# Waves I:

Introduction to Waves - Traveling Waves

In these lecture notes on waves, our goal is to understand the physical behavior of waves - waves on guitar strings, sound waves in air, and also in dense media - such as vibrating guitar bodies, guitar necks, *etc*. Here we provide a brief review of the mathematics necessary to describe the behavior of such waves.

In general, a *traveling* wave is a *disturbance* that propagates in a medium (*e.g.* air, water, a guitar string, *etc*.) as a function of time, carrying with it energy, E and momentum p. As the traveling wave propagates, say along the x-axis, if the nature of the local disturbance - the displacement of atoms or molecules from their equilibrium positions - associated with the traveling wave at a given point, x at a given time, t is *transverse* (*i.e.* perpendicular) to the direction of propagation (say in the y-direction), as in the case of traveling waves on a string, we call such waves *transverse traveling waves*. In contrast to this, sound waves propagating *e.g.* in air or water (a fluid) are *longitudinal traveling waves* - the local disturbance (*i.e.* displacement from equilibrium position of air or water molecules) at a given point, x at a given time, t is *longitudinal* (*i.e.* parallel) to the direction of propagation.

In the figure shown below, we show a time sequence of the propagation of a transverse traveling wave, plotting the transverse displacement, $y(x,t) = y_o \exp\{-(x-v_x t)^2\}$ for a gaussian-shaped transverse traveling wave, for $t = -5$ seconds, $t = 0$ and $t = +5$ seconds. Here, we have chosen the amplitude, y_o (= transverse *displacement* from the equilibrium, $y = 0$ position) of this transverse wave, in *SI* (*i.e. mksa {meters-kilograms-seconds-ampere* units}) to be $y_o = 1.0$ *m*, the *longitudinal* propagation velocity of the transverse traveling wave (the x-velocity of the wave as it propagates along the x-axis) to be $v_x = +1$ *m/sec*. Note that this wave propagates in the $+x$ direction as time increases.

y(x,t) vs. x

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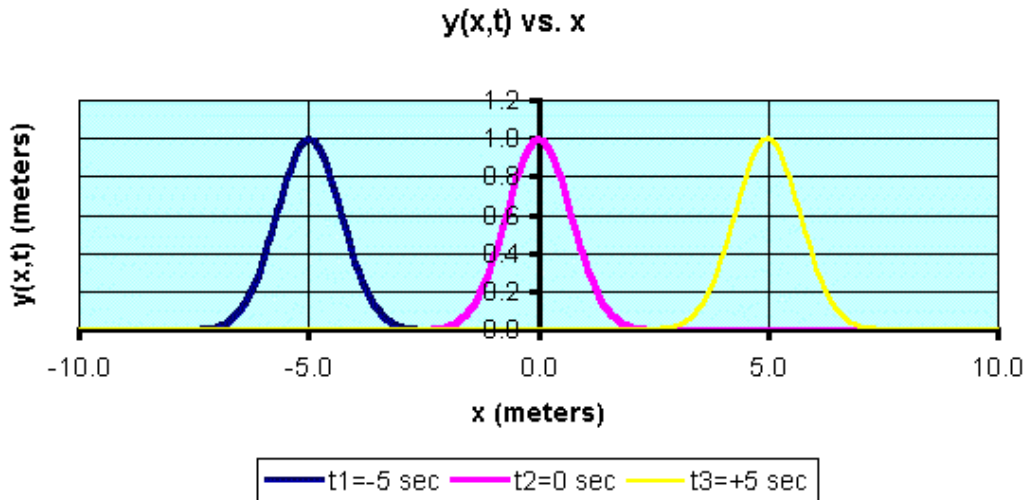
Alternatively, we could have instead chosen a gaussian transverse wave propagating in the −x direction as time increases. Mathematically, such a wave would be described as $y(x, t) = y_o\, exp\{-(x + v_x t)^2\}$. The longitudinal velocity of such a left-moving wave on the x-axis is $v_x = -1.0\ m/sec$.

In general, when we speak of arbitrarily-shaped, propagating transverse traveling waves, for an arbitrarily-shaped *right-moving* transverse traveling wave, described by the function $y(x, t) = f(x - v_x t)$, the x-position of a given point on the waveform *increases* as time, t increases. This wave propagates with longitudinal wave velocity $+v_x\ m/sec$ in the +x-direction. For an arbitrarily-shaped, *left-moving* transverse traveling wave, described by the function $y(x, t) = f(x + v_x t)$, the x-position of a given point on the waveform *decreases* as time increases. This wave propagates with longitudinal wave velocity $-v_x$ m/sec in the −x-direction.

