{i}}(\vec{r},\omega) = \mathrm{Im}\left\{\tilde{\vec{z}}_a(\vec{r},\omega)\right\}$ and $\vec{y}_a^{\mathrm{i}}(\vec{r},\omega) = \mathrm{Im}\left\{\tilde{\vec{y}}_a(\vec{r},\omega)\right\}$, are associated with ***non-propagating sound energy*** – *i.e.* sound energy that only "sloshes" back and forth {locally} 2× during each cycle of oscillation.

Acoustic Energy Densities:

Energy, $W(t)$ (SI units: *Joules*) and energy ***density***, $w(\vec{r},t)$ (SI units *Joules* /m^3) are ***always purely real***, ***scalar*** physical quantities. They also are ***additive*** in nature, in that different ***kinds*** of energy/energy density – such as potential, kinetic, *etc*. energies can/must be ***linearly*** added together to obtain *e.g*. the ***total*** energy of a system – since ***total*** energy $W_{tot}(t) = \int_v w_{tot}(\vec{r},t)\,dV$ (global) and ***total*** energy density $w_{tot}(\vec{r},t)$ (local) are ***conserved*** quantities.

The ***total instantaneous***, ***physical*** acoustic energy ***density***, $w_a^{tot}(\vec{r},t)$ is the linear sum of the ***instantaneous***, ***physical*** acoustic potential and kinetic energy densities (***time-domain*** quantities):

$$w_a^{tot}(\vec{r},t) = w_a^{potl}(\vec{r},t) + w_a^{kin}(\vec{r},t)$$

where:

$$w_a^{potl}(\vec{r},t) = \frac{1}{2}\rho_o \frac{p^2(\vec{r},t)}{z_o^2} = \frac{1}{2}\frac{1}{\rho_o c^2} p^2(\vec{r},t)$$

and:

$$w_a^{kin}(\vec{r},t) = \frac{1}{2}\rho_o u^2(\vec{r},t) = \frac{1}{2}\rho_o \vec{u}(\vec{r},t) \bullet \vec{u}(\vec{r},t)$$

$$= \frac{1}{2}\rho_o\left[u_x^2(\vec{r},t) + u_y^2(\vec{r},t) + u_z^2(\vec{r},t)\right]$$

$z_o \equiv \rho_o c$ = characteristic longitudinal specific acoustic impedance of "free-air"

Note that the ***instantaneous***, ***physical*** acoustic potential and kinetic energy densities,

$$w_a^{potl}(\vec{r},t) = \tfrac{1}{2}\tfrac{1}{\rho_o c^2} p^2(\vec{r},t) \quad \text{and:} \quad w_a^{kin}(\vec{r},t) = \tfrac{1}{2}\rho_o u^2(\vec{r},t)$$

are respectively analogous *e.g.* to the instantaneous mechanical potential and kinetic energies for a simple 1-D mass-spring system:

$$W_{mech}^{potl}(t) = \tfrac{1}{2}kx^2(t) \quad \text{and:} \quad W_{mech}^{kin}(t) = \tfrac{1}{2}mv^2(t).$$

For a harmonic sound field, the ***instantaneous***, ***physical*** purely real scalar over-pressure is:

$$p(\vec{r},t) = p_o(\vec{r},\omega)\cos\left(\omega t + \varphi_p(\vec{r},\omega)\right) = p_o(\vec{r},\omega)\cos\left(\omega t + \varphi_p(\vec{r},\omega)\right)$$

and the $k = x, y,$ or z components of ***instantaneous***, ***physical*** 3-D vector particle velocity are:

$$u_k(\vec{r},t) = u_{o_k}(\vec{r},\omega)\cos\left(\omega t + \varphi_{u_k}(\vec{r},\omega)\right)$$

Hence: $$w_a^{potl}(\vec{r},t) = \frac{1}{2}\frac{1}{\rho_o c^2} p^2(\vec{r},t) = \frac{1}{2}\frac{1}{\rho_o c^2} p_o^2(\vec{r},\omega)\cos^2\left(\omega t + \varphi_p(\vec{r},\omega)\right)$$

and:

$$\begin{aligned} w_a^{kin}(\vec{r},t) &= \frac{1}{2}\rho_o u^2(\vec{r},t) = \frac{1}{2}\rho_o \vec{u}(\vec{r},t)\bullet\vec{u}(\vec{r},t) \\ &= \frac{1}{2}\rho_o\left[u_x^2(\vec{r},t) + u_y^2(\vec{r},t) + u_z^2(\vec{r},t)\right] \\ &= \frac{1}{2}\rho_o\begin{bmatrix} u_{o_x}^2(\vec{r},\omega)\cos^2\left(\omega t + \varphi_{u_x}(\vec{r},\omega)\right) \\ +u_{o_y}^2(\vec{r},\omega)\cos^2\left(\omega t + \varphi_{u_y}(\vec{r},\omega)\right) \\ +u_{o_z}^2(\vec{r},\omega)\cos^2\left(\omega t + \varphi_{u_z}(\vec{r},\omega)\right) \end{bmatrix} \end{aligned}$$

Now, we can also equivalently write these expressions in terms of complex ***instantaneous/time-domain*** over-pressure and 3-D particle velocity as:

$$w_a^{potl}(\vec{r},t) = \frac{1}{2}\frac{1}{\rho_o c^2}\operatorname{Re}\left\{\tilde{p}^2(\vec{r},t)\right\} = \frac{1}{2}\frac{1}{\rho_o c^2}\operatorname{Re}\left\{p_o^2(\vec{r},\omega)e^{2i\left(\omega t + \varphi_p(\vec{r},\omega)\right)}\right\}$$

and:

$$\begin{aligned} w_a^{kin}(\vec{r},t) &= \frac{1}{2}\rho_o \operatorname{Re}\left[\tilde{u}_x^2(\vec{r},t) + \tilde{u}_y^2(\vec{r},t) + \tilde{u}_z^2(\vec{r},t)\right] \\ &= \frac{1}{2}\rho_o \operatorname{Re}\begin{bmatrix} u_{o_x}^2(\vec{r},\omega)e^{2i\left(\omega t + \varphi_{u_x}(\vec{r},\omega)\right)} \\ +u_{o_y}^2(\vec{r},\omega)e^{2i\left(\omega t + \varphi_{u_y}(\vec{r},\omega)\right)} \\ +u_{o_z}^2(\vec{r},\omega)e^{2i\left(\omega t + \varphi_{u_z}(\vec{r},\omega)\right)} \end{bmatrix} \end{aligned}$$

The time averages of the ***instantaneous*** acoustic potential and kinetic energy densities are:

$$\left\langle w_a^{potl}(\vec{r},t)\right\rangle_t = \frac{1}{2}\frac{1}{\rho_o c^2}\left\langle p^2(\vec{r},t)\right\rangle_t = \frac{1}{2}\frac{1}{\rho_o c^2} p_o^2(\vec{r},\omega)\underbrace{\left\langle \cos^2\left(\omega t+\varphi_p(\vec{r},\omega)\right)\right\rangle_t}_{=1/2}$$

$$= \frac{1}{2}\frac{1}{\rho_o c^2} p_o^2(\vec{r},\omega)\cdot\frac{1}{\tau}\underbrace{\int_{t=0}^{t=\tau}\cos^2\left(\omega t+\varphi_p(\vec{r},\omega)\right)dt}_{=\tau/2} = \frac{1}{4}\frac{1}{\rho_o c^2} p_o^2(\vec{r},\omega)$$

$$\left\langle w_a^{kin}(\vec{r},t)\right\rangle_t = \frac{1}{2}\rho_o\left[u_x^2(\vec{r},t)+u_y^2(\vec{r},t)+u_z^2(\vec{r},t)\right]$$

and:

$$= \frac{1}{2}\rho_o\begin{bmatrix} u_{o_x}^2(\vec{r},\omega)\left\langle \cos^2\left(\omega t+\varphi_{u_x}(\vec{r},\omega)\right)\right\rangle_t \\ +u_{o_y}^2(\vec{r},\omega)\left\langle \cos^2\left(\omega t+\varphi_{u_y}(\vec{r},\omega)\right)\right\rangle_t \\ +u_{o_z}^2(\vec{r},\omega)\left\langle \cos^2\left(\omega t+\varphi_{u_z}(\vec{r},\omega)\right)\right\rangle_t \end{bmatrix}$$

$$= \frac{1}{4}\rho_o\left[u_{o_x}^2(\vec{r},\omega)+u_{o_y}^2(\vec{r},\omega)+u_{o_z}^2(\vec{r},\omega)\right]$$

Thus, the time average of the ***instantaneous*** total acoustic energy density is:

$$\left\langle w_a^{tot}(\vec{r},t)\right\rangle_t = \left\langle w_a^{potl}(\vec{r},t)\right\rangle_t + \left\langle w_a^{kin}(\vec{r},t)\right\rangle_t$$

$$= \frac{1}{4}\frac{1}{\rho_o c^2} p_o^2(\vec{r},\omega) + \frac{1}{4}\rho_o\left[u_{o_x}^2(\vec{r},\omega)+u_{o_y}^2(\vec{r},\omega)+u_{o_z}^2(\vec{r},\omega)\right]$$

For a harmonic sound field: $\tilde{p}(\vec{r},t)=\tilde{p}_o(\vec{r},\omega)e^{i\omega t}$ and $\tilde{\vec{u}}(\vec{r},t)=\tilde{\vec{u}}_o(\vec{r},\omega)e^{i\omega t}$, hence we can "complexify" these relations to obtain the ***purely real***, ***frequency-domain*** acoustic potential, kinetic and total energy densities – note that they remain ***purely real*** quantities:

$$w_a^{potl}(\vec{r},\omega) = \frac{1}{4}\frac{1}{\rho_o c^2}\left|\tilde{p}(\vec{r},t)\right|^2 = \frac{1}{4}\frac{1}{\rho_o c^2}\left|\tilde{p}(\vec{r},\omega)\right|^2 = \left\langle w_a^{potl}(\vec{r},t)\right\rangle_t$$

$$w_a^{kin}(\vec{r},\omega) = \frac{1}{4}\rho_o\left|\tilde{\vec{u}}(\vec{r},t)\right|^2 = \frac{1}{4}\rho_o\tilde{\vec{u}}(\vec{r},t)\bullet\tilde{\vec{u}}^*(\vec{r},t) = \frac{1}{4}\rho_o\left[\left|\tilde{u}_x(\vec{r},t)\right|^2+\left|\tilde{u}_y(\vec{r},t)\right|^2+\left|\tilde{u}_z(\vec{r},t)\right|^2\right]$$

$$= \frac{1}{4}\rho_o\left|\tilde{\vec{u}}(\vec{r},\omega)\right|^2 = \frac{1}{4}\rho_o\tilde{\vec{u}}(\vec{r},\omega)\bullet\tilde{\vec{u}}^*(\vec{r},\omega) = \frac{1}{4}\rho_o\left[\left|\tilde{u}_x(\vec{r},\omega)\right|^2+\left|\tilde{u}_y(\vec{r},\omega)\right|^2+\left|\tilde{u}_z(\vec{r},\omega)\right|^2\right]$$

$$= \left\langle w_a^{kin}(\vec{r},t)\right\rangle_t$$

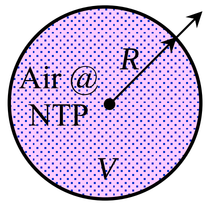
$$w_a^{tot}(\vec{r},\omega) = w_a^{potl}(\vec{r},\omega)+w_a^{kin}(\vec{r},\omega)$$

$$= \frac{1}{4}\frac{1}{\rho_o c^2}\left|\tilde{p}(\vec{r},\omega)\right|^2 + \frac{1}{4}\rho_o\left|\tilde{\vec{u}}(\vec{r},\omega)\right|^2$$

$$= \left\langle w_a^{potl}(\vec{r},t)\right\rangle_t + \left\langle w_a^{kin}(\vec{r},t)\right\rangle_t = \left\langle w_a^{tot}(\vec{r},t)\right\rangle_t$$

<u>Conservation of Energy</u>:

Consider the following "gedanken" experiment: An idealized, infinitesimally-thin spherical shell of radius *R* contains air @ NTP, but has a vacuum on the outside the spherical shell, as shown in the figure below. The infinitesimally-thin spherical shell is made of piezoelectric material, such that when an AC voltage $V_{FG}(t) = V_{FG}^{o}\cos(\omega t)$ is applied to the piezoelectric material *e.g.* using a sine-wave function generator, the spherical shell mechanically vibrates/ oscillates back and forth in the radial $(\hat{r})$ direction with angular frequency $\omega = 2\pi f$.



The instantaneous mechanical power $P_{mech}(t)$ associated with the mechanically oscillating spherical shell is equal to the instantaneous time rate of change of the {total} mechanical work done on the spherical shell: $P_{mech}(t) = \partial W_{mech}(t)/\partial t$. Neglecting any/all dissipative/frictional/loss effects, by conservation of energy, the instantaneous mechanical power $P_{mech}(t)$ being produced at/on the surface of the spherical shell must equal the instantaneous acoustical power $P_a(t)$ associated with instantaneous acoustic energy flowing into the interior volume *V* of the spherical shell {through the surface *S* of the spherical shell of radius *R*} via the relation: $P_a(t) = -\oint_S \vec{I}_a(\vec{r},t) \bullet \vec{da}$, where the vector area element $\vec{da} = \hat{n}da$, where $\hat{n}$ is the outward-pointing unit normal to the surface *S*, which means that $\hat{n} \parallel \hat{r}$, Since the instantaneous acoustical power flowing ***<u>into</u>*** the sphere ***<u>must</u>*** be a positive quantity, *i.e.* $P_a(t) \geq 0$, hence the need for the – sign in the above formula, since we are considering acoustical energy flowing {radially} ***<u>into</u>*** the volume *V* of the spherical shell ***<u>through</u>*** the enclosing surface *S* of radius *R*.

However, using the divergence theorem: $P_a(t) = -\oint_S \vec{I}_a(\vec{r},t) \bullet \vec{da} = -\int_V \vec{\nabla} \bullet \vec{I}_a(\vec{r},t)\, d\tau$. The ***instantaneous, physical <u>time</u>-<u>domain</u>*** vector acoustic intensity $\vec{I}_a(\vec{r},t) \equiv p(\vec{r},t) \cdot \vec{u}(\vec{r},t)$, thus the divergence of the instantaneous vector acoustic intensity, $\vec{\nabla} \bullet \vec{I}_a(\vec{r},t) = \vec{\nabla} \bullet \left[p(\vec{r},t) \cdot \vec{u}(\vec{r},t) \right]$.