Note that the *argument*, $(x \pm v_x t)$ of the arbitrary function, $f(x \pm v_x t)$ that mathematically describes the transverse wave - as a function of x, for a given time, t, propagating with longitudinal wave velocity $\pm v_x$ in the x-direction is a *constant*. For example, if $(x - v_x t) = $ constant $= K$, then as the time, t increases, then x must also *increase*, so that $(x - v_x t) = K$. If $(x + v_x t) = K'$, then as the time, t increases, then x must *decrease*, so that $(x - v_x t) = K'$.

In the above figure with $v_x = +1\ m/sec$, for $t = -5, 0$, and $+5$ seconds, the *peak* of the transverse displacement of the right-moving waveform in each case was at $(x - v_x t) = K = -5, 0$, and $+5$ meters, respectively. However, for a *different* x-position along the waveform, say at $x = -6, -1$, and $+4$ meters, at times $t = -5, 0$, and $+5$ seconds, respectively, the argument of <u>this</u> function, $f(x - v_x t)$ describing a right-moving gaussian-shaped transverse traveling wave has $(x - v_x t) = K' = -1$. Note also that (here) the argument, $(x - v_x t)$ of the function $f(x - v_x t)$ that mathematically describes the transverse displacement of the waveform as a function of position, x and time, t has dimensions of *length* (here, in *meters*).

Note that for a transverse traveling wave propagating *e.g.* in the +x-direction, the actual motion of the displacement of the string is in the *y*-direction, transverse (*i.e.* perpendicular) to the x-axis of the string. Thus, for an infinitesimally small segment of the string, of length *dx*, the motion of that portion of the string is only up and down, in the y-direction as the transverse wave disturbance passes by, propagating along the string in the x-direction. If we imagine taking a snapshot at time $t = 0$, then for a transverse wave propagating in the +x-direction, the *leading* edge of the waveform is moving up, away from the x-axis at that instant in time, and the *trailing* edge of the waveform is moving down, back towards the x-axis, at that instant in time. The *peak* of the waveform at that instant in time is stationary - it is not moving up or down at all.

Thus, we can describe the transverse motion of the string as a function of position and time with the notion of a *transverse* velocity, $u_y(x, t)$ of the string at any given point, x and time, t. Formally, the transverse velocity, $u_y(x, t)$ of the string at any given point, x and time, t is the time-rate of change of the y-position of the string at the point x at time t.

Mathematically, from calculus, this is stated as the derivative of $y(x, t)$ with respect to time, t - *i.e.* the time-rate of change of the transverse displacement, $y(x,t)$. The transverse velocity, $u_y(x, t)$ at the point x at time t is defined as:

$$u_y(x,t) \equiv \frac{dy(x,t)}{dt}$$

However, again from calculus, by the *chain-rule of differentiation*, we can write $dy(x,t)/dt = df(x - v_x t)/dt$, thus:

$$u_y(x,t) \equiv \frac{df(x - v_x t)}{dt} = \frac{df(x - v_x t)}{d(x - v_x t)} * \frac{d(x - v_x t)}{dt} = \frac{\partial y(x,t)}{\partial x} * (-v_x) = -v_x \frac{\partial y(x,t)}{\partial x}$$

Here, we used the fact that the *total* derivative, $df(x - v_x t)/d(x - v_x t)$ = *partial* derivative = $\partial f(x - v_x t)/\partial(x - v_x t) = \partial y(x, t)/\partial x$ for *time t held fixed* (*i.e.* a constant), since the *longitudinal* velocity of the wave, v_x is a constant. The *longitudinal* velocity of the wave, v_x is defined as the $-ve$ of the time-rate of change of the right-moving waveform, *i.e.* $v_x \equiv -d(x - v_x t)/dt$. Note that the *local* slope of the string, $m(x, t) = \partial y(x, t)/\partial x \cong \Delta y(x, t)/\Delta x$ = change in y per change in x, at the point, x at a given time, t.

Thus, we see that the *transverse* velocity of the string, $u_y(x, t)$ at the point x and time t is the *product* of (the *local* slope of the string, $m(x, t) = \partial y(x, t)/\partial x$ at the point, x at *that* time, t) *and* (the negative of the *longitudinal* velocity of the transverse wave, $-v_x$). Note that the *partial* derivative of $y(x, t)$ with respect to x, $\partial y(x, t)/\partial x$ means differentiating $y(x, t)$ with respect to x (*i.e.* the change in y per change in x = "rise" over "run") *while holding the time, t fixed*. Thus, the (local) slope of the string, $m(x, t) = \partial y(x, t)/\partial x$ at the point, x is a *snapshot* of the string at the time, t. A right-moving, traveling *triangle* wave is shown in the figure below, which illustrates these concepts.
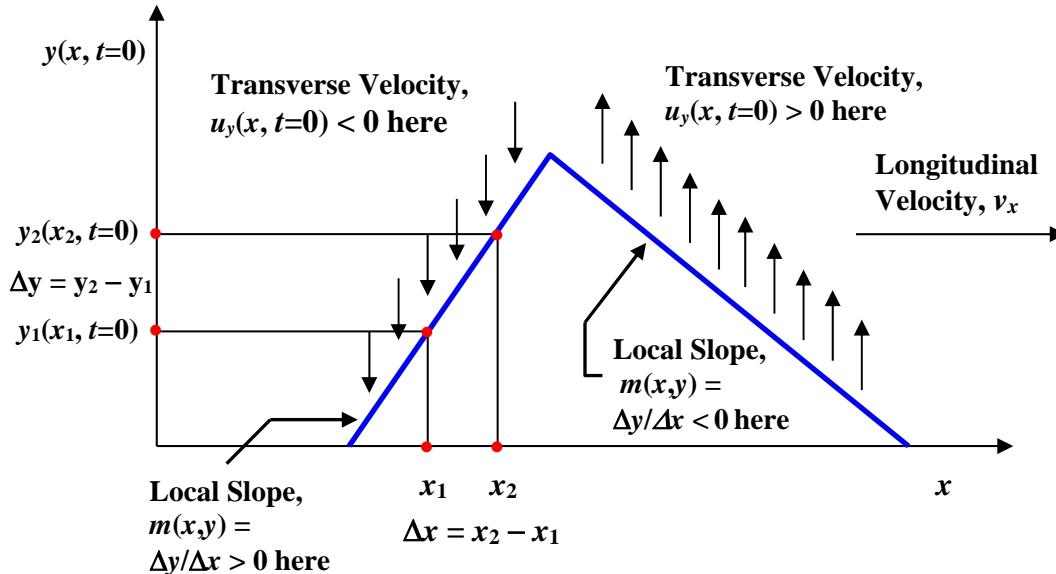
Note that by convention, a right-moving (left-moving) traveling wave has *longitudinal* velocity, $v_x > 0$ ($v_x < 0$), respectively.

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For the right-moving gaussian transverse wave, $y(x, t) = y_o \exp\{-(x - v_x t)^2\}$, with amplitude, $y_o = 1.0$ *m*, and *longitudinal* wave velocity, $v_x = +1.0$ *m/sec* the following plot shows the transverse velocity of this wave, $u_y(x, t) = - (\partial y(x, t)/\partial x) v_x$ as a function of the position, x on the string, for $t = -5$, 0 and $+5$ seconds. Reproducing this plot is left as an exercise for the interested reader - see exercise(s) at the end of these lecture notes.
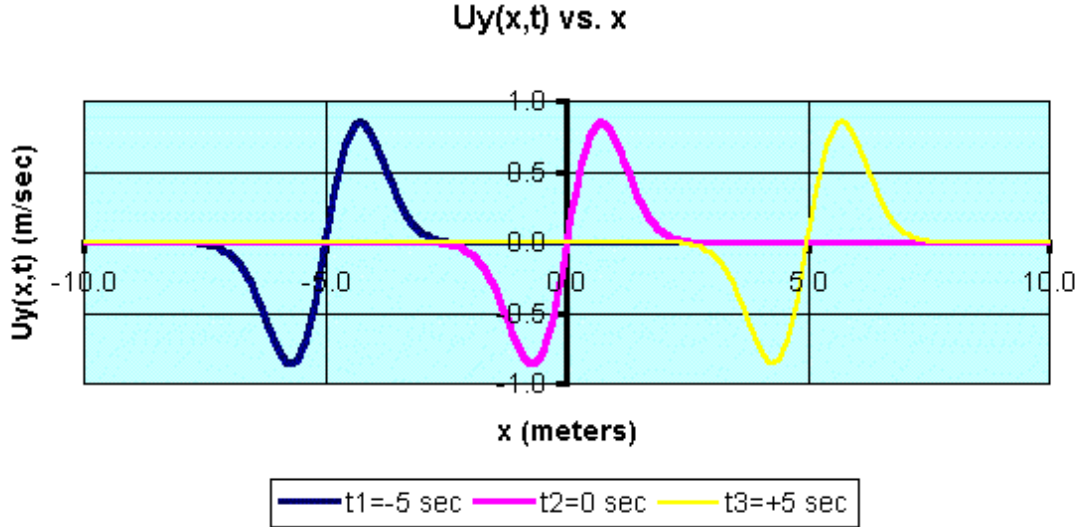
Harmonic Traveling Waves