Note that since the SI units of vector acoustic intensity $\vec{I}_a$ are $Watts/m^2$, the SI units of $\vec{\nabla} \bullet \vec{I}_a$ are $Watts/m^3$, *i.e* physically, $-\vec{\nabla} \bullet \vec{I}_a$ is an acoustic power ***<u>density</u>***. Hence, the scalar ***instantaneous, physical <u>time</u>-<u>domain</u>*** acoustic power ***<u>density</u>*** $\rho_a^P(\vec{r},t)$ is:

$$\rho_a^P(\vec{r},t) = -\vec{\nabla} \bullet \vec{I}_a(\vec{r},t) = -\vec{\nabla} \bullet \left[p(\vec{r},t) \cdot \vec{u}(\vec{r},t) \right] \quad (Watts/m^3)$$

with:

$$P_a(t) = \int_V \rho_a^P(\vec{r},t)\, d\tau = -\int_V \vec{\nabla} \bullet \vec{I}_a(\vec{r},t)\, d\tau \;\; \geq 0 \;\; (Watts)$$

Next, we use the following product rule from vector calculus: $\vec{\nabla}\bullet\left(f\vec{A}\right)=\vec{\nabla}f\bullet\vec{A}+f\left(\vec{\nabla}\bullet\vec{A}\right)$. Thus: $\vec{\nabla}\bullet\left(p\vec{u}\right)=\vec{\nabla}p\bullet\vec{u}+p\left(\vec{\nabla}\bullet\vec{u}\right)$. For "everyday" sound fields, with $SPL's \ll 134\,dB$ (*i.e.* $|p| \ll 100\,Pascals$), using the {linearized} Euler's equation: $\vec{\nabla}p \simeq -\rho_o\,\partial\vec{u}/\partial t$, and using the {linearized} mass continuity equation: $\vec{\nabla}\bullet\vec{u} \simeq -\frac{1}{\rho_o}\partial\rho/\partial t$. However, from the {linearized} adiabatic relationship between pressure, *p* and mass density, ρ we also have: $\partial\rho/\partial t \simeq \frac{1}{c^2}\partial p/\partial t$. Thus:

$$\rho_a^P=-\vec{\nabla}\bullet\vec{I}_a=-\vec{\nabla}\bullet\left(p\vec{u}\right)=-\vec{\nabla}p\bullet\vec{u}-p\left(\vec{\nabla}\bullet\vec{u}\right)=+\rho_o\frac{\partial\vec{u}}{\partial t}\bullet\vec{u}+\frac{1}{\rho_o c^2}p\frac{\partial p}{\partial t}$$

Now, note that: $$\frac{\partial u^2}{\partial t}=\frac{\partial\left(\vec{u}\bullet\vec{u}\right)}{\partial t}=\frac{\partial\vec{u}}{\partial t}\bullet\vec{u}+\vec{u}\bullet\frac{\partial\vec{u}}{\partial t}=2\frac{\partial\vec{u}}{\partial t}\bullet\vec{u}=2\vec{u}\bullet\frac{\partial\vec{u}}{\partial t}$$

and that: $$\frac{\partial p^2}{\partial t}=\frac{\partial\left(p\cdot p\right)}{\partial t}=\frac{\partial p}{\partial t}p+p\frac{\partial p}{\partial t}=2\frac{\partial p}{\partial t}p=2p\frac{\partial p}{\partial t}$$

Hence, we see that:

$$\rho_a^P\equiv-\vec{\nabla}\bullet\vec{I}_a=-\vec{\nabla}\bullet\left(p\vec{u}\right)=-\vec{\nabla}p\bullet\vec{u}-p\left(\vec{\nabla}\bullet\vec{u}\right)=\frac{1}{2}\rho_o\frac{\partial u^2}{\partial t}+\frac{1}{2\rho_o c^2}\frac{\partial p^2}{\partial t}=\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_o u^2\right)+\frac{\partial}{\partial t}\left(\frac{p^2}{2\rho_o c^2}\right)$$

But the instantaneous acoustic kinetic, potential and total energy densities, respectively are:

$$w_a^{kin}\left(\vec{r},t\right)=\frac{1}{2}\rho_o u^2\left(\vec{r},t\right),\quad w_a^{potl}\left(\vec{r},t\right)=\frac{1}{2}\frac{p^2\left(\vec{r},t\right)}{\rho_o c^2},\quad\text{and:}\quad w_a^{tot}\left(\vec{r},t\right)=w_a^{potl}\left(\vec{r},t\right)+w_a^{kin}\left(\vec{r},t\right)$$

Thus, we see that the {negative} divergence of the ***instantaneous <u>time</u>-<u>domain</u>*** vector acoustic intensity, *i.e.* the ***instantaneous <u>time</u>-<u>domain</u>*** acoustic power ***<u>density</u>*** is:

$$\boxed{\rho_a^P\left(\vec{r},t\right)=-\vec{\nabla}\bullet\vec{I}_a\left(\vec{r},t\right)=\frac{\partial w_a^{kin}\left(\vec{r},t\right)}{\partial t}+\frac{\partial w_a^{potl}\left(\vec{r},t\right)}{\partial t}=\frac{\partial w_a^{tot}\left(\vec{r},t\right)}{\partial t}\ \left(Watts/m^3\right)}$$

The explicit forms of the ***instantaneous <u>time</u>-<u>domain</u>*** potential and kinetic energy densities are:

$$w_a^{potl}\left(\vec{r},t\right)=\frac{1}{2}\frac{1}{\rho_o c^2}p^2\left(\vec{r},t\right)=\frac{1}{2}\frac{1}{\rho_o c^2}p_o^2\left(\vec{r},\omega\right)\cos^2\left(\omega t+\varphi_p\left(\vec{r},\omega\right)\right)$$

and:

$$w_a^{kin}\left(\vec{r},t\right)=\frac{1}{2}\rho_o u^2\left(\vec{r},t\right)=\frac{1}{2}\rho_o\vec{u}\left(\vec{r},t\right)\bullet\vec{u}\left(\vec{r},t\right)=\frac{1}{2}\rho_o\begin{bmatrix}u_{o_x}^2\left(\vec{r},\omega\right)\cos^2\left(\omega t+\varphi_{u_x}\left(\vec{r},\omega\right)\right)\\+u_{o_y}^2\left(\vec{r},\omega\right)\cos^2\left(\omega t+\varphi_{u_y}\left(\vec{r},\omega\right)\right)\\+u_{o_z}^2\left(\vec{r},\omega\right)\cos^2\left(\omega t+\varphi_{u_z}\left(\vec{r},\omega\right)\right)\end{bmatrix}$$

Hence:

$$\frac{\partial w_a^{potl}(\vec{r},t)}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\frac{1}{\rho_o c^2} p_o^2(\vec{r},\omega)\cos^2\left(\omega t + \varphi_p(\vec{r},\omega)\right)\right)$$
$$= -\frac{\omega}{\rho_o c^2} p_o^2(\vec{r},\omega)\sin\left(\omega t + \varphi_p(\vec{r},\omega)\right)\cos\left(\omega t + \varphi_p(\vec{r},\omega)\right)$$

and:

$$\frac{\partial w_a^{kin}(\vec{r},t)}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\rho_o\begin{bmatrix} u_{o_x}^2(\vec{r},\omega)\cos^2\left(\omega t + \varphi_{u_x}(\vec{r},\omega)\right) \\ +u_{o_y}^2(\vec{r},\omega)\cos^2\left(\omega t + \varphi_{u_y}(\vec{r},\omega)\right) \\ +u_{o_z}^2(\vec{r},\omega)\cos^2\left(\omega t + \varphi_{u_z}(\vec{r},\omega)\right)\end{bmatrix}\right)$$
$$= -\omega\rho_o\begin{bmatrix} u_{o_x}^2(\vec{r},\omega)\sin\left(\omega t + \varphi_{u_x}(\vec{r},\omega)\right)\cos\left(\omega t + \varphi_{u_x}(\vec{r},\omega)\right) \\ +u_{o_y}^2(\vec{r},\omega)\sin\left(\omega t + \varphi_{u_y}(\vec{r},\omega)\right)\cos\left(\omega t + \varphi_{u_y}(\vec{r},\omega)\right) \\ +u_{o_z}^2(\vec{r},\omega)\sin\left(\omega t + \varphi_{u_z}(\vec{r},\omega)\right)\cos\left(\omega t + \varphi_{u_z}(\vec{r},\omega)\right)\end{bmatrix}$$

Using the trigonometric relation: $\sin\Theta\cos\Theta = \frac{1}{2}\sin 2\Theta$ with: $\Theta \equiv \omega t + \varphi$, we can rewrite the above relations as:

$$\frac{\partial w_a^{potl}(\vec{r},t)}{\partial t} = -\frac{\omega}{2\rho_o c^2} p_o^2(\vec{r},\omega)\sin 2\left(\omega t + \varphi_p(\vec{r},\omega)\right)$$

and:
$$\frac{\partial w_a^{kin}(\vec{r},t)}{\partial t} = -\frac{1}{2}\omega\rho_o\begin{bmatrix} u_{o_x}^2(\vec{r},\omega)\sin 2\left(\omega t + \varphi_{u_x}(\vec{r},\omega)\right) \\ +u_{o_y}^2(\vec{r},\omega)\sin 2\left(\omega t + \varphi_{u_y}(\vec{r},\omega)\right) \\ +u_{o_z}^2(\vec{r},\omega)\sin 2\left(\omega t + \varphi_{u_z}(\vec{r},\omega)\right)\end{bmatrix}$$

Thus, the ***instantaneous, physical <u>time</u>-<u>domain</u>*** acoustic power density can be seen to be an ***<u>oscillatory</u>*** function (*i.e.* oscillating above/below $\rho_a^P(\vec{r},t) = 0$) at angular frequency $2\omega = 2\cdot 2\pi f = 4\pi f$, *i.e.* 2× per cycle of oscillation:

$$\rho_a^P(\vec{r},t) = -\vec{\nabla}\bullet\vec{I}_a(\vec{r},t) = \frac{\partial w_a^{kin}(\vec{r},t)}{\partial t} + \frac{\partial w_a^{potl}(\vec{r},t)}{\partial t} = \frac{\partial w_a^{tot}(\vec{r},t)}{\partial t}$$
$$= -\frac{1}{2}\omega\rho_o\begin{bmatrix} u_{o_x}^2(\vec{r},\omega)\sin 2\left(\omega t + \varphi_{u_x}(\vec{r},\omega)\right) \\ +u_{o_y}^2(\vec{r},\omega)\sin 2\left(\omega t + \varphi_{u_y}(\vec{r},\omega)\right) \\ +u_{o_z}^2(\vec{r},\omega)\sin 2\left(\omega t + \varphi_{u_z}(\vec{r},\omega)\right)\end{bmatrix} - \frac{\omega}{2\rho_o c^2} p_o^2(\vec{r},\omega)\sin 2\left(\omega t + \varphi_p(\vec{r},\omega)\right)$$

Note that this is ***<u>not</u>*** the kind of behavior we associate with ***<u>propagating</u>*** sound radiation – rather it is that which we associate with the ***<u>non</u>-<u>propagating</u>*** portion of the acoustic energy!

It can thus be seen that since the ***time*** ***averages*** of the ***time*** ***rates*** ***of*** ***change*** of the individual ***instantaneous, physical time-domain*** acoustic potential and kinetic energy densities, and thus the ***time*** ***average*** of the ***instantaneous, physical time-domain*** total acoustic energy density are all ***zero***:

$$\left\langle \frac{\partial w_a^{potl}(\vec{r},t)}{\partial t} \right\rangle_t = -\frac{\omega}{2\rho_o c^2} p_o^2(\vec{r},\omega) \left\langle \sin 2\left(\omega t + \varphi_p(\vec{r},\omega)\right)\right\rangle_t$$

$$= -\frac{\omega}{2\rho_o c^2} p_o^2(\vec{r},\omega) \cdot \frac{1}{\tau} \underbrace{\int_{t=0}^{t=\tau} \sin 2\left(\omega t + \varphi_p(\vec{r},\omega)\right) dt}_{=0} = 0$$

and:

$$\left\langle \frac{\partial w_a^{kin}(\vec{r},t)}{\partial t} \right\rangle_t = -\frac{1}{2}\omega\rho_o \begin{bmatrix} u_{o_x}^2(\vec{r},\omega) \left\langle \sin 2\left(\omega t + \varphi_{u_x}(\vec{r},\omega)\right)\right\rangle_t \\ +u_{o_y}^2(\vec{r},\omega) \left\langle \sin 2\left(\omega t + \varphi_{u_y}(\vec{r},\omega)\right)\right\rangle_t \\ +u_{o_z}^2(\vec{r},\omega) \left\langle \sin 2\left(\omega t + \varphi_{u_z}(\vec{r},\omega)\right)\right\rangle_t \end{bmatrix} = 0$$

and:

$$\left\langle \frac{\partial w_a^{kin}(\vec{r},t)}{\partial t} \right\rangle_t + \left\langle \frac{\partial w_a^{potl}(\vec{r},t)}{\partial t} \right\rangle_t = \left\langle \frac{\partial w_a^{tot}(\vec{r},t)}{\partial t} \right\rangle_t = 0$$

that we also see/learn that the ***time*** ***average*** of the ***instantaneous, physical time-domain*** acoustic power ***density***, which is equal to the ***time*** ***average*** of the {negative} divergence of the ***instantaneous, physical time-domain*** vector acoustic intensity, is also ***zero***:

$$\left\langle \rho_a^P(\vec{r},t)\right\rangle_t = -\left\langle \vec{\nabla}\bullet\vec{I}_a(\vec{r},t)\right\rangle_t = \left\langle \frac{\partial w_a^{kin}(\vec{r},t)}{\partial t} \right\rangle_t + \left\langle \frac{\partial w_a^{potl}(\vec{r},t)}{\partial t} \right\rangle_t = \left\langle \frac{\partial w_a^{tot}(\vec{r},t)}{\partial t} \right\rangle_t = 0$$

It is instructive to also work this out for the complex ***frequency-domain*** case – *e.g.* for a harmonic/monochromatic sound field, because these relations must ***also*** hold for complex $\tilde{\vec{I}}_a(\vec{r},\omega) \equiv \tfrac{1}{2}\tilde{p}(\vec{r},\omega)\cdot\tilde{\vec{u}}^*(\vec{r},\omega) = \tfrac{1}{2}\tilde{p}(\vec{r},t)\cdot\tilde{\vec{u}}^*(\vec{r},t)$: $\vec{\nabla}\bullet\tilde{\vec{I}}_a = \tfrac{1}{2}\vec{\nabla}\bullet\left(\tilde{p}\tilde{\vec{u}}^*\right) = \tfrac{1}{2}\vec{\nabla}\tilde{p}\bullet\tilde{\vec{u}}^* + \tfrac{1}{2}\tilde{p}\left(\vec{\nabla}\bullet\tilde{\vec{u}}^*\right)$, where the complex ***time-domain*** $\tilde{p}(\vec{r},t) = \tilde{p}(\vec{r},\omega)\cdot e^{i\omega t}$ and: $\tilde{\vec{u}}(\vec{r},t) = \tilde{\vec{u}}(\vec{r},\omega)\cdot e^{i\omega t}$, thus: $\tilde{\vec{u}}^*(\vec{r},t) = \tilde{\vec{u}}^*(\vec{r},\omega)\cdot e^{-i\omega t}$. Using the complex ***time-domain*** relations: $\vec{\nabla}\tilde{p} \simeq -\rho_o\,\partial\tilde{\vec{u}}/\partial t$, $\vec{\nabla}\bullet\tilde{\vec{u}}^* \simeq -\tfrac{1}{\rho_o}\partial\tilde{\rho}^*/\partial t$ and: $\partial\tilde{\rho}^*/\partial t \simeq \tfrac{1}{c^2}\partial\tilde{p}^*/\partial t$ we have:

$$\tilde{\rho}_a^P = -\vec{\nabla}\bullet\tilde{\vec{I}}_a = -\tfrac{1}{2}\vec{\nabla}\bullet\left(\tilde{p}\tilde{\vec{u}}^*\right) = -\tfrac{1}{2}\vec{\nabla}\tilde{p}\bullet\tilde{\vec{u}}^* - \tfrac{1}{2}\tilde{p}\left(\vec{\nabla}\bullet\tilde{\vec{u}}^*\right) = +\frac{1}{2}\rho_o\frac{\partial\tilde{\vec{u}}}{\partial t}\bullet\tilde{\vec{u}}^* + \frac{1}{2}\frac{1}{\rho_o c^2}\tilde{p}\frac{\partial\tilde{p}^*}{\partial t}$$