Having discussed the general properties of traveling waves, we now focus specifically on *harmonic* traveling waves - i.e. waves that repeat themselves periodically in space and in time. Harmonic traveling waves are sinuosoids - and can be described either by sine or cosine functions - e.g. right-moving harmonic traveling waves can be mathematically described as:

$$y(x,t) = A\sin\left[2\pi\left(\tfrac{x}{\lambda}-\tfrac{t}{\tau}\right)\right] = A\sin\left(kx - \omega t\right)$$

or:

$$y(x,t) = A\cos\left[2\pi\left(\tfrac{x}{\lambda}-\tfrac{t}{\tau}\right)\right] = A\cos\left(kx - \omega t\right)$$

where the *amplitude*, A (= transverse displacement from equilibrium position, $y = 0$) has units of length (*e.g. meters*), the *wavelength*, λ (in *meters*) which is the spatial repeat distance of a harmonic traveling wave; the *period*, τ (units of time, *e.g.* seconds) is the repeat time of the harmonic traveling wave. Related quantities are the *frequency*, $f = 1/\tau$ (units of *cycles per second*, or *Hertz*), the *angular frequency*, $\omega = 2\pi f$ (units of *radians/second*, often abbreviated as *rad/sec*), and the *wavenumber*, $k = 2\pi/\lambda$ (units of inverse length, *e.g. inverse meters* $= 1/m = m^{-1}$) of the harmonic traveling wave. The longitudinal wave *speed*, $|v_x|$ (= the *magnitude* of the longitudinal velocity, v_x) is related to the frequency and wavelength of the harmonic traveling wave by the relation $|v_x| = f\lambda$. Since $\omega = 2\pi f$ and $k = 2\pi/\lambda$, we also have $|v_x| = (2\pi f)*(\lambda/2\pi) = \omega/k$.

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Snapshots of the transverse displacement, $y(x,t)$ *vs.* position, x for a right-moving sine-type harmonic traveling wave are shown in the figure below, for $t = -5$, 0 and +5 seconds. The transverse displacement, $y(x, t) = A \sin(kx - \omega t) = A \sin[2\pi(x/\lambda - ft)]$, with amplitude, $A = 1.0$ *m*, wavelength, $\lambda = 4.0$ *m*, longitudinal velocity $v_x = +1.0$ *m/sec* and thus $f = |v_x|/\lambda = 1/4 = 0.25$ *Hz*.

Y(x,t) = A sin[kx-wt] vs x
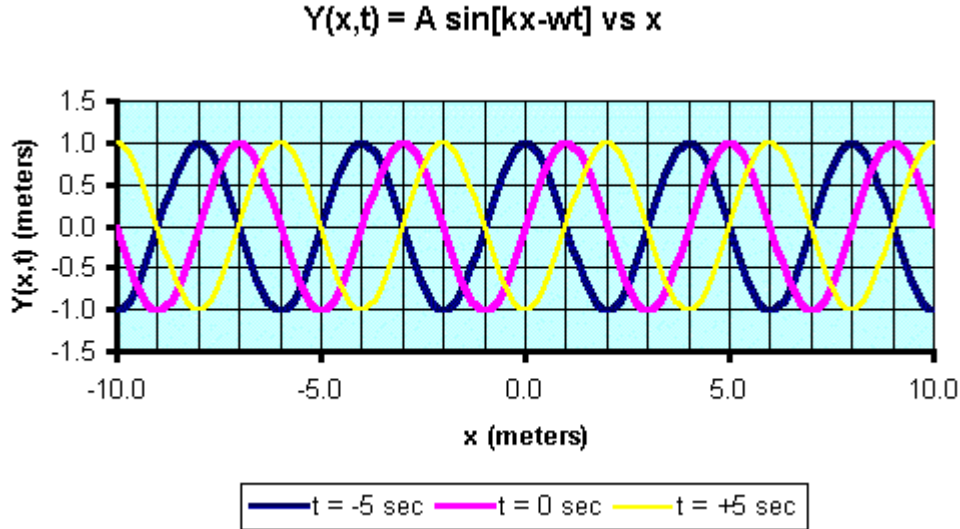
The three sinusoidal curves in this figure may seem a bit confusing at first glance. Consider the crest at x ($t = -5$ *sec*) $= -8.0$ *m* associated with the snapshot of the dark blue sinusoidal traveling wave at $t = -5$ *sec*. Five seconds later, this crest (along with the rest of the sinusoidal traveling wave) has propagated to the *right*, a distance of $\Delta x = |v_x|\Delta t = 1$ *m/sec**5 *sec* = 5.0 *m*. Thus, this same crest is now located at x ($t = 0$ *sec*) $= -8.0 + 5.0$ *m* $= -3.0$ *m*. This is the crest located at $x(t = 0$ *sec*) $= -3.0$ *m* on the magenta curve. Five seconds after this, at $t = +5.0$ *sec*, this same crest has propagated to the *right* another distance of $\Delta x = |v_x|\Delta t = 1$ *m/sec**5 *sec* = 5.0 m. This crest is now at $x(t = +5$ *sec*) $= -3.0 + 5.0$ *m* $= +2.0$ *m*, *i.e.* the crest located at $x(t = +5.0$ *sec*) $= +2.0$ *m* on the yellow curve.