For a ***harmonic*** sound field we have: $\tilde{p}(\vec{r},t)=\tilde{p}(\vec{r},\omega)\cdot e^{i\omega t}$, $\tilde{p}^*(\vec{r},t)=\tilde{p}^*(\vec{r},\omega)\cdot e^{-i\omega t}$ and $\tilde{\vec{u}}(\vec{r},t)=\tilde{\vec{u}}(\vec{r},\omega)\cdot e^{i\omega t}$, $\tilde{\vec{u}}^*(\vec{r},t)=\tilde{\vec{u}}^*(\vec{r},\omega)\cdot e^{-i\omega t}$. Thus, for a ***harmonic*** sound field we ***also*** have: $\partial\vec{u}(\vec{r},t)/\partial t=i\omega\cdot\vec{u}(\vec{r},t)$ and: $\partial\tilde{p}^*(\vec{r},t)/\partial t=-i\omega\cdot\tilde{p}^*(\vec{r},t)$, and thus we see for the complex ***frequency-domain***, for a ***harmonic*** sound field, that:

$$\tilde{\rho}_a^P=-\vec{\nabla}\bullet\tilde{\vec{I}}_a=i\omega\left(\frac{1}{2}\rho_o\tilde{\vec{u}}\bullet\tilde{\vec{u}}^*\right)-i\omega\left(\frac{1}{2}\frac{\tilde{p}\cdot\tilde{p}^*}{\rho_o c^2}\right)=i\omega\left(\frac{1}{2}\rho_o\left|\tilde{\vec{u}}\right|^2\right)-i\omega\left(\frac{1}{2}\frac{\left|\tilde{p}\right|^2}{\rho_o c^2}\right)$$

However, the ***frequency-domain*** versions of the acoustic kinetic and potential energy densities are, respectively:

$$\begin{aligned}w_a^{kin}(\vec{r},\omega)&=\frac{1}{4}\rho_o\left|\tilde{\vec{u}}(\vec{r},t)\right|^2=\frac{1}{4}\rho_o\tilde{\vec{u}}(\vec{r},t)\bullet\tilde{\vec{u}}^*(\vec{r},t)=\frac{1}{4}\rho_o\left[\left|\tilde{u}_x(\vec{r},t)\right|^2+\left|\tilde{u}_y(\vec{r},t)\right|^2+\left|\tilde{u}_z(\vec{r},t)\right|^2\right]\\&=\frac{1}{4}\rho_o\left|\tilde{\vec{u}}(\vec{r},\omega)\right|^2=\frac{1}{4}\rho_o\tilde{\vec{u}}(\vec{r},\omega)\bullet\tilde{\vec{u}}^*(\vec{r},\omega)=\frac{1}{4}\rho_o\left[\left|\tilde{u}_x(\vec{r},\omega)\right|^2+\left|\tilde{u}_y(\vec{r},\omega)\right|^2+\left|\tilde{u}_z(\vec{r},\omega)\right|^2\right]\\&=\left\langle w_a^{kin}(\vec{r},t)\right\rangle_t\end{aligned}$$

$$w_a^{potl}(\vec{r},\omega)=\frac{1}{4}\frac{1}{\rho_o c^2}\left|\tilde{p}(\vec{r},t)\right|^2=\frac{1}{4}\frac{1}{\rho_o c^2}\left|\tilde{p}(\vec{r},\omega)\right|^2=\left\langle w_a^{potl}(\vec{r},t)\right\rangle_t$$

Thus, we see that for a ***harmonic*** sound field, in the complex ***frequency-domain*** representation we have the relation:

$$\tilde{\rho}_a^P(\vec{r},\omega)\equiv-\vec{\nabla}\bullet\tilde{\vec{I}}_a(\vec{r},\omega)=2i\omega\left(\frac{1}{4}\rho_o\left|\tilde{\vec{u}}(\vec{r},\omega)\right|^2\right)-2i\omega\left(\frac{1}{4}\frac{\left|\tilde{p}(\vec{r},\omega)\right|^2}{\rho_o c^2}\right)=2i\omega\left[w_a^{kin}(\vec{r},\omega)-w_a^{potl}(\vec{r},\omega)\right]$$

i.e.

$$\boxed{\begin{aligned}\tilde{\rho}_a^P(\vec{r},\omega)&=-\vec{\nabla}\bullet\tilde{\vec{I}}_a(\vec{r},\omega)=2i\omega\left[w_a^{kin}(\vec{r},\omega)-w_a^{potl}(\vec{r},\omega)\right]\\&=\frac{\partial w_a^{kin}(\vec{r},\omega)}{\partial t}+\frac{\partial w_a^{potl}(\vec{r},\omega)}{\partial t}=\frac{\partial w_a^{tot}(\vec{r},\omega)}{\partial t}\ \left(Watts/m^3\right)\end{aligned}}$$

Now, since energy densities such as w_a^{potl}, w_a^{kin} are ***always/must be*** purely ***real***, ***positive*** quantities, we see that since $\tilde{\rho}_a^P(\vec{r},\omega)\equiv-\vec{\nabla}\bullet\tilde{\vec{I}}_a(\vec{r},\omega)$ is a ***purely imaginary*** quantity for a harmonic sound field, this means that:

$$\boxed{\mathrm{Re}\left\{\tilde{\rho}_a^P(\vec{r},\omega)\right\}=\mathrm{Re}\left\{-\vec{\nabla}\bullet\tilde{\vec{I}}_a(\vec{r},\omega)\right\}=0\ \left(Watts/m^3\right)}$$

and that:

$$\boxed{\mathrm{Im}\left\{\tilde{\rho}_a^P(\vec{r},\omega)\right\}=\mathrm{Im}\left\{-\vec{\nabla}\bullet\tilde{\vec{I}}_a(\vec{r},\omega)\right\}=2\omega\left[w_a^{kin}(\vec{r},\omega)-w_a^{potl}(\vec{r},\omega)\right]\ \left(Watts/m^3\right)}$$

Since the real (imaginary) parts of the complex ***frequency-domain*** vector acoustic sound intensity $\tilde{\vec{I}}_a(\vec{r},\omega)$ are physically associated with ***propagating*** sound/sound radiation (***non-propagating*** acoustic energy, locally sloshing back and forth each cycle) respectively, we see/learn that for ***propagating sound waves***, the complex ***frequency-domain*** acoustic power density, equal to the {negative} divergence of the complex ***frequency-domain*** vector acoustic intensity is ***zero***:

Active Acoustic Power Density: $\boxed{\mathrm{Re}\{\tilde{\wp}_a^P(\vec{r},\omega)\} = \mathrm{Re}\{-\vec{\nabla}\bullet\tilde{\vec{I}}_a(\vec{r},\omega)\} = 0}$

whereas for the ***non-propagating*** portion of the acoustic energy, the complex ***frequency-domain*** acoustic power density, equal to the {negative} divergence of the complex ***frequency-domain*** vector acoustic intensity is ***non-zero***, it is:

Reactive Acoustic Power Density: $\boxed{\mathrm{Im}\{\tilde{\wp}_a^P(\vec{r},\omega)\} = \mathrm{Im}\{-\vec{\nabla}\bullet\tilde{\vec{I}}_a(\vec{r},\omega)\} = 2\omega\left[w_a^{kin}(\vec{r},\omega) - w_a^{potl}(\vec{r},\omega)\right]}$

For a harmonic sound field, we can thus also consider the physical meaning of the ***complex frequency-domain*** representation of the mechanical and acoustic power:

$$\boxed{\begin{aligned}\tilde{P}_a(\omega) &= \int_V \tilde{\wp}_a^P(\vec{r},\omega)\,d\tau = -\int_V \vec{\nabla}\bullet\tilde{\vec{I}}_a(\vec{r},\omega)\,d\tau = -\oint_S \tilde{\vec{I}}_a(\vec{r},\omega)\bullet\overrightarrow{da} \\ &= 2i\omega\int_V\left[w_a^{kin}(\vec{r},\omega) - w_a^{potl}(\vec{r},\omega)\right]d\tau \ \ (Watts)\end{aligned}}$$

Again, since energy densities are ***always*** purely ***real***, ***positive*** quantities, we see/learn that the complex ***frequency-domain*** representation of the mechanical and acoustic powers are such that:

Active Acoustic Power: $\boxed{\mathrm{Re}\{\tilde{P}_a(\omega)\} = 0}$

and:

Reactive Acoustic Power: $\boxed{\mathrm{Im}\{\tilde{P}_a(\omega)\} = 2\omega\int_V\left[w_a^{kin}(\vec{r},\omega) - w_a^{potl}(\vec{r},\omega)\right]d\tau}$

The above results for ***active*** *vs.* ***reactive*** power have important physical ramifications as to how the mechanical energy is able to be injected into and/or extracted from the acoustical system!

Since we have discussed the nature of the {negative} divergence of the vector acoustic intensity, $-\vec{\nabla}\bullet\vec{I}_a$, for completeness' sake, we also discuss the nature of the curl of the vector acoustic intensity, $\vec{\nabla}\times\vec{I}_a$, the ***vorticity*** associated with 3-D acoustic intensity.

In the ***instantaneous***, ***physical time-domain*** representation: $\vec{\nabla}\times\vec{I}_a(\vec{r},t) = \vec{\nabla}\times\left(p(\vec{r},t)\cdot\vec{u}(\vec{r},t)\right)$.

Using another product rule from vector calculus: $\vec{\nabla}\times\left(f\vec{A}\right) = f\left(\vec{\nabla}\times\vec{A}\right) - \vec{A}\times\vec{\nabla}f$, we have:

$$\vec{\nabla}\times\vec{I}_a = \vec{\nabla}\times\left(p\cdot\vec{u}\right) = p\underbrace{\left(\cancel{\vec{\nabla}\times\vec{u}}\right)}_{=\,0} - \vec{u}\times\vec{\nabla}p = -\vec{u}\times\vec{\nabla}p$$

where $\vec{\nabla}\times\vec{u}=0$ for an inviscid (*i.e.* dissipationless) fluid – no vorticity exists, such as is the case for {still/calm} air @ NTP. The {linearized} Euler equation is: $-\vec{\nabla}p \simeq \rho_o\,\partial\vec{u}/\partial t$, thus:

$$\vec{\nabla}\times\vec{I}_a=\vec{\nabla}\times\left(p\cdot\vec{u}\right)=-\vec{u}\times\vec{\nabla}p\simeq\rho_o\vec{u}\times\frac{\partial\vec{u}}{\partial t}$$

The ***instantaneous***, ***physical <u>time</u>-<u>domain</u>*** representation of the vector particle velocity is:

$$\vec{u}\left(\vec{r},t\right)=u_x\left(\vec{r},t\right)\hat{x}+u_y\left(\vec{r},t\right)\hat{y}+u_z\left(\vec{r},t\right)\hat{z}$$
$$=u_{o_x}\left(\vec{r},\omega\right)\cos\left(\omega t+\varphi_{u_x}\left(\vec{r},\omega\right)\right)\hat{x}+u_{o_y}\left(\vec{r},\omega\right)\cos\left(\omega t+\varphi_{u_y}\left(\vec{r},\omega\right)\right)\hat{y}+u_{o_z}\left(\vec{r},\omega\right)\cos\left(\omega t+\varphi_{u_z}\left(\vec{r},\omega\right)\right)\hat{z}$$

and thus the ***<u>time</u> <u>rate</u> <u>of</u> <u>change</u>*** of the ***instantaneous***, ***physical <u>time</u>-<u>domain</u>*** representation of the vector particle velocity is:

$$\frac{\partial\vec{u}\left(\vec{r},t\right)}{\partial t}=\frac{\partial\vec{u}_x\left(\vec{r},t\right)}{\partial t}\hat{x}+\frac{\partial\vec{u}_y\left(\vec{r},t\right)}{\partial t}\hat{y}+\frac{\partial\vec{u}_z\left(\vec{r},t\right)}{\partial t}\hat{z}$$
$$=-\omega\left[u_{o_x}\left(\vec{r},\omega\right)\sin\left(\omega t+\varphi_{u_x}\left(\vec{r},\omega\right)\right)\hat{x}+u_{o_y}\left(\vec{r},\omega\right)\sin\left(\omega t+\varphi_{u_y}\left(\vec{r},\omega\right)\right)\hat{y}+u_{o_z}\left(\vec{r},\omega\right)\sin\left(\omega t+\varphi_{u_z}\left(\vec{r},\omega\right)\right)\hat{z}\right]$$

Thus, we need to work out the vector cross product: $\vec{u}\times\partial\vec{u}/\partial t$. Recall that the cross product of two arbitrary vectors $\vec{A}$ and $\vec{B}$ is defined as:

$$\vec{A}\times\vec{B}=\left(A_x\hat{x}+A_y\hat{y}+A_z\hat{z}\right)\times\left(B_x\hat{x}+B_y\hat{y}+B_z\hat{z}\right)=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\A_x&A_y&A_z\\B_x&B_y&B_z\end{vmatrix}$$
$$=\left(A_yB_z-A_zB_y\right)\hat{x}+\left(A_zB_x-A_xB_z\right)\hat{y}+\left(A_xB_y-A_yB_x\right)\hat{z}$$

Thus, temporarily suppressing $\left(\vec{r},t\right)$ and $\left(\vec{r},\omega\right)$ arguments for the sake of notational clarity:

$$\begin{aligned}\vec{u}\times\frac{\partial\vec{u}}{\partial t}=&-\omega\left[u_{o_y}u_{o_z}\cos\left(\omega t+\varphi_{u_y}\right)\sin\left(\omega t+\varphi_{u_z}\right)-u_{o_z}u_{o_y}\cos\left(\omega t+\varphi_{u_z}\right)\sin\left(\omega t+\varphi_{u_y}\right)\right]\hat{x}\\&-\omega\left[u_{o_z}u_{o_x}\cos\left(\omega t+\varphi_{u_z}\right)\sin\left(\omega t+\varphi_{u_x}\right)-u_{o_x}u_{o_z}\cos\left(\omega t+\varphi_{u_x}\right)\sin\left(\omega t+\varphi_{u_z}\right)\right]\hat{y}\\&-\omega\left[u_{o_x}u_{o_y}\cos\left(\omega t+\varphi_{u_x}\right)\sin\left(\omega t+\varphi_{u_y}\right)-u_{o_y}u_{o_x}\cos\left(\omega t+\varphi_{u_y}\right)\sin\left(\omega t+\varphi_{u_x}\right)\right]\hat{z}\end{aligned}$$

$$\begin{aligned}=&-\omega\cdot u_{o_y}u_{o_z}\left[\cos\left(\omega t+\varphi_{u_y}\right)\sin\left(\omega t+\varphi_{u_z}\right)-\cos\left(\omega t+\varphi_{u_z}\right)\sin\left(\omega t+\varphi_{u_y}\right)\right]\hat{x}\\&-\omega\cdot u_{o_z}u_{o_x}\left[\cos\left(\omega t+\varphi_{u_z}\right)\sin\left(\omega t+\varphi_{u_x}\right)-\cos\left(\omega t+\varphi_{u_x}\right)\sin\left(\omega t+\varphi_{u_z}\right)\right]\hat{y}\\&-\omega\cdot u_{o_x}u_{o_y}\left[\cos\left(\omega t+\varphi_{u_x}\right)\sin\left(\omega t+\varphi_{u_y}\right)-\cos\left(\omega t+\varphi_{u_y}\right)\sin\left(\omega t+\varphi_{u_x}\right)\right]\hat{z}\end{aligned}$$

Defining: $\Theta_{u_k} \equiv \omega t + \varphi_{u_k}$, $k = x, y, z$ we can rewrite this relation as:

$$\begin{aligned}
\vec{u} \times \frac{\partial \vec{u}}{\partial t} = &-\omega \cdot u_{o_y} u_{o_z} \left[\cos\Theta_{u_y} \sin\Theta_{u_z} - \cos\Theta_{u_z} \sin\Theta_{u_y} \right] \hat{x} \\
&-\omega \cdot u_{o_z} u_{o_x} \left[\cos\Theta_{u_z} \sin\Theta_{u_x} - \cos\Theta_{u_x} \sin\Theta_{u_z} \right] \hat{y} \\
&-\omega \cdot u_{o_x} u_{o_y} \left[\cos\Theta_{u_x} \sin\Theta_{u_y} - \cos\Theta_{u_y} \sin\Theta_{u_x} \right] \hat{z}
\end{aligned}$$

$$\begin{aligned}
= &\ \ \omega \cdot u_{o_y} u_{o_z} \left[\sin\Theta_{u_y} \cos\Theta_{u_z} - \cos\Theta_{u_y} \sin\Theta_{u_z} \right] \hat{x} \\
&+\omega \cdot u_{o_z} u_{o_x} \left[\sin\Theta_{u_z} \cos\Theta_{u_x} - \cos\Theta_{u_z} \sin\Theta_{u_x} \right] \hat{y} \\
&+\omega \cdot u_{o_x} u_{o_y} \left[\sin\Theta_{u_x} \cos\Theta_{u_y} - \cos\Theta_{u_x} \sin\Theta_{u_y} \right] \hat{z}
\end{aligned}$$

Using the trigonometric relation: $\sin(A-B) = \cos A \sin B - \sin A \cos B$, and noting that: $\Theta_{u_k} - \Theta_{u_j} \equiv (\omega t + \varphi_{u_k}) - (\omega t + \varphi_{u_j}) = \varphi_{u_k} - \varphi_{u_j} \equiv \Delta\varphi_{u_k - u_j}$, for $k \neq j = x, y, z$ this relation can be further rewritten as:

$$\vec{u} \times \frac{\partial \vec{u}}{\partial t} = \omega \cdot \left[u_{o_y} u_{o_z} \sin\Delta\varphi_{u_y - u_z} \hat{x} + u_{o_z} u_{o_x} \sin\Delta\varphi_{u_z - u_x} \hat{y} + u_{o_x} u_{o_y} \sin\Delta\varphi_{u_x - u_y} \hat{z} \right]$$

or, more explicitly:

$$\vec{\nabla} \times \vec{I}_a(\vec{r},t) \simeq \rho_o \vec{u}(\vec{r},t) \times \frac{\partial \vec{u}(\vec{r},t)}{\partial t} = \omega\rho_o \cdot \begin{bmatrix} u_{o_y}(\vec{r},\omega) \cdot u_{o_z}(\vec{r},\omega) \sin\Delta\varphi_{u_y - u_z}(\vec{r},\omega) \hat{x} \\ +u_{o_z}(\vec{r},\omega) \cdot u_{o_x}(\vec{r},\omega) \sin\Delta\varphi_{u_z - u_x}(\vec{r},\omega) \hat{y} \\ +u_{o_x}(\vec{r},\omega) \cdot u_{o_y}(\vec{r},\omega) \sin\Delta\varphi_{u_x - u_y}(\vec{r},\omega) \hat{z} \end{bmatrix}$$

Thus, we see that the curl of the ***instantaneous***, ***physical*** ***time***-***domain*** vector acoustic intensity, $\vec{\nabla} \times \vec{I}_a(\vec{r},t)$ has ***no*** explicit time dependence, and note also that $\vec{\nabla} \times \vec{I}_a(\vec{r},t)$ has ***no*** dependence on the over-pressure, $p(\vec{r},t)$. It does depend on the {sine of} the $k \neq j = x, y, z$ component phase ***differences*** in particle velocity, $\Delta\varphi_{u_k - u_j}(\vec{r},\omega)$ and bi-linear products of $k \neq j = x, y, z$ ***frequency***-***domain*** particle velocity amplitudes $u_{o_k}(\vec{r},\omega) \cdot u_{o_j}(\vec{r},\omega)$. If ***all*** phase difference components $\Delta\varphi_{u_k - u_j}(\vec{r},\omega) = 0^o, \pm 180^o$, then $\vec{\nabla} \times \vec{I}_a(\vec{r},t) = 0$. Maximum vorticity associated with the 3-D vector acoustic intensity occurs when $\Delta\varphi_{u_k - u_j}(\vec{r},\omega) = \pm 90^o$.

Note that the SI units of $\vec{\nabla} \times \vec{I}_a(\vec{r},t)$, as for $\vec{\nabla} \bullet \vec{I}_a(\vec{r},t)$ are $Watts/m^3$. From Stoke's theorem: $\int_S \left(\vec{\nabla} \times \vec{I}_a(\vec{r},t)\right) \bullet \overrightarrow{da} = \oint_C \vec{I}_a(\vec{r},t) \bullet \overrightarrow{d\ell}$ where *S* {here} is an ***open*** surface is enclosed {*i.e*. bounded} by the contour *C*. The SI units of $\int_S \left(\vec{\nabla} \times \vec{I}_a(\vec{r},t)\right) \bullet \overrightarrow{da} = \oint_C \vec{I}_a(\vec{r},t) \bullet \overrightarrow{d\ell}$ are thus $Watts/m$.

Again, it is instructive to work this out for the complex ***frequency-domain*** case – *e.g.* for a harmonic/monochromatic sound field, because these relations must ***also*** hold for complex $\tilde{\vec{I}}_a(\vec{r},\omega) \equiv \frac{1}{2}\tilde{p}(\vec{r},\omega)\cdot\tilde{\vec{u}}^*(\vec{r},\omega) = \frac{1}{2}\tilde{p}(\vec{r},t)\cdot\tilde{\vec{u}}^*(\vec{r},t)$:

$$\vec{\nabla}\times\tilde{\vec{I}}_a = \tfrac{1}{2}\vec{\nabla}\times\left(\tilde{p}\tilde{\vec{u}}^*\right) = \tfrac{1}{2}\tilde{p}\underbrace{\left(\vec{\nabla}\times\tilde{\vec{u}}^*\right)}_{=0} - \tfrac{1}{2}\tilde{\vec{u}}^*\times\vec{\nabla}\tilde{p} = -\tfrac{1}{2}\tilde{\vec{u}}^*\times\vec{\nabla}\tilde{p}$$

where the complex ***time-domain*** $\tilde{p}(\vec{r},t) = \tilde{p}(\vec{r},\omega)\cdot e^{i\omega t}$ and: $\tilde{\vec{u}}(\vec{r},t) = \tilde{\vec{u}}(\vec{r},\omega)\cdot e^{i\omega t}$, thus: $\tilde{\vec{u}}^*(\vec{r},t) = \tilde{\vec{u}}^*(\vec{r},\omega)\cdot e^{-i\omega t}$. Using the complex ***time-domain*** relations $\vec{\nabla}\tilde{p} \simeq -\rho_o\,\partial\tilde{\vec{u}}/\partial t$, $\partial\tilde{\vec{u}}/\partial t = i\omega\cdot\tilde{\vec{u}}$, hence: $\partial\tilde{\vec{u}}^*/\partial t = -i\omega\cdot\tilde{\vec{u}}^*$, we have:

$$\vec{\nabla}\times\tilde{\vec{I}}_a = \tfrac{1}{2}\vec{\nabla}\times\left(\tilde{p}\tilde{\vec{u}}^*\right) = -\tfrac{1}{2}\tilde{\vec{u}}^*\times\vec{\nabla}\tilde{p} = \tfrac{1}{2}\rho_o\tilde{\vec{u}}^*\times\frac{\partial\tilde{\vec{u}}}{\partial t} = \tfrac{1}{2}i\omega\rho_o\left(\tilde{\vec{u}}^*\times\tilde{\vec{u}}\right)$$

Now:

$$\tilde{\vec{u}}(\vec{r},t) = \tilde{\vec{u}}(\vec{r},\omega)\cdot e^{i\omega t} = \left[u_{o_x}(\vec{r},\omega)e^{i\varphi_{u_x}(\vec{r},\omega)}\hat{x} + u_{o_y}(\vec{r},\omega)e^{i\varphi_{u_y}(\vec{r},\omega)}\hat{y} + u_{o_z}(\vec{r},\omega)e^{i\varphi_{u_z}(\vec{r},\omega)}\hat{z}\right]\cdot e^{i\omega t}$$

And:

$$\tilde{\vec{u}}^*(\vec{r},t) = \tilde{\vec{u}}^*(\vec{r},\omega)\cdot e^{-i\omega t} = \left[u_{o_x}(\vec{r},\omega)e^{-i\varphi_{u_x}(\vec{r},\omega)}\hat{x} + u_{o_y}(\vec{r},\omega)e^{-i\varphi_{u_y}(\vec{r},\omega)}\hat{y} + u_{o_z}(\vec{r},\omega)e^{-i\varphi_{u_z}(\vec{r},\omega)}\hat{z}\right]\cdot e^{-i\omega t}$$

Thus, we need to work out the complex cross product $\tilde{\vec{u}}^*\times\tilde{\vec{u}} = -\tilde{\vec{u}}\times\tilde{\vec{u}}^*$ (*n.b.* which is ***not*** zero – because $\tilde{\vec{u}}^* \not\parallel \tilde{\vec{u}}$!):

$$\begin{aligned}
\tilde{\vec{u}}\times\tilde{\vec{u}}^* &= e^{i\omega t}\cdot\left[u_{o_x}e^{i\varphi_{u_x}}\hat{x} + u_{o_y}e^{i\varphi_{u_y}}\hat{y} + u_{o_z}e^{i\varphi_{u_z}}\hat{z}\right]\times\left[u_{o_x}e^{-i\varphi_{u_x}}\hat{x} + u_{o_y}e^{-i\varphi_{u_y}}\hat{y} + u_{o_z}e^{-i\varphi_{u_z}}\hat{z}\right]\cdot e^{-i\omega t} \\
&= \left[u_{o_x}e^{i\varphi_{u_x}}\hat{x} + u_{o_y}e^{i\varphi_{u_y}}\hat{y} + u_{o_z}e^{i\varphi_{u_z}}\hat{z}\right]\times\left[u_{o_x}e^{-i\varphi_{u_x}}\hat{x} + u_{o_y}e^{-i\varphi_{u_y}}\hat{y} + u_{o_z}e^{-i\varphi_{u_z}}\hat{z}\right] \\
&= \begin{bmatrix} u_{o_y}u_{o_z}\left(e^{i\Delta\varphi_{u_y-u_z}} - e^{-i\Delta\varphi_{u_y-u_z}}\right)\hat{x} \\ +u_{o_z}u_{o_x}\left(e^{i\Delta\varphi_{u_z-u_x}} - e^{-i\Delta\varphi_{u_z-u_x}}\right)\hat{y} \\ +u_{o_x}u_{o_y}\left(e^{i\Delta\varphi_{u_x-u_y}} - e^{-i\Delta\varphi_{u_x-u_y}}\right)\hat{z} \end{bmatrix} \qquad \left\{\text{use: } \sin\theta = \tfrac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right\} \\
&= 2i\left[u_{o_y}u_{o_z}\sin\Delta\varphi_{u_y-u_z}\hat{x} + u_{o_z}u_{o_x}\sin\Delta\varphi_{u_z-u_x}\hat{y} + u_{o_x}u_{o_y}\sin\Delta\varphi_{u_x-u_y}\hat{z}\right]
\end{aligned}$$

Hence:

$$\begin{aligned}
\vec{\nabla}\times\tilde{\vec{I}}_a &\simeq \tfrac{1}{2}\rho_o\tilde{\vec{u}}^*\times\frac{\partial\tilde{\vec{u}}}{\partial t} = \tfrac{1}{2}i\omega\rho_o\cdot\left(\tilde{\vec{u}}^*\times\tilde{\vec{u}}\right) = -\tfrac{1}{2}i\omega\rho_o\cdot\left(\tilde{\vec{u}}\times\tilde{\vec{u}}^*\right) \\
&= -\tfrac{1}{2}i\cdot 2i\omega\rho_o\left[u_{o_y}u_{o_z}\sin\Delta\varphi_{u_y-u_z}\hat{x} + u_{o_z}u_{o_x}\sin\Delta\varphi_{u_z-u_x}\hat{y} + u_{o_x}u_{o_y}\sin\Delta\varphi_{u_x-u_y}\hat{z}\right] \\
&= \omega\rho_o\left[u_{o_y}u_{o_z}\sin\Delta\varphi_{u_y-u_z}\hat{x} + u_{o_z}u_{o_x}\sin\Delta\varphi_{u_z-u_x}\hat{y} + u_{o_x}u_{o_y}\sin\Delta\varphi_{u_x-u_y}\hat{z}\right]
\end{aligned}$$

Note that this result is ***identical*** to that obtained for the curl of the ***physical purely real instantaneous time-domain*** 3-D vector acoustic intensity, $\vec{\nabla}\times\vec{I}_a(\vec{r},t)$.

Hence we see/learn that:

$$\operatorname{Re}\left\{\vec{\nabla}\times\tilde{\vec{I}}_a\right\} \simeq \tfrac{1}{2}\rho_o\tilde{\vec{u}}^* \times \frac{\partial\tilde{\vec{u}}}{\partial t} = \tfrac{1}{2}i\omega\rho_o\cdot\left(\tilde{\vec{u}}^*\times\tilde{\vec{u}}\right) = -\tfrac{1}{2}i\omega\rho_o\cdot\left(\tilde{\vec{u}}\times\tilde{\vec{u}}^*\right)$$
$$= \omega\rho_o\left[u_{o_y}u_{o_z}\sin\Delta\varphi_{u_y-u_z}\hat{x} + u_{o_z}u_{o_x}\sin\Delta\varphi_{u_z-u_x}\hat{y} + u_{o_x}u_{o_y}\sin\Delta\varphi_{u_x-u_y}\hat{z}\right]$$

and that: $$\operatorname{Im}\left\{\vec{\nabla}\times\tilde{\vec{I}}_a\right\} = 0$$

The above results make it clear that the curl of the 3-D vector acoustic intensity – the ***vorticity*** associated with 3-D vector acoustic intensity is associated ***only*** with ***propagating*** sound radiation.

The Complex Acoustic 3-D Velocity of Energy Flow:

In Physics 406 Lecture Notes 12 (*p*. 3-6) we will see/learn that for a 1-D monochromatic traveling wave propagating in "free air", the complex longitudinal acoustic ***specific*** impedance, – a physical property associated with the medium ("free air") in which the 1-D monochromatic traveling wave is propagating in – is a ***purely real constant***: $\tilde{z}_a(\vec{r},\omega) = \rho_o c \equiv z_o$. The so-called ***characteristic*** longitudinal ***specific*** impedance of "free air", $z_o \equiv \rho_o c \simeq 415\ \Omega_a$ @ NTP. Note that the constant z_o is thus independent of ***position*** $\vec{r}$ and of {angular} ***frequency*** $\omega = 2\pi f$.

For an arbitrary complex harmonic sound field $\tilde{\vec{S}}(\vec{r},t)$, but one with "everyday" sound pressure levels $SPL(\vec{r}) \ll 134\,dB$ ($\left|\tilde{p}(\vec{r})\right| \ll 100\,Pascals$) in air @ NTP, in general the 3-D vector ***specific*** acoustic impedance $\tilde{\vec{z}}_a(\vec{r},\omega)$ ***will*** be ***complex***, and depend on both position in space, $\vec{r}$ and {angular} frequency, $\omega = 2\pi f$. Thus, the above ***specific*** acoustic impedance relation for the "free air" case $\tilde{z}_a(\vec{r},\omega) = \rho_o c \equiv z_o$ can be generalized to $\tilde{\vec{z}}_a(\vec{r},\omega) = \rho_o\tilde{\vec{c}}_a(\vec{r},\omega)$, where $\tilde{\vec{c}}_a(\vec{r},\omega)$ (m/s) is the ***complex acoustic 3-D velocity of energy flow***. Note that, like $\tilde{\vec{z}}_a(\vec{r},\omega)$, $\tilde{\vec{c}}_a(\vec{r},\omega)$ is manifestly a ***frequency-domain*** quantity, since it is simply-related to $\tilde{\vec{z}}_a(\vec{r},\omega)$ by the equilibrium mass density of the medium $\rho_o\left(kg/m^3\right)$, a scalar quantity.

The ***complex acoustic 3-D vector velocity of energy flow*** $\tilde{\vec{c}}_a(\vec{r},\omega)$ must ***not*** be confused with the ***purely real***, ***scalar adiabatic/thermodynamic speed*** of sound of the medium c, nor with the ***purely real***, ***scalar phase speed*** of propagation of the wave $v_\varphi(\vec{r},\omega) \equiv \omega/k(\vec{r},\omega)$, nor with the ***purely real***, ***3-D vector group velocity*** $\vec{v}_g(\vec{r},\omega) \equiv \left[d\vec{k}(\vec{r},\omega)/d\omega\right]^{-1}$. In various wave-type physics situations, there can in fact be several/many different definitions /kinds of propagation speeds…. {See *e.g*. S.C. Bloch, "Eighth Velocity of Light", *p*. 538-549, Am. J. Phys., Vol. 45, No. 6, June 1977}.