The transverse *velocity*, $u_y(x,t)$ of a sine-type harmonic traveling wave can be obtained from the transverse *displacement*, $y(x, t)$. Since velocity (units = *m/sec*) is the *change* of position per unit *change* in time, the transverse velocity, $u_y(x,t)$ is the *derivative*, d/dt of position with respect to time. Then $u_y(x,t) = d/dt\,(y(x,t)) = dy(x, t)/dt = d/dt(A \sin[kx - \omega t])$ $= -\omega A \cos[kx - \omega t]$, since the derivative, d/dt of the $\sin(u(t))$ function is $d/dt(\sin u(t)) = d(\sin(u(t))/dt = \cos u^* du(t)/dt$, by the *chain-rule of differentiation*, where $u(t) = [kx - \omega t]$, thus $du(t)/dt = -\omega$. Snapshots of the transverse velocity, $u_y(x, t)$ as a function of position, x, for $t = -5$, 0 and +5 seconds are shown in the figure below. Again, for the crest of $u_y(x = -7.0$ *m*, $t = -5$ *sec*) at $x(t = -5$ *sec*) associated with the snapshot of the dark blue sinusoidal traveling wave at $t = -5$ sec, after 5 seconds, at $t = 0$ sec, this crest associated with the dark blue $u_y(x,t)$ curve has also propagated to the right a distance $\Delta x = |v_x|\Delta t = 1$ *m/sec**5 *sec* = 5.0 *m*. Thus, the velocity crest is now at $x(t = 0$ *sec*) $= -7.0 + 5.0 = -2.0$ *m*, on the magenta curve; and another $\Delta t = 5$ seconds later, this velocity crest is at +3.0 *m* on the yellow curve.