Since complex 3-D vector ***specific*** acoustic admittance is the ***reciprocal*** of the complex 3-D vector ***specific*** acoustic impedance, we have the paired reciprocal relations:

$$\tilde{\vec{z}}_a(\vec{r},\omega)=\frac{\tilde{p}(\vec{r},\omega)}{\tilde{\vec{u}}(\vec{r},\omega)}=\frac{1}{\tilde{\vec{y}}_a(\vec{r},\omega)}=\rho_o\tilde{\vec{c}}_a(\vec{r},\omega)\ \left(\Omega_a\right) \text{ and: } \tilde{\vec{y}}_a(\vec{r},\omega)=\frac{\tilde{\vec{u}}(\vec{r},\omega)}{\tilde{p}(\vec{r},\omega)}=\frac{1}{\tilde{\vec{z}}_a(\vec{r},\omega)}=\frac{1}{\rho_o\tilde{\vec{c}}_a(\vec{r},\omega)}\ \left(\Omega_a^{-1}\right).$$

However, we have the complex ***frequency-domain*** relation:

$$\tilde{\vec{I}}_a(\vec{r},\omega)\equiv\frac{1}{2}\tilde{p}(\vec{r},\omega)\cdot\tilde{\vec{u}}^*(\vec{r},\omega)=\frac{1}{2}\frac{\tilde{p}(\vec{r},\omega)\cdot\left(\tilde{\vec{u}}(\vec{r},\omega)\bullet\tilde{\vec{u}}^*(\vec{r},\omega)\right)}{\tilde{\vec{u}}(\vec{r},\omega)}$$

$$=\frac{1}{2}\left|\tilde{\vec{u}}(\vec{r},\omega)\right|^2\cdot\left(\frac{\tilde{p}(\vec{r},\omega)}{\tilde{\vec{u}}(\vec{r},\omega)}\right)=\frac{1}{2}\left|\tilde{\vec{u}}(\vec{r},\omega)\right|^2\cdot\tilde{\vec{z}}_a(\vec{r},\omega)=\frac{1}{2}\rho_o\left|\tilde{\vec{u}}(\vec{r},\omega)\right|^2\cdot\tilde{\vec{c}}_a(\vec{r},\omega)$$

But, the {***purely real***} ***frequency-domain*** acoustic ***kinetic*** energy density $w_a^{kin}(\vec{r},\omega)=\tfrac{1}{4}\rho_o\left|\tilde{\vec{u}}(\vec{r},\omega)\right|^2$, hence: $\tilde{\vec{I}}_a(\vec{r},\omega)=2\tilde{\vec{c}}_a(\vec{r},\omega)\cdot w_a^{kin}(\vec{r},\omega)$.

However, we ***also*** have the complex ***frequency-domain*** relation:

$$\tilde{\vec{I}}_a(\vec{r},\omega)\equiv\frac{1}{2}\tilde{p}(\vec{r},\omega)\cdot\tilde{\vec{u}}^*(\vec{r},\omega)=\frac{1}{2}\frac{\left(\tilde{p}(\vec{r},\omega)\cdot\tilde{p}^*(\vec{r},\omega)\right)\cdot\tilde{\vec{u}}^*(\vec{r},\omega)}{\tilde{p}^*(\vec{r},\omega)}$$

$$=\frac{1}{2}\left|\tilde{p}(\vec{r},\omega)\right|^2\cdot\left(\frac{\tilde{\vec{u}}^*(\vec{r},\omega)}{\tilde{p}^*(\vec{r},\omega)}\right)=\frac{1}{2}\left|\tilde{p}(\vec{r},\omega)\right|^2\cdot\tilde{\vec{y}}_a^*(\vec{r},\omega)=\frac{1}{2}\frac{\left|\tilde{p}(\vec{r},\omega)\right|^2}{\rho_o\tilde{\vec{c}}_a^*(\vec{r},\omega)}$$

$$=\frac{1}{2}\frac{\left|\tilde{p}(\vec{r},\omega)\right|^2}{\rho_o\left|\tilde{\vec{c}}_a(\vec{r},\omega)\right|^2}\cdot\tilde{\vec{c}}_a(\vec{r},\omega)=\frac{1}{2}\frac{\left|\tilde{p}(\vec{r},\omega)\right|^2}{\rho_o c^2}\cdot\left(\frac{c^2}{\left|\tilde{\vec{c}}_a(\vec{r},\omega)\right|^2}\right)\cdot\tilde{\vec{c}}_a(\vec{r},\omega)$$

But, the {***purely real***} ***frequency-domain*** acoustic ***potential*** energy density $w_a^{potl}(\vec{r},\omega)=\tfrac{1}{4}\left|\tilde{p}(\vec{r},\omega)\right|^2/\rho_o c^2$, hence: $\tilde{\vec{I}}_a(\vec{r},\omega)=2\tilde{\vec{c}}_a(\vec{r},\omega)\cdot\left(c^2/\left|\tilde{\vec{c}}_a(\vec{r},\omega)\right|^2\right)\cdot w_a^{potl}(\vec{r},\omega)$.

Thus, we see that:

$$\boxed{\tilde{\vec{I}}_a(\vec{r},\omega)=2\tilde{\vec{c}}_a(\vec{r},\omega)\cdot w_a^{kin}(\vec{r},\omega)=2\left(\frac{c^2}{\left|\tilde{\vec{c}}_a(\vec{r},\omega)\right|^2}\right)\cdot\tilde{\vec{c}}_a(\vec{r},\omega)\cdot w_a^{potl}(\vec{r},\omega)}$$

Note also that: $\tilde{\vec{I}}_a(\vec{r},\omega)\parallel\tilde{\vec{z}}_a(\vec{r},\omega)\parallel\tilde{\vec{c}}_a(\vec{r},\omega)$, and additionally note that the ***ratio*** of acoustic ***potential*** to ***kinetic*** energy density is:

$$\boxed{\frac{w_a^{potl}(\vec{r},\omega)}{w_a^{kin}(\vec{r},\omega)}=\frac{\left|\tilde{\vec{c}}_a(\vec{r},\omega)\right|^2}{c^2}=\frac{\left|\tilde{\vec{z}}_a(\vec{r},\omega)\right|^2}{z_o^2}}$$

As mentioned above, for the "free-field" propagation of monochromatic plane traveling waves $\tilde{z}_a(\vec{r},\omega)=\rho_o c \equiv z_o$, hence for the "free-field" propagation of monochromatic plane traveling waves, we see/learn that: $w_a^{potl}(\vec{r},\omega)/w_a^{kin}(\vec{r},\omega)=\left|\tilde{\tilde{c}}_a(\vec{r},\omega)\right|^2/c^2=\left|\tilde{\tilde{z}}_a(\vec{r},\omega)\right|^2/z_o^2=1$. Note that in general these 3 ratios are ***not*** equal to unity for an ***arbitrary*** complex sound field.

We can also define a physically-related quantity, known as the ***complex 3-D vector acoustic "index of refraction" of energy flow***:

$$\boxed{\tilde{\tilde{n}}_a(\vec{r},\omega)\equiv\frac{\tilde{\tilde{c}}_a(\vec{r},\omega)}{c}=\frac{\tilde{\tilde{z}}_a(\vec{r},\omega)}{z_o}} \quad \text{thus:} \quad \boxed{\left|\tilde{\tilde{n}}_a(\vec{r},\omega)\right|^2=\frac{\left|\tilde{\tilde{c}}_a(\vec{r},\omega)\right|^2}{c^2}=\frac{\left|\tilde{\tilde{z}}_a(\vec{r},\omega)\right|^2}{z_o^2}=\frac{w_a^{potl}(\vec{r},\omega)}{w_a^{kin}(\vec{r},\omega)}}$$

Hence for the "free-field" propagation of monochromatic plane traveling waves, we see/learn that: $\left|\tilde{\tilde{n}}_a(\vec{r},\omega)\right|^2=w_a^{potl}(\vec{r},\omega)/w_a^{kin}(\vec{r},\omega)=\left|\tilde{\tilde{c}}_a(\vec{r},\omega)\right|^2/c^2=\left|\tilde{\tilde{z}}_a(\vec{r},\omega)\right|^2/z_o^2=1$.

Since the {***purely real***} ***frequency-domain*** acoustic total energy density is the sum of acoustic ***kinetic*** and ***potential*** energy densities, we see that:

$$w_a^{tot}(\vec{r},\omega)=w_a^{kin}(\vec{r},\omega)+w_a^{potl}(\vec{r},\omega)=\frac{\tilde{\tilde{I}}_a(\vec{r},\omega)}{2\tilde{\tilde{c}}_a(\vec{r},\omega)}+\frac{\tilde{\tilde{I}}_a(\vec{r},\omega)}{2\left(c^2/\left|\tilde{\tilde{c}}_a(\vec{r},\omega)\right|^2\right)\cdot\tilde{\tilde{c}}_a(\vec{r},\omega)}$$

$$=\frac{\tilde{\tilde{I}}_a(\vec{r},\omega)}{2\tilde{\tilde{c}}_a(\vec{r},\omega)}+\frac{\tilde{\tilde{I}}_a(\vec{r},\omega)\cdot\left(\left|\tilde{\tilde{c}}_a(\vec{r},\omega)\right|^2/c^2\right)}{2\tilde{\tilde{c}}_a(\vec{r},\omega)}=\frac{1}{2}\left[1+\frac{\left|\tilde{\tilde{c}}_a(\vec{r},\omega)\right|^2}{c^2}\right]\cdot\frac{\tilde{\tilde{I}}_a(\vec{r},\omega)}{\tilde{\tilde{c}}_a(\vec{r},\omega)}$$

Thus, we ***also*** see that:

$$\boxed{\tilde{\tilde{I}}_a(\vec{r},\omega)=\frac{\tilde{\tilde{c}}_a(\vec{r},\omega)\cdot w_a^{tot}(\vec{r},\omega)}{\tfrac{1}{2}\left[1+\left|\tilde{\tilde{c}}_a(\vec{r},\omega)\right|^2/c^2\right]}} \quad \text{where:} \quad \boxed{\tilde{\tilde{c}}_a(\vec{r},\omega)=\frac{\tilde{\tilde{z}}_a(\vec{r},\omega)}{\rho_o}}$$

or equivalently:

$$\boxed{\tilde{\tilde{I}}_a(\vec{r},\omega)=\frac{\tilde{\tilde{n}}_a(\vec{r},\omega)\cdot c\cdot w_a^{tot}(\vec{r},\omega)}{\tfrac{1}{2}\left[1+\left|\tilde{\tilde{n}}_a(\vec{r},\omega)\right|^2\right]}} \quad \text{where:} \quad \boxed{\tilde{\tilde{n}}_a(\vec{r},\omega)\equiv\frac{\tilde{\tilde{c}}_a(\vec{r},\omega)}{c}=\frac{\tilde{\tilde{z}}_a(\vec{r},\omega)}{z_o}}$$

From energy {density} conservation: $w_a^{tot}(\vec{r},\omega)=w_a^{kin}(\vec{r},\omega)+w_a^{potl}(\vec{r},\omega)$, we can also define {***purely real***} ***frequency-domain fractional*** acoustic kinetic and potential energy densities:

$$\boxed{f_a^{kin}(\vec{r},\omega)\equiv w_a^{kin}(\vec{r},\omega)/w_a^{tot}(\vec{r},\omega)} \quad \text{and:} \quad \boxed{f_a^{potl}(\vec{r},\omega)\equiv w_a^{potl}(\vec{r},\omega)/w_a^{tot}(\vec{r},\omega)}$$

with: $$\boxed{f_a^{kin}(\vec{r},\omega)+f_a^{potl}(\vec{r},\omega)=w_a^{kin}(\vec{r},\omega)/w_a^{tot}(\vec{r},\omega)+w_a^{potl}(\vec{r},\omega)/w_a^{tot}(\vec{r},\omega)=1}$$

Using the relation: $\left|\tilde{\tilde{n}}_a(\vec{r},\omega)\right|^2 = w_a^{potl}(\vec{r},\omega)/w_a^{kin}(\vec{r},\omega)$, we see that:

$$\boxed{\left|\tilde{\tilde{n}}_a(\vec{r},\omega)\right|^2 = \frac{w_a^{potl}(\vec{r},\omega)}{w_a^{kin}(\vec{r},\omega)} = \frac{w_a^{potl}(\vec{r},\omega)/w_a^{tot}(\vec{r},\omega)}{w_a^{kin}(\vec{r},\omega)/w_a^{tot}(\vec{r},\omega)} = \frac{f_a^{potl}(\vec{r},\omega)}{f_a^{kin}(\vec{r},\omega)}}$$

Or equivalently: $\boxed{f_a^{potl}(\vec{r},\omega) = \left|\tilde{\tilde{n}}_a(\vec{r},\omega)\right|^2 f_a^{kin}(\vec{r},\omega)}$ and/or: $\boxed{w_a^{potl}(\vec{r},\omega) = \left|\tilde{\tilde{n}}_a(\vec{r},\omega)\right|^2 w_a^{kin}(\vec{r},\omega)}$

As we have discussed before, the physical meaning of the ***real*** parts of $\tilde{\tilde{I}}_a(\vec{r},\omega)$, $\tilde{\tilde{z}}_a(\vec{r},\omega)$ and hence the ***real*** parts of $\tilde{\tilde{c}}_a(\vec{r},\omega)$ and $\tilde{\tilde{n}}_a(\vec{r},\omega)$ are associated with ***propagating*** sound radiation, whereas the ***imaginary*** parts of $\tilde{\tilde{I}}_a(\vec{r},\omega)$, $\tilde{\tilde{z}}_a(\vec{r},\omega)$ and hence the ***imaginary*** parts of $\tilde{\tilde{c}}_a(\vec{r},\omega)$ and $\tilde{\tilde{n}}_a(\vec{r},\omega)$ are associated with ***non-propagating*** acoustic energy, locally sloshing back and forth each cycle of oscillation.

Since energy and energy densities are ***additive*** scalar, ***purely real***, positive quantities, we have from energy conservation:

$$w_a^{tot}(\vec{r},\omega) = w_a^{kin}(\vec{r},\omega) + w_a^{potl}(\vec{r},\omega)$$

But we ***also*** have the ***additive*** scalar, ***purely real*** relation: $w_a^{tot}(\vec{r},\omega) = w_a^{rad}(\vec{r},\omega) + w_a^{virt}(\vec{r},\omega)$ where $w_a^{rad}(\vec{r},\omega)$ is the acoustic energy density associated with ***propagating*** sound radiation and $w_a^{virt}(\vec{r},\omega)$ is the acoustic energy density associated with ***non-propagating*** acoustic energy, locally sloshing back and forth each cycle of oscillation.

We can thus define: $f_a^{rad}(\vec{r},\omega) \equiv w_a^{rad}(\vec{r},\omega)/w_a^{tot}(\vec{r},\omega)$ and $f_a^{virt}(\vec{r},\omega) \equiv w_a^{virt}(\vec{r},\omega)/w_a^{tot}(\vec{r},\omega)$ as the {***purely real***} ***frequency-domain fractional*** energy densities associated with ***propagating*** sound radiation and ***non-propagating*** acoustic energy, respectively.

Note also that we must have: $f_a^{rad}(\vec{r},\omega) + f_a^{virt}(\vec{r},\omega) = \left[w_a^{rad}(\vec{r},\omega) + w_a^{virt}(\vec{r},\omega)\right]/w_a^{tot}(\vec{r},\omega) = 1$.

Now:

$$1 = \frac{\tilde{\tilde{I}}_a(\vec{r},\omega)}{\tilde{\tilde{I}}_a(\vec{r},\omega)} = \frac{\cancel{\tilde{\tilde{n}}_a(\vec{r},\omega)} \cdot \cancel{c} \cdot w_a^{tot}(\vec{r},\omega)}{\cancel{\tfrac{1}{2}}\left[1+\left|\tilde{\tilde{n}}_a(\vec{r},\omega)\right|^2\right]} \Bigg/ \frac{\cancel{\tilde{\tilde{n}}_a(\vec{r},\omega)} \cdot \cancel{c} \cdot w_a^{tot}(\vec{r},\omega)}{\cancel{\tfrac{1}{2}}\left[1+\left|\tilde{\tilde{n}}_a(\vec{r},\omega)\right|^2\right]} = \frac{\left[1+\left|\tilde{\tilde{n}}_a(\vec{r},\omega)\right|^2\right]}{\left[1+\left|\tilde{\tilde{n}}_a(\vec{r},\omega)\right|^2\right]} \Bigg/ \frac{w_a^{tot}(\vec{r},\omega)}{w_a^{tot}(\vec{r},\omega)}$$

Or:

$$1 = \frac{w_a^{tot}(\vec{r},\omega)}{w_a^{tot}(\vec{r},\omega)} = \frac{\left[1+\left|\tilde{\tilde{n}}_a(\vec{r},\omega)\right|^2\right]}{\left[1+\left|\tilde{\tilde{n}}_a(\vec{r},\omega)\right|^2\right]}$$

We use $w_a^{tot}(\vec{r},\omega) = w_a^{rad}(\vec{r},\omega) + w_a^{virt}(\vec{r},\omega)$ for the numerator of the LHS of this relation, and write $|\tilde{\tilde{n}}_a(\vec{r},\omega)|^2 = \text{Re}\{\tilde{\tilde{n}}_a(\vec{r},\omega)\}^2 + \text{Im}\{\tilde{\tilde{n}}_a(\vec{r},\omega)\}^2 \equiv n_a^{r\,2}(\vec{r},\omega) + n_a^{i\,2}(\vec{r},\omega)$ for the numerator of the RHS of this relation:

$$\frac{w_a^{rad}(\vec{r},\omega) + w_a^{virt}(\vec{r},\omega)}{w_a^{tot}(\vec{r},\omega)} = f_a^{rad}(\vec{r},\omega) + f_a^{virt}(\vec{r},\omega) = \frac{\left[1 + n_a^{r\,2}(\vec{r},\omega) + n_a^{i\,2}(\vec{r},\omega)\right]}{\left[1 + |\tilde{\tilde{n}}_a(\vec{r},\omega)|^2\right]}$$

Then: $$\left[1 + |\tilde{\tilde{n}}_a(\vec{r},\omega)|^2\right]\left[f_a^{rad}(\vec{r},\omega) + f_a^{virt}(\vec{r},\omega)\right] = \left[1 + n_a^{r\,2}(\vec{r},\omega) + n_a^{i\,2}(\vec{r},\omega)\right]$$

Or: $$\underbrace{\left[f_a^{rad}(\vec{r},\omega) + f_a^{virt}(\vec{r},\omega)\right]}_{\equiv 1} + |\tilde{\tilde{n}}_a(\vec{r},\omega)|^2\left[f_a^{rad}(\vec{r},\omega) + f_a^{virt}(\vec{r},\omega)\right] = \left[1 + n_a^{r\,2}(\vec{r},\omega) + n_a^{i\,2}(\vec{r},\omega)\right]$$

Thus: $$\cancel{1} + |\tilde{\tilde{n}}_a(\vec{r},\omega)|^2\left[f_a^{rad}(\vec{r},\omega) + f_a^{virt}(\vec{r},\omega)\right] = \cancel{1} + n_a^{r\,2}(\vec{r},\omega) + n_a^{i\,2}(\vec{r},\omega)$$

Or: $$|\tilde{\tilde{n}}_a(\vec{r},\omega)|^2\left[f_a^{rad}(\vec{r},\omega) + f_a^{virt}(\vec{r},\omega)\right] = n_a^{r\,2}(\vec{r},\omega) + n_a^{i\,2}(\vec{r},\omega)$$

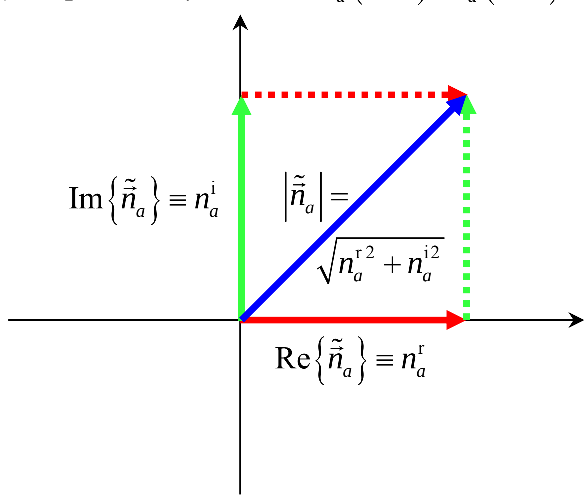
Or: $$\boxed{f_a^{rad}(\vec{r},\omega) + f_a^{virt}(\vec{r},\omega) = \frac{n_a^{r\,2}(\vec{r},\omega)}{|\tilde{\tilde{n}}_a(\vec{r},\omega)|^2} + \frac{n_a^{i\,2}(\vec{r},\omega)}{|\tilde{\tilde{n}}_a(\vec{r},\omega)|^2} = \frac{n_a^{r\,2}(\vec{r},\omega) + n_a^{i\,2}(\vec{r},\omega)}{|\tilde{\tilde{n}}_a(\vec{r},\omega)|^2} = 1}$$

Thus we see that:

$$\boxed{f_a^{rad}(\vec{r},\omega) \equiv \frac{w_a^{rad}(\vec{r},\omega)}{w_a^{tot}(\vec{r},\omega)} = \frac{n_a^{r\,2}(\vec{r},\omega)}{|\tilde{\tilde{n}}_a(\vec{r},\omega)|^2} = \frac{c_a^{r\,2}(\vec{r},\omega)}{|\tilde{\tilde{c}}_a(\vec{r},\omega)|^2} = \frac{z_a^{r\,2}(\vec{r},\omega)}{|\tilde{\tilde{z}}_a(\vec{r},\omega)|^2} = \frac{I_a^{r\,2}(\vec{r},\omega)}{|\tilde{\tilde{I}}_a(\vec{r},\omega)|^2}}$$

$$\boxed{f_a^{virt}(\vec{r},\omega) \equiv \frac{w_a^{virt}(\vec{r},\omega)}{w_a^{tot}(\vec{r},\omega)} = \frac{n_a^{i\,2}(\vec{r},\omega)}{|\tilde{\tilde{n}}_a(\vec{r},\omega)|^2} = \frac{c_a^{i\,2}(\vec{r},\omega)}{|\tilde{\tilde{c}}_a(\vec{r},\omega)|^2} = \frac{z_a^{i\,2}(\vec{r},\omega)}{|\tilde{\tilde{z}}_a(\vec{r},\omega)|^2} = \frac{I_a^{i\,2}(\vec{r},\omega)}{|\tilde{\tilde{I}}_a(\vec{r},\omega)|^2}}$$

Physical understanding of these relations can also be gained from the phasor diagram of the ***frequency-domain*** $\tilde{\tilde{n}}_a(\vec{r},\omega)$ {or equivalently, that for $\tilde{\tilde{c}}_a(\vec{r},\omega)$, $\tilde{\tilde{z}}_a(\vec{r},\omega)$ or $\tilde{\tilde{I}}_a(\vec{r},\omega)$}:



While the real (imaginary) parts of the ***frequency-domain*** 3-D vector $\tilde{\vec{n}}_a(\vec{r},\omega)$ amplitude (or equivalently, the real (imaginary) parts of the ***frequency-domain*** 3-D vector $\tilde{\vec{c}}_a(\vec{r},\omega)$, $\tilde{\vec{z}}_a(\vec{r},\omega)$ or $\tilde{\vec{I}}_a(\vec{r},\omega)$ "amplitudes") ***are*** physically associated with ***propagating*** sound radiation (***non-propagating*** acoustic energy), respectively, the ***fractional*** amounts of ***propagating*** versus ***non-propagating*** ***purely real***, ***scalar*** acoustic energy density ($f_a^{rad}(\vec{r},\omega)$ *vs.* $f_a^{virt}(\vec{r},\omega)$) ***must*** be based on the ***additive*** component^2 nature associated with the right triangle relation of the above phasor diagram for $\tilde{\vec{n}}_a(\vec{r},\omega)$ - *i.e.* Pythagoras' theorem:

$$\left|\tilde{\vec{n}}_a\right|^2 = n_a^{r\,2} + n_a^{i\,2} \;\Rightarrow\; f_a^{rad} + f_a^{virt} = \frac{n_a^{r\,2}}{\left|\tilde{\vec{n}}_a\right|^2} + \frac{n_a^{i\,2}}{\left|\tilde{\vec{n}}_a\right|^2}$$

Next, we explore relations between $\left\{ f_a^{kin}(\vec{r},\omega), f_a^{potl}(\vec{r},\omega) \right\}$ and $\left\{ f_a^{rad}(\vec{r},\omega), f_a^{virt}(\vec{r},\omega) \right\}$, or equivalently between $\left\{ w_a^{kin}(\vec{r},\omega), w_a^{potl}(\vec{r},\omega) \right\}$ and $\left\{ w_a^{rad}(\vec{r},\omega), w_a^{virt}(\vec{r},\omega) \right\}$. From conservation of energy, we have:

$$w_a^{tot}(\vec{r},\omega) = w_a^{kin}(\vec{r},\omega) + w_a^{potl}(\vec{r},\omega) = w_a^{rad}(\vec{r},\omega) + w_a^{virt}(\vec{r},\omega)$$

Or equivalently: $\quad 1 = f_a^{kin}(\vec{r},\omega) + f_a^{potl}(\vec{r},\omega) = f_a^{rad}(\vec{r},\omega) + f_a^{virt}(\vec{r},\omega)$

Using the relation: $f_a^{potl}(\vec{r},\omega) = \left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2 f_a^{kin}(\vec{r},\omega)$ we can rewrite the above relation as:

$$1 = f_a^{kin}(\vec{r},\omega) + \left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2 f_a^{kin}(\vec{r},\omega) = f_a^{rad}(\vec{r},\omega) + f_a^{virt}(\vec{r},\omega)$$

Thus: $$\left[1 + \left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2\right] f_a^{kin}(\vec{r},\omega) = f_a^{rad}(\vec{r},\omega) + f_a^{virt}(\vec{r},\omega)$$

Or: $$\boxed{f_a^{kin}(\vec{r},\omega) = \frac{1}{\left[1 + \left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2\right]} \underbrace{\left[f_a^{rad}(\vec{r},\omega) + f_a^{virt}(\vec{r},\omega) \right]}_{=1} = \frac{1}{\left[1 + \left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2\right]}}$$

Or equivalently:

$$\boxed{w_a^{kin}(\vec{r},\omega) = \frac{1}{\left[1 + \left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2\right]} \underbrace{\left[w_a^{rad}(\vec{r},\omega) + w_a^{virt}(\vec{r},\omega) \right]}_{= w_a^{tot}(\vec{r},\omega)} = \frac{1}{\left[1 + \left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2\right]} w_a^{tot}(\vec{r},\omega)}$$

Using the (same) relation for: $f_a^{kin}(\vec{r},\omega) = f_a^{potl}(\vec{r},\omega) \Big/ \left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2$, we also have:

$$1 = f_a^{potl}(\vec{r},\omega) \Big/ \left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2 + f_a^{potl}(\vec{r},\omega) = f_a^{rad}(\vec{r},\omega) + f_a^{virt}(\vec{r},\omega)$$

Thus: $\left[\dfrac{1}{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}+1\right] f_a^{potl}(\vec{r},\omega)=\left[\dfrac{1+\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}\right] f_a^{potl}(\vec{r},\omega)=f_a^{rad}(\vec{r},\omega)+f_a^{virt}(\vec{r},\omega)$

Or:

$$f_a^{potl}(\vec{r},\omega)=\left[\frac{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}{1+\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}\right]\underbrace{\left[f_a^{rad}(\vec{r},\omega)+f_a^{virt}(\vec{r},\omega)\right]}_{=1}=\left[\frac{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}{1+\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}\right]$$

Or equivalently:

$$w_a^{potl}(\vec{r},\omega)=\left[\frac{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}{1+\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}\right]\underbrace{\left[w_a^{rad}(\vec{r},\omega)+w_a^{virt}(\vec{r},\omega)\right]}_{=w_a^{tot}(\vec{r},\omega)}=\left[\frac{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}{1+\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}\right]w_a^{tot}(\vec{r},\omega)$$

Then as a check: $f_a^{kin}(\vec{r},\omega)+f_a^{potl}(\vec{r},\omega)=\dfrac{1}{1+\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}+\dfrac{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}{1+\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}=\dfrac{1+\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}{1+\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}=1\ \surd$

Thus, we have obtained the relations:

$$f_a^{kin}(\vec{r},\omega)=\frac{1}{\left[1+\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2\right]}\underbrace{\left[f_a^{rad}(\vec{r},\omega)+f_a^{virt}(\vec{r},\omega)\right]}_{=1}=\frac{1}{\left[1+\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2\right]}$$

and:

$$f_a^{potl}(\vec{r},\omega)=\left[\frac{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}{1+\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}\right]\underbrace{\left[f_a^{rad}(\vec{r},\omega)+f_a^{virt}(\vec{r},\omega)\right]}_{=1}=\left[\frac{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}{1+\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}\right]$$

Or equivalently:

$$w_a^{kin}(\vec{r},\omega)=\frac{1}{\left[1+\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2\right]}\underbrace{\left[w_a^{rad}(\vec{r},\omega)+w_a^{virt}(\vec{r},\omega)\right]}_{=w_a^{tot}(\vec{r},\omega)}=\frac{1}{\left[1+\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2\right]}w_a^{tot}(\vec{r},\omega)$$

$$w_a^{potl}(\vec{r},\omega)=\left[\frac{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}{1+\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}\right]\underbrace{\left[w_a^{rad}(\vec{r},\omega)+w_a^{virt}(\vec{r},\omega)\right]}_{=w_a^{tot}(\vec{r},\omega)}=\left[\frac{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}{1+\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}\right]w_a^{tot}(\vec{r},\omega)$$

Physically, from these relations we learn that the energy densities associated with ***propagating*** sound radiation and ***non-propagating*** acoustical energy {locally sloshing back and forth each cycle of oscillation} contribute ***linear-proportionally*** to the acoustic kinetic and potential energy densities.