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Uy(x,t) = - wA cos[kx-wt] vs. x
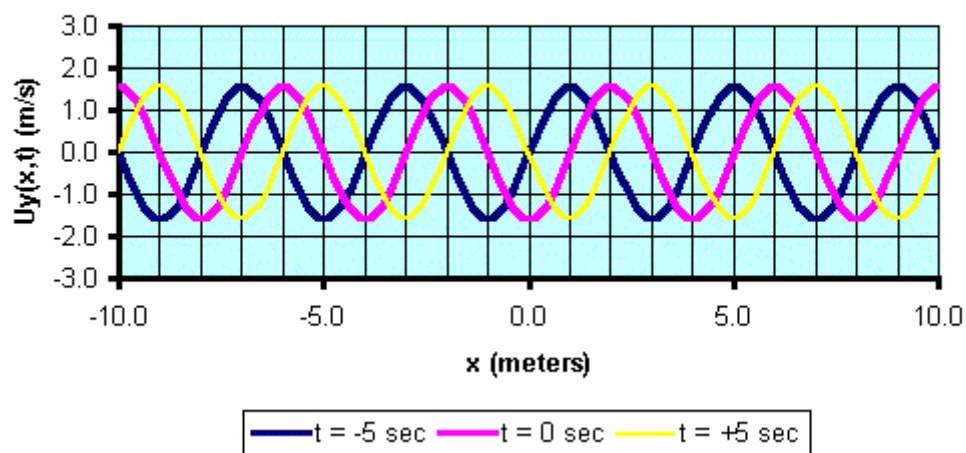
Note that the transverse velocity, $u_y(x, t)$ of a traveling harmonic wave is 90º out of phase with the transverse displacement, $y(x,t)$ of the wave (*i.e.* ¼ of a cycle). This is because the cosine and sine functions are related to each other by a phase angle of $\delta = 90º$; $cos\ \theta = sin(\theta + 90º)$, and $sin\ \theta = cos(\theta - 90º)$, where θ is an arbitrary angle. These relations can be derived from the *angle-addition formulae* for the sine and cosine functions:
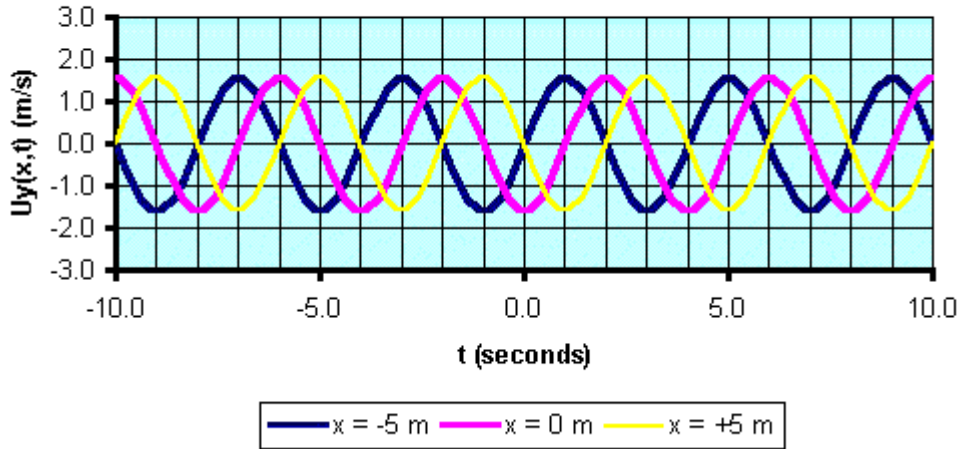
$$sin(A \pm B) = sin\ A\ cos\ B \pm sin\ B\ cos\ A \ \ and: \ \ cos(A \pm B) = cos\ A\ cos\ B \mp sin\ A\ sin\ B.$$

Alternatively, we can show the transverse displacement, $y(x,t)$ and the transverse velocity, $u_y(x, t)$ for the above sine-type harmonic traveling wave for fixed position, $x = -5$, 0 and +5 *meters*, as a function of time, t:

y(x,t) = A sin[kx-wt] vs. t

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Uy(x,t) = - wA cos [kx-wt] vs. t

x = -5 m   x = 0 m   x = +5 m
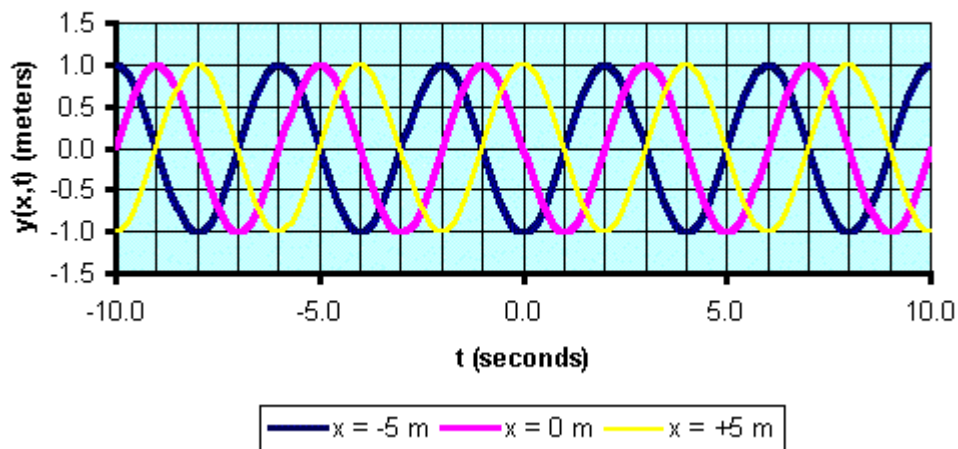
For the type of transverse waves we are used to dealing with on *e.g.* stringed instruments, such as the guitar, or violin, in order for propagation of transverse waves to occur on a string, the string must be stretched, with a tension (holding force), *T*. The mksa units of tension, *T* are the same as that for force, *F* - namely *Newtons* of force, or *Newtons* of tension. One Newton of force, by (Isaac) Newton's second law, $F = ma$, is equal to the force, *F* associated with accelerating a mass, $m = 1$ *kilogram* (*kg*) by an acceleration, $a = 1$ *m/sec²*. Thus, one *Newton* = 1 kg m/*sec²*.