We can equivalently write these relations in matrix form as:

$$\begin{pmatrix} f_a^{kin}(\vec{r},\omega) \\ f_a^{potl}(\vec{r},\omega) \end{pmatrix} = \frac{1}{\left[1+\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2\right]} \begin{pmatrix} 1 & 1 \\ \left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2 & \left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2 \end{pmatrix} \begin{pmatrix} f_a^{rad}(\vec{r},\omega) \\ f_a^{virt}(\vec{r},\omega) \end{pmatrix}$$

Note that the determinant of this 2×2 matrix is ***zero***, *i.e.* this 2×2 matrix is ***singular***, thus it has ***no*** inverse…

However, since: $f_a^{rad}(\vec{r},\omega) \equiv \dfrac{w_a^{rad}(\vec{r},\omega)}{w_a^{tot}(\vec{r},\omega)} = \dfrac{n_a^{r\,2}(\vec{r},\omega)}{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}$ and: $f_a^{virt}(\vec{r},\omega) \equiv \dfrac{w_a^{virt}(\vec{r},\omega)}{w_a^{tot}(\vec{r},\omega)} = \dfrac{n_a^{i\,2}(\vec{r},\omega)}{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}$

Or, equivalently: $w_a^{rad}(\vec{r},\omega) = \dfrac{n_a^{r\,2}(\vec{r},\omega)}{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2} w_a^{tot}(\vec{r},\omega)$ and: $w_a^{virt}(\vec{r},\omega) = \dfrac{n_a^{i\,2}(\vec{r},\omega)}{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2} w_a^{tot}(\vec{r},\omega)$

Then, using: $w_a^{tot}(\vec{r},\omega) = w_a^{kin}(\vec{r},\omega) + w_a^{potl}(\vec{r},\omega)$ on the RHS of both relations, we have:

$$w_a^{rad}(\vec{r},\omega) = \frac{n_a^{r\,2}(\vec{r},\omega)}{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2} w_a^{tot}(\vec{r},\omega) = \frac{n_a^{r\,2}(\vec{r},\omega)}{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}\left[w_a^{kin}(\vec{r},\omega) + w_a^{potl}(\vec{r},\omega)\right]$$

and:

$$w_a^{virt}(\vec{r},\omega) = \frac{n_a^{i\,2}(\vec{r},\omega)}{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2} w_a^{tot}(\vec{r},\omega) = \frac{n_a^{i\,2}(\vec{r},\omega)}{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}\left[w_a^{kin}(\vec{r},\omega) + w_a^{potl}(\vec{r},\omega)\right]$$

Dividing both sides of these relations by $w_a^{tot}(\vec{r},\omega)$, we equivalently have:

$$f_a^{rad}(\vec{r},\omega) = \frac{n_a^{r\,2}(\vec{r},\omega)}{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}\underbrace{\left[f_a^{kin}(\vec{r},\omega) + f_a^{potl}(\vec{r},\omega)\right]}_{=1} = \frac{n_a^{r\,2}(\vec{r},\omega)}{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}$$

and:

$$f_a^{virt}(\vec{r},\omega) = \frac{n_a^{i\,2}(\vec{r},\omega)}{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}\underbrace{\left[f_a^{kin}(\vec{r},\omega) + f_a^{potl}(\vec{r},\omega)\right]}_{=1} = \frac{n_a^{i\,2}(\vec{r},\omega)}{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}$$

Then as a check, we can easily see that:

$$f_a^{rad}(\vec{r},\omega) + f_a^{virt}(\vec{r},\omega) = \frac{n_a^{r\,2}(\vec{r},\omega)}{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2} + \frac{n_a^{i\,2}(\vec{r},\omega)}{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2} = \frac{n_a^{r\,2}(\vec{r},\omega) + n_a^{i\,2}(\vec{r},\omega)}{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2} = \frac{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2}{\left|\tilde{\vec{n}}_a(\vec{r},\omega)\right|^2} = 1 \quad \surd$$

We can equivalently write these relations in matrix form as:

$$\begin{pmatrix} f_a^{rad}(\vec{r},\omega) \\ f_a^{virt}(\vec{r},\omega) \end{pmatrix} = \frac{1}{\left|\tilde{\tilde{n}}_a(\vec{r},\omega)\right|^2} \begin{pmatrix} n_a^{r\,2}(\vec{r},\omega) & n_a^{r\,2}(\vec{r},\omega) \\ n_a^{i\,2}(\vec{r},\omega) & n_a^{i\,2}(\vec{r},\omega) \end{pmatrix} \begin{pmatrix} f_a^{kin}(\vec{r},\omega) \\ f_a^{potl}(\vec{r},\omega) \end{pmatrix}$$

Note that the determinant of this 2×2 matrix is ***zero***, *i.e.* this 2×2 matrix is ***singular***, thus it has ***no*** inverse…

Physically, we learn from these relations that the acoustic kinetic and potential energy densities contribute ***linear-proportionally*** to the energy densities associated with ***propagating*** sound radiation and ***non-propagating*** acoustical energy {locally sloshing back and forth 2× each cycle of oscillation}.

We can now see/understand better why the 2×2 matrices in both cases are singular – because the physics associated with how (kinetic *vs*. potential) energy density is parceled out into (propagating *vs*. non-propagating) energy density {and/or *vice-versa*} is ***independent*** of the ***form/types*** of the energy densities!

Conservation of Acoustic Linear Momentum:

In the linearized "every-day" sound field regime (sound pressure levels $SPL(\vec{r}) \ll 134\,dB$, corresponding to $\left|\tilde{p}(\vec{r})\right| \ll 100\,Pascals$, and neglecting dissipative effects), if a complex ***harmonic*** sound field $\tilde{S}(\vec{r},t;\omega)$ exists, with: $\tilde{p}(\vec{r},t;\omega) = \tilde{p}(\vec{r},\omega)e^{i\omega t} = \left|\tilde{p}(\vec{r},\omega)\right| e^{i\varphi_p(\vec{r},\omega)} \cdot e^{i\omega t}$ and: $\tilde{\vec{u}}(\vec{r},t;\omega) = \tilde{\vec{u}}(\vec{r},\omega)e^{i\omega t} = \left[\left|\tilde{u}_x(\vec{r},\omega)\right| e^{i\varphi_{u_x}(\vec{r},\omega)}\hat{x} + \left|\tilde{u}_y(\vec{r},\omega)\right| e^{i\varphi_{u_y}(\vec{r},\omega)}\hat{y} + \left|\tilde{u}_z(\vec{r},\omega)\right| e^{i\varphi_{u_z}(\vec{r},\omega)}\hat{z}\right] \cdot e^{i\omega t}$ in a volume *V* (with associated enclosing surface *S*) of interest, if no acousto-mechanical excitation sources (or sinks) exist within the volume *V*/enclosing surface *S*, the net/total complex acoustic force acting on the air within *V*, by Newton's 2nd law is: $\tilde{\vec{F}}_a(t;\omega) = d\tilde{\vec{G}}_a(t;\omega)/dt$ $(Newtons)$ where $\tilde{\vec{G}}_a(t;\omega)$ $(Newton\text{-}sec = kg\text{-}m/sec)$ is the total complex acoustic linear momentum contained within the volume *V*/enclosing surface *S*.

We can write: $\tilde{\vec{F}}_a(t;\omega) = \int_V \tilde{\vec{f}}_a(\vec{r},t;\omega)\,d\tau$, where $\tilde{\vec{f}}_a(\vec{r},t;\omega)$ (N/m^3) is the complex acoustic force density at $(\vec{r},t;\omega)$. Likewise: $\tilde{\vec{G}}_a(t;\omega) = \int_V \tilde{\vec{g}}_a(\vec{r},t;\omega)\,d\tau$, where $\tilde{\vec{g}}_a(\vec{r},t;\omega)$ $(N\text{-}sec/m^3 = kg/m^2\text{-}sec)$ is the complex acoustic linear momentum density at $(\vec{r},t;\omega)$. Thus: $\tilde{\vec{f}}_a(\vec{r},t;\omega) = \partial\tilde{\vec{g}}_a(\vec{r},t;\omega)/\partial t$.

It can be shown that the complex acoustic force density $\tilde{\vec{f}}_a(\vec{r},t;\omega) = \partial\tilde{\vec{g}}_a(\vec{r},t;\omega)/\partial t = -\vec{\nabla}\cdot\tilde{\vec{T}}_a(\vec{r},t;\omega)$, where $\tilde{\vec{T}}_a(\vec{r},t;\omega)$ is the so-called complex acoustic stress tensor, a Hermitian rank-2 tensor (a 3×3 matrix), *i.e.* $\tilde{\vec{T}}_a(\vec{r},t;\omega) = \tilde{\vec{T}}_a^{\dagger}(\vec{r},t;\omega)$

where $\tilde{\tilde{T}}_a^\dagger(\vec{r},t;\omega)$ is the conjugate transpose of $\tilde{\tilde{T}}_a(\vec{r},t;\omega)$. The acoustic stress tensor $\tilde{\tilde{T}}_a(\vec{r},t;\omega)$ has SI units of pressure – *i.e.* $N/m^2 = Pascals$ {*n.b.* same units as energy density $(Joules/m^3)$}. Temporarily suppressing the space-time-frequency argument $(\vec{r},t;\omega)$, the acoustic stress tensor $\tilde{\tilde{T}}_a$ is constructed as follows (*n.b.* stress tensors are {always} expressed in Cartesian (*x*, *y*, *z*) components}: $\tilde{\tilde{T}}_a = \frac{1}{2}\rho_o\tilde{\vec{u}}\otimes\tilde{\vec{u}}^* - \mathcal{L}_a\hat{1}$ where the symbol $\otimes$ represents the tensor product (*aka* outer, or dyadic product), the Lagrangian acoustic density $\mathcal{L}_a \equiv w_a^{kin} - w_a^{potl}$ $(Joules/m^3)$ {*n.b.* a purely real quantity) and $\hat{1}$ is the 3×3 unit matrix.

For reference purposes, the tensor/outer/dyadic product of any two arbitrary 3-D space vectors $\vec{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$ and $\vec{b} = b_x\hat{x} + b_y\hat{y} + b_z\hat{z}$, representing each of the two 3-D space vectors as 3×1 column matrices: $\underline{a} = \begin{pmatrix} a_x\hat{x} \\ a_y\hat{y} \\ a_z\hat{z}\end{pmatrix}$ and $\underline{b} = \begin{pmatrix} b_x\hat{x} \\ b_y\hat{y} \\ b_z\hat{z}\end{pmatrix}$ is:

$$\vec{a}\otimes\vec{b} = \underline{a}\,\underline{b}^T = \begin{pmatrix} a_x\hat{x} \\ a_y\hat{y} \\ a_z\hat{z}\end{pmatrix}\begin{pmatrix} b_x\hat{x} & b_y\hat{y} & b_z\hat{z}\end{pmatrix} = \begin{pmatrix} a_xb_x\hat{x}\hat{x} & a_xb_y\hat{x}\hat{y} & a_xb_z\hat{x}\hat{z} \\ a_yb_x\hat{y}\hat{x} & a_yb_y\hat{y}\hat{y} & a_yb_z\hat{y}\hat{z} \\ a_zb_x\hat{z}\hat{x} & a_zb_y\hat{z}\hat{y} & a_zb_z\hat{z}\hat{z}\end{pmatrix}$$

The *i-j*th component of the complex acoustic stress tensor (*i*, *j* = 1, 2, 3 {= *x*, *y*, *z*}) is thus:

$$\boxed{\tilde{T}_{a_{ij}} = \tfrac{1}{2}\rho_o\tilde{u}_i\tilde{u}_j^*\hat{i}\,\hat{j} - \tfrac{1}{4}\rho_o\left(\tfrac{|\tilde{p}|^2}{z_o^2} + |\tilde{u}|^2\right)\delta_{ij}\hat{i}\,\hat{j} \;=\; \tfrac{1}{2}\rho_o\tilde{u}_i\tilde{u}_j^*\hat{i}\,\hat{j} - \tfrac{1}{4}\rho_o\left(\tfrac{|\tilde{p}|^2}{\rho_o^2c^2} + |\tilde{u}|^2\right)\delta_{ij}\hat{i}\,\hat{j}}$$

where: $\delta_{ij} = \begin{cases}1 \text{ if } i=j \\ 0 \text{ if } i\neq j\end{cases}$ is the Kroenecker δ-function. Explicitly writing out $\tilde{\tilde{T}}_a$ in matrix form:

$$\boxed{\tilde{\tilde{T}}_a = \tfrac{1}{4}\rho_o\begin{pmatrix} \left[|\tilde{u}_x|^2 - |\tilde{u}_y|^2 - |\tilde{u}_z|^2 + \tfrac{|\tilde{p}|^2}{\rho_o^2c^2}\right]\hat{x}\hat{x} & 2\tilde{u}_x\cdot u_y^*\hat{x}\hat{y} & 2\tilde{u}_x\cdot u_z^*\hat{x}\hat{z} \\ 2\tilde{u}_y\cdot u_x^*\hat{y}\hat{x} & \left[|\tilde{u}_y|^2 - |\tilde{u}_z|^2 - |\tilde{u}_x|^2 + \tfrac{|\tilde{p}|^2}{\rho_o^2c^2}\right]\hat{y}\hat{y} & 2\tilde{u}_y\cdot u_z^*\hat{y}\hat{z} \\ 2\tilde{u}_z\cdot u_x^*\hat{z}\hat{x} & 2\tilde{u}_z\cdot u_y^*\hat{z}\hat{y} & \left[|\tilde{u}_z|^2 - |\tilde{u}_x|^2 - |\tilde{u}_y|^2 + \tfrac{|\tilde{p}|^2}{\rho_o^2c^2}\right]\hat{z}\hat{z}\end{pmatrix}}$$

The 3 diagonal elements of the acoustic pressure tensor $\tilde{T}_{a_{ii}}\hat{i}\,\hat{i}$ {*n.b.* purely real quantities} are physically interpreted as *pressures* acting in the *i*th direction on a surface with outward-pointing unit normal in the *i*th direction, whereas the 6 off-diagonal elements $\tilde{T}_{a_{ij}}\hat{i}\,\hat{j} = \tilde{T}_{a_{ji}}^*\hat{j}\,\hat{i}$ {*n.b.* in general complex} are physically interpreted as *shears* – an areal force density (*i.e.* force per unit area) acting in the *i*th direction on a surface with outward-pointing unit normal in the *j*th direction.

Then:

$$-\vec{\nabla}\bullet\tilde{\vec{\vec{T}}}_a = -\tfrac{1}{4}\rho_o\left(\frac{\partial}{\partial x}\hat{x} \quad \frac{\partial}{\partial y}\hat{y} \quad \frac{\partial}{\partial z}\hat{z}\right)\times$$

$$\begin{pmatrix} \left[|\tilde{u}_x|^2-|\tilde{u}_y|^2-|\tilde{u}_z|^2+\frac{|\tilde{p}|^2}{\rho_o^2c^2}\right]\hat{x}\hat{x} & 2\tilde{u}_x\cdot u_y^*\hat{x}\hat{y} & 2\tilde{u}_x\cdot u_z^*\hat{x}\hat{z} \\ 2\tilde{u}_y\cdot u_x^*\hat{y}\hat{x} & \left[|\tilde{u}_y|^2-|\tilde{u}_z|^2-|\tilde{u}_x|^2+\frac{|\tilde{p}|^2}{\rho_o^2c^2}\right]\hat{y}\hat{y} & 2\tilde{u}_y\cdot u_z^*\hat{y}\hat{z} \\ 2\tilde{u}_z\cdot u_x^*\hat{z}\hat{x} & 2\tilde{u}_z\cdot u_y^*\hat{z}\hat{y} & \left[|\tilde{u}_z|^2-|\tilde{u}_x|^2-|\tilde{u}_y|^2+\frac{|\tilde{p}|^2}{\rho_o^2c^2}\right]\hat{z}\hat{z} \end{pmatrix}$$

$$= -\tfrac{1}{4}\rho_o\begin{pmatrix} \frac{\partial}{\partial x}\left[|\tilde{u}_x|^2-|\tilde{u}_y|^2-|\tilde{u}_z|^2+\frac{|\tilde{p}|^2}{\rho_o^2c^2}\right]\hat{x}+\frac{\partial}{\partial y}\left[2\tilde{u}_y\cdot u_x^*\right]\hat{x}+\frac{\partial}{\partial z}\left[2\tilde{u}_z\cdot u_x^*\right]\hat{x} \\ \frac{\partial}{\partial x}\left[2\tilde{u}_x\cdot u_y^*\right]\hat{y}+\frac{\partial}{\partial y}\left[|\tilde{u}_y|^2-|\tilde{u}_z|^2-|\tilde{u}_x|^2+\frac{|\tilde{p}|^2}{\rho_o^2c^2}\right]\hat{y}+\frac{\partial}{\partial z}\left[2\tilde{u}_z\cdot u_y^*\right]\hat{y} \\ \frac{\partial}{\partial x}\left[2\tilde{u}_x\cdot u_z^*\right]\hat{z}+\frac{\partial}{\partial y}\left[2\tilde{u}_y\cdot u_z^*\right]\hat{z}+\frac{\partial}{\partial z}\left[|\tilde{u}_z|^2-|\tilde{u}_x|^2-|\tilde{u}_y|^2+\frac{|\tilde{p}|^2}{\rho_o^2c^2}\right]\hat{z} \end{pmatrix}$$

After some more work on carrying out the *–ve* divergence of $\tilde{\vec{\vec{T}}}_a$, one {indeed} obtains:

$$\boxed{-\vec{\nabla}\bullet\tilde{\vec{\vec{T}}}_a = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\tfrac{1}{2}\tilde{p}\cdot\tilde{\vec{u}}^*\right) = \frac{1}{c^2}\frac{\partial\tilde{\vec{I}}_a}{\partial t}}$$

But the complex acoustic force density $\tilde{\vec{f}}_a = \partial\tilde{\vec{g}}_a/\partial t = -\vec{\nabla}\bullet\tilde{\vec{\vec{T}}}_a = \tfrac{1}{c^2}\partial\tilde{\vec{I}}_a/\partial t \ \left(N/m^3\right)$, thus we see that the complex acoustic linear momentum density is related to the complex acoustic intensity via: $\boxed{\tilde{\vec{g}}_a = \tilde{\vec{I}}_a/c^2}$, the acoustic analog of that *e.g.* for propagation of *EM* waves in electrodynamics, where the *EM* field linear momentum density $\tilde{\vec{g}}_{EM}\ \left(N\text{-}sec/m^3 = kg/m^2\text{-}sec\right)$ is related to Poynting's vector $\tilde{\vec{S}}_{EM} = \tfrac{1}{2}\tilde{E}\times\tilde{H}^*\left(Watts/m^2\right)$ via: $\boxed{\tilde{\vec{g}}_{EM} = \tilde{\vec{S}}_{EM}/c^2}$.