The Wave Equation

There are many ways to derive the wave equation - a so-called 2nd order linear, homogeneous differential equation, which describes the propagation of waves associated with a physical system, for which there are no dissipative losses. For *small amplitude* transverse waves on a string, we can consider the balance of forces and accelerations associated with an infinitesimally small segment of the string, of length *dx*, as shown in the figure below, for a snapshot in time, *e.g.* at $t = 0$.

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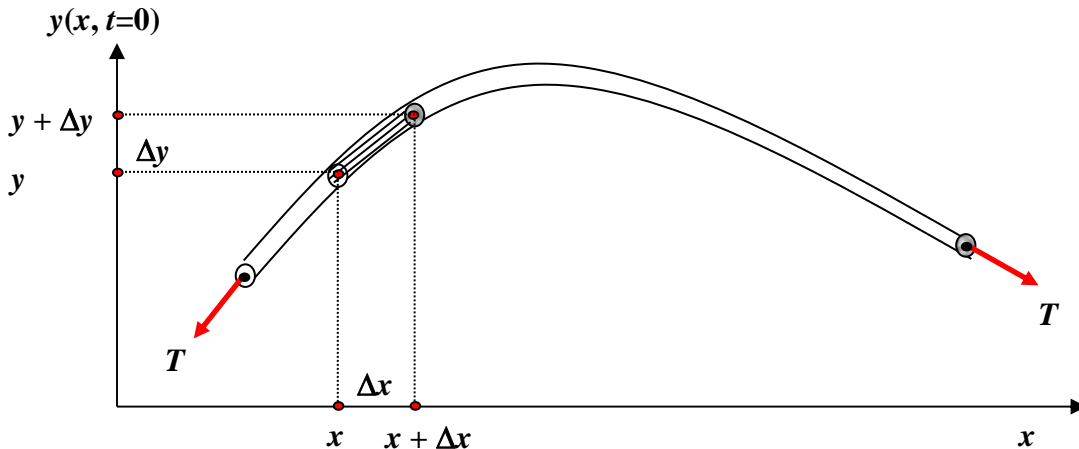
The force(s) acting on an infinitesimally small segment of the string, at $t = 0$ are shown in the figure below:
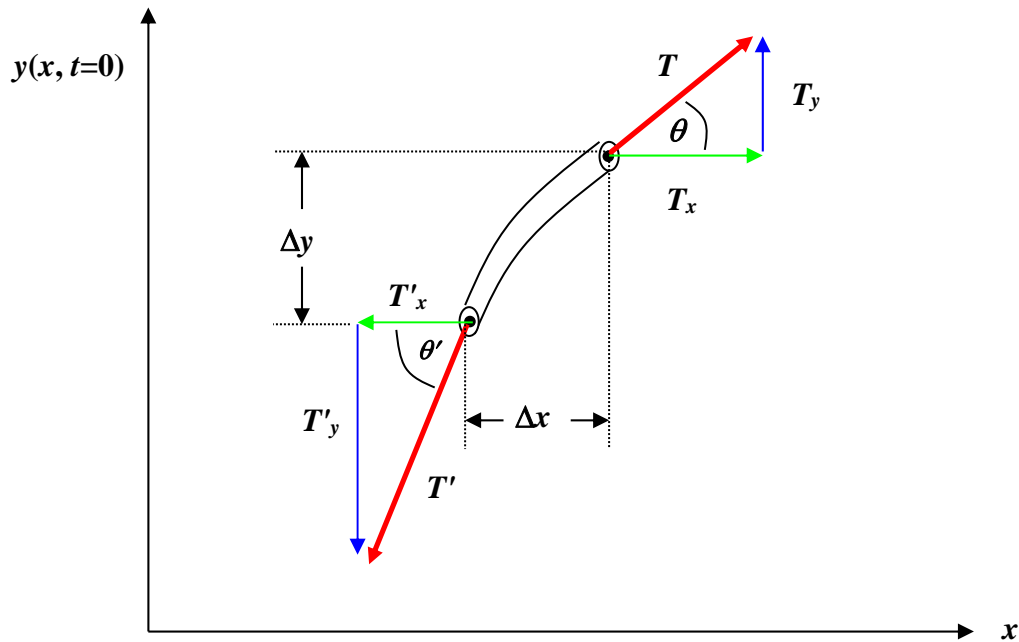
In the above figure, we have resolved the (vector) tensions, T and T' acting on each end of the infinitesimal segment of the string, into their x- and y-components. On the upper right-hand end of the string segment, $T_x = +|T| \cos \theta$, and $T_y = +|T| \sin \theta$. Note that the *magnitude* of the tension T, $|T| = (T_x^2 + T_y^2)^{1/2}$, since $\cos^2 \theta + \sin^2 \theta = 1$. On the lower left-hand end of the string segment, $T'_x = -|T'| \cos \theta'$, and $T'_y = -|T'| \sin \theta'$, with the *magnitude* of the tension T', $|T'| = (T'^2_x + T'^2_y)^{1/2}$, again since $\cos^2 \theta' + \sin^2 \theta' = 1$.