So: $\tilde{\vec{I}}_a = c^2\tilde{\vec{g}}_a$, but we also have the relation: $\tilde{\vec{I}}_a = \dfrac{\tilde{\vec{c}}_a\cdot w_a^{tot}}{\tfrac{1}{2}\left[1+\left|\tilde{\vec{c}}_a\right|^2/c^2\right]}$ where: $\tilde{\vec{c}}_a = \dfrac{\tilde{\vec{z}}_a}{\rho_o}$, thus:

$$c^2\tilde{\vec{g}}_a = \frac{\tilde{\vec{c}}_a\cdot w_a^{tot}}{\tfrac{1}{2}\left[1+\left|\tilde{\vec{c}}_a\right|^2/c^2\right]} \text{ or:}$$

$$w_a^{tot} = \tfrac{1}{2}\left[c^2+\left|\tilde{\vec{c}}_a\right|^2\right]\frac{\tilde{\vec{g}}_a}{\tilde{\vec{c}}_a} = \tfrac{1}{2}\left[c^2+\left|\tilde{\vec{c}}_a\right|^2\right]\frac{\tilde{\vec{g}}_a}{\tilde{\vec{c}}_a}\bullet\frac{\tilde{\vec{c}}_a^*}{\tilde{\vec{c}}_a^*} = \tfrac{1}{2}\left[c^2+\left|\tilde{\vec{c}}_a\right|^2\right]\frac{\tilde{\vec{g}}_a\bullet\tilde{\vec{c}}_a^*}{\left|\tilde{\vec{c}}_a\right|^2} = \tfrac{1}{2}\left[1+c^2/\left|\tilde{\vec{c}}_a\right|^2\right]\tilde{\vec{g}}_a\bullet\tilde{\vec{c}}_a^*$$

But: $w_a^{tot} = w_a^{potl} + w_a^{kin} = \frac{1}{4}\frac{|\tilde{p}|^2}{\rho_o c^2} + \frac{1}{4}\left|\tilde{\vec{u}}\right|^2 = \frac{1}{2}\left[1 + \frac{c^2}{|\tilde{\vec{c}}_a|^2}\right]\tilde{\vec{g}}_a \bullet \tilde{\vec{c}}_a^* = \frac{1}{2}\tilde{\vec{g}}_a \bullet \tilde{\vec{c}}_a^* + \frac{1}{2}\frac{c^2}{|\tilde{\vec{c}}_a|^2}\tilde{\vec{g}}_a \bullet \tilde{\vec{c}}_a^*$. If we {tentatively} assign: $w_a^{potl} = \frac{1}{4}\frac{|\tilde{p}|^2}{\rho_o c^2} = \frac{1}{2}\tilde{\vec{g}}_a \bullet \tilde{\vec{c}}_a^*$ and: $w_a^{kin} = \frac{1}{4}\left|\tilde{\vec{u}}\right|^2 = \frac{1}{2}\frac{c^2}{|\tilde{\vec{c}}_a|^2}\tilde{\vec{g}}_a \bullet \tilde{\vec{c}}_a^*$, then we can check these relations with the ratio: $\frac{w_a^{potl}}{w_a^{kin}} = \frac{\left|\tilde{\vec{c}}_a\right|^2}{c^2} = \frac{\left|\tilde{\vec{z}}_a\right|^2}{z_o^2} = \frac{\frac{1}{2}\tilde{\vec{g}}_a \bullet \tilde{\vec{c}}_a^*}{\frac{1}{2}\frac{c^2}{|\tilde{\vec{c}}_a|^2}\tilde{\vec{g}}_a \bullet \tilde{\vec{c}}_a^*} = \frac{\left|\tilde{\vec{c}}_a\right|^2}{c^2}$ √.

Thus, we also have the relations:

$$w_a^{tot} = w_a^{potl} + w_a^{kin} = \frac{1}{4}\frac{|\tilde{p}|^2}{\rho_o c^2} + \frac{1}{4}\left|\tilde{\vec{u}}\right|^2 = \frac{1}{2}\left[1 + \frac{c^2}{|\tilde{\vec{c}}_a|^2}\right]\tilde{\vec{g}}_a \bullet \tilde{\vec{c}}_a^* = \frac{1}{2}\tilde{\vec{g}}_a \bullet \tilde{\vec{c}}_a^* + \frac{1}{2}\frac{c^2}{|\tilde{\vec{c}}_a|^2}\tilde{\vec{g}}_a \bullet \tilde{\vec{c}}_a^*$$

where:

$$w_a^{potl} = \frac{1}{4}\frac{|\tilde{p}|^2}{\rho_o c^2} = \frac{1}{2}\tilde{\vec{g}}_a \bullet \tilde{\vec{c}}_a^* \quad \text{and:} \quad w_a^{kin} = \frac{1}{4}\left|\tilde{\vec{u}}\right|^2 = \frac{1}{2}\frac{c^2}{|\tilde{\vec{c}}_a|^2}\tilde{\vec{g}}_a \bullet \tilde{\vec{c}}_a^*, \quad \text{with:} \quad \frac{w_a^{potl}}{w_a^{kin}} = \frac{\left|\tilde{\vec{c}}_a\right|^2}{c^2} = \frac{\left|\tilde{\vec{z}}_a\right|^2}{z_o^2}.$$

We also have the relations:

$$w_a^{rad} = \frac{n_a^{r\,2}}{\left|\tilde{\vec{n}}_a\right|^2} w_a^{tot} = \frac{c_a^{r\,2}}{\left|\tilde{\vec{c}}_a\right|^2} w_a^{tot} \quad \text{and:} \quad w_a^{virt} = \frac{n_a^{i\,2}}{\left|\tilde{\vec{n}}_a\right|^2} w_a^{tot} = \frac{c_a^{i\,2}}{\left|\tilde{\vec{c}}_a\right|^2} w_a^{tot} \quad \text{where:} \quad \tilde{\vec{n}}_a \equiv \frac{\tilde{\vec{c}}_a}{c} = \frac{\tilde{\vec{z}}_a}{z_o}.$$

Thus, we also see that:

$$w_a^{rad} = \frac{c_a^{r\,2}}{\left|\tilde{\vec{c}}_a\right|^2} w_a^{tot} = \frac{1}{2}\frac{c_a^{r\,2}}{\left|\tilde{\vec{c}}_a\right|^2}\left[1 + \frac{c^2}{|\tilde{\vec{c}}_a|^2}\right]\tilde{\vec{g}}_a \bullet \tilde{\vec{c}}_a^* \quad \text{and:} \quad w_a^{virt} = \frac{c_a^{i\,2}}{\left|\tilde{\vec{c}}_a\right|^2} w_a^{tot} = \frac{1}{2}\frac{c_a^{i\,2}}{\left|\tilde{\vec{c}}_a\right|^2}\left[1 + \frac{c^2}{|\tilde{\vec{c}}_a|^2}\right]\tilde{\vec{g}}_a \bullet \tilde{\vec{c}}_a^*$$

Acoustic Angular Momentum Density and Conservation of Acoustic Angular Momentum:

We can now also define the complex acoustic angular momentum density:

$$\tilde{\vec{\ell}}_a(\vec{r},t;\omega) = \vec{r} \times \tilde{\vec{g}}_a(\vec{r},t;\omega) \quad (N\text{-}sec/m^2 = kg/m\text{-}sec)$$

Note that this quantity is defined with respect to the local origin ϑ associated with $\vec{r}$.

The complex acoustic force density $\tilde{\vec{f}}_a(\vec{r},t;\omega) = \partial\tilde{\vec{g}}_a(\vec{r},t;\omega)/\partial t = -\vec{\nabla}\cdot\tilde{\overleftrightarrow{T}}_a(\vec{r},t;\omega)$.

The complex acoustic torque density $\tilde{\vec{\eta}}_a(\vec{r},t;\omega) \equiv \vec{r} \times \tilde{\vec{f}}_a(\vec{r},t;\omega)$ (N/m^2), but the complex acoustic torque density is also equal to the time rate of change of the complex acoustic angular momentum density, *i.e.* $\tilde{\vec{\eta}}_a(\vec{r},t;\omega) = \partial\tilde{\vec{\ell}}_a(\vec{r},t;\omega)/\partial t$. Since $\tilde{\vec{\ell}}_a(\vec{r},t;\omega) = \vec{r} \times \tilde{\vec{g}}_a(\vec{r},t;\omega)$, if $\vec{r}$ is a ***constant*** vector {*i.e.* $\vec{r} \neq fcn(t)$ }, then:

$$\tilde{\vec{\eta}}_a(\vec{r},t;\omega) = \frac{\partial \tilde{\vec{\ell}}_a(\vec{r},t;\omega)}{\partial t} = \frac{\partial\left(\vec{r}\times\tilde{\vec{g}}_a(\vec{r},t;\omega)\right)}{\partial t} = \vec{r}\times\frac{\partial \tilde{\vec{g}}_a(\vec{r},t;\omega)}{\partial t} \quad \left(N/m^2\right)$$

Hence {for $\vec{r} \neq fcn(t)$ }, we see that the complex acoustic torque density is:

$$\tilde{\vec{\eta}}_a(\vec{r},t;\omega) = \vec{r}\times\tilde{\vec{f}}_a(\vec{r},t;\omega) = \frac{\partial \tilde{\vec{\ell}}_a(\vec{r},t;\omega)}{\partial t} = \vec{r}\times\frac{\partial \tilde{\vec{g}}_a(\vec{r},t;\omega)}{\partial t} = -\vec{r}\times\left(\vec{\nabla}\cdot\tilde{\overset{\leftrightarrow}{T}}_a(\vec{r},t;\omega)\right) \quad \left(N/m^2\right)$$

This relation is in fact an expression for conservation of angular momentum associated with the complex acoustic field. We can {perhaps} see this more clearly in an equivalent manner:

We can write the k^{th} component {k = 1, 2, 3 (= x, y, z)} of the complex acoustic angular momentum density $\tilde{\ell}_{a_k}(\vec{r},t;\omega)$ and its associated cross product $\left(\vec{r}\times\tilde{\vec{g}}_a(\vec{r},t;\omega)\right)_k$, temporarily suppressing the $(\vec{r},t;\omega)$ argument(s), as: $\tilde{\ell}_{a_k} = \tfrac{1}{2}\in_{kij}\left(x^i\tilde{g}_a^j - x^j\tilde{g}_a^i\right)$ where: $\vec{r} = x\hat{x}+y\hat{y}+z\hat{z} = x^1\hat{e}^1+x^2\hat{e}^2+x^3\hat{e}^3$ = ***constant*** vector, and $\in_{kij}$ is the so-called totally antisymmetric rank-3 Ricci tensor, whose properties are such that: $\in_{kij}=+1$ for kij = ***even*** permutation of (1, 2, 3), $\in_{kij}=-1$ for kij = ***odd*** permutation of (1, 2, 3) and $\in_{kij}=0$ for kij = not a permutation of (1, 2, 3). The notation {***here***} adopts the so-called "Einstein summation convention", *i.e.* that repeated indices are summed over. Hence, in $\tilde{\ell}_{a_k} = \tfrac{1}{2}\in_{kij}\left(x^i\tilde{g}_a^j - x^j\tilde{g}_a^i\right)$, we must sum over {both} indices i, j = 1, 2, 3 (= x, y, z) in this expression.

It is then also possible to relate $\tilde{\ell}_{a_k}$ to a new, rank-two anti-symmetric tensor $\tilde{L}_a^{ij}$ via: $\tilde{\ell}_{a_k} = \tfrac{1}{2}\in_{kij}\tilde{L}_a^{ij}$. Conservation of complex acoustic angular momentum density (*i.e.* conservation of complex angular momentum in differential form) can then be expressed as the –*ve* divergence of a rank-3 complex acoustic "moment-of-a-force-density" tensor $\tilde{M}_a^{kij}$:

$$\frac{\partial \tilde{L}_a^{ij}}{\partial t} = -\left(\frac{\partial \tilde{M}_a^{1ij}}{\partial x^1}+\frac{\partial \tilde{M}_a^{2ij}}{\partial x^2}+\frac{\partial \tilde{M}_a^{3ij}}{\partial x^2}\right) \equiv -\partial_\ell \tilde{M}_a^{\ell ij} \quad \text{where: } \partial_k \equiv \frac{\partial}{\partial x^1}+\frac{\partial}{\partial x^2}+\frac{\partial}{\partial x^2}$$

The rank-3 complex acoustic "moment-of-a-force-density" tensor $\tilde{M}_a^{\ell ij}$ is related to the rank-2 complex acoustic stress tensor $\tilde{\overset{\leftrightarrow}{T}}_a$ {for $\vec{r} \neq fcn(t)$ } via: $\tilde{M}_a^{\ell ij} \equiv \tilde{T}_a^{\ell i}x^j - \tilde{T}_a^{\ell j}x^i$.

Then in integral form, conservation of complex acoustic angular momentum can be written as:

$$\begin{aligned}\frac{d}{dt}\int_V \tilde{\vec{\ell}}_a(\vec{r},t;\omega)\,d\tau &= \tfrac{1}{2}\in_{kij}\hat{e}^k\frac{d}{dt}\int_V \tilde{L}_a^{ij}(\vec{r},t;\omega)\,d\tau \\ &= -\tfrac{1}{2}\in_{kij}\hat{e}^k\int_V \partial_\ell\tilde{M}_a^{\ell ij}(\vec{r},t;\omega)\,d\tau = -\tfrac{1}{2}\in_{kij}\hat{e}^k\oint_S n_\ell\tilde{M}_a^{\ell ij}(\vec{r},t;\omega)\,da\end{aligned}$$

where we have used the divergence theorem in the last term, and $\hat{n}_\ell \equiv n_\ell\hat{e}^\ell$ is the ℓ^{th} outward pointing unit normal { ℓ = 1, 2, 3 (= x, y, z)} associated with surface S, enclosing volume V.

An explicit experimental demonstration of the acoustic angular momentum density/acoustic torque was published in the paper "Circularly Polarized Acoustic Field: The Number Theory Connection", M.R. Schroeder, Acustica **75**, *p*. 94-98 (1991). It's a nice acoustic analog of *EM* wave angular momentum density/*EM* wave torque: see *e.g*. "Mechanical Detection and Measurement of the Angular Momentum of Light", R.A. Beth, Physical Review **50**, *p*. 115−125 (1936). See also "A Radiation Torque Experiment", P.J. Allen, Am. J. Phys. **34**, *p*. 1185−1192 (1966).

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