In the horizontal (x-direction), for small-amplitude transverse waves, there is no *net* force acting on the string - the horizontal tension components T_x and T'_x balance (*i.e.* cancel) each other. Newton's second law, $F_x = ma_x$ says that if there were a net force, F_x in the x-direction, then this string segment, of mass, m would accelerate in that direction, with acceleration, a_x. Since the string segment does not accelerate *longitudinally* for *transverse* waves, there is no net *longitudinal* force acting on the string segment. Mathematically, this is stated as $F_x = T_x + T'_x = 0$, *i.e.* $T_x = -T'_x$, or equivalently, $|T| \cos \theta = +|T'| \cos \theta'$.

In the vertical (y-direction), for small-amplitude transverse waves, there *is* a net restoring force, F_y acting on the string, which acts in such a way as to move the string back towards its equilibrium ($y = 0$) position, for a non-zero transverse displacement of the string, $y(x,t)$. This net transverse restoring force must be present, otherwise the string wouldn't vibrate - transversely! The vertical tension components, T_y and T'_y therefore do not cancel each other completely. Here, Newton's second law, $F_y = ma_y = T_y + T'_y \neq 0$ says that there is a net acceleration, a_y in the vertical (y-direction), transverse to the axis of the string. Then $F_y = ma_y = T_y + T'_y = |T| \sin \theta - |T'| \sin \theta' \neq 0$.

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Now if the transverse waves on the string have small amplitude, this means that the angles, θ and θ' are quite small. Then in this regime, the small-angle approximation holds, such that $\sin \theta \cong \tan \theta \cong \theta$, and $\sin \theta' \cong \tan \theta' \cong \theta'$. But $\tan \theta \cong |\partial y(x+\Delta x, t=0)/\partial x|$ and $\tan \theta' \cong |\partial y(x, t=0)/\partial x|$. Thus, the *magnitude* of the *slope* of the string, $|\partial y(x, t=0)/\partial x|$ is small everywhere on the string for small amplitudes. In other words, for all x, $0 \leq x \leq L$, where L is the total length of the string, then $|\partial y(x, t=0)/\partial x| << 1$. Then for the snapshot of the string at $t = 0$:

$$F_y = ma_y = T_y + T'_y = |T| \sin \theta - |T'| \sin \theta'$$
$$\cong |T| \, \partial y(x,t=0)/\partial x \, (x+\Delta x) - |T| \, \partial y(x,t=0)/\partial x \, (x)$$
$$= |T| \, [\partial y(x,t=0)/\partial x \, (x+\Delta x) - \partial y(x,t=0)/\partial x \, (x)]$$

This relation holds for *any* time t, not just for $t = 0$. The quantity in the square brackets:

$$[\partial y(x,t=0)/\partial x \, (x+\Delta x) - \partial y(x,t=0)/\partial x \, (x)] = \Delta x \, \partial/\partial x \, (\partial y(x,t)/\partial x) = \Delta x \, (\partial^2 y(x,t)/\partial x^2)$$

by the definition of a double derivative. Thus, $F_y = ma_y \cong |\mathrm{T}| \, \Delta \mathrm{x} \, (\partial^2 \mathrm{y}(\mathrm{x},\mathrm{t})/\partial \mathrm{x}^2)$.

If the string has mass per unit length, $\mu = M/L$ (kg/m), where M = total mass of the string (in *kilograms*) and L = total length of the string (in *meters*), then, for small amplitude waves on the string (where $\Delta y << \Delta x$) the infinitesimal string segment, of length $\Delta L = (\Delta x^2 + \Delta y^2)^{1/2} \cong \Delta x$ has a mass $m \cong \mu \, \Delta x$ (kg).

Now the (transverse) acceleration in the y-direction, is $a_y(x, t)$ at the point x, for a given time, t, which by the definition of an acceleration, is the change in the velocity per unit change in time, is given by:

$$a_y(x, t) = \partial u_y(x,t)/\partial t = \partial/\partial t(\partial y(x,t)/\partial t) = \partial^2 y(x,t)/\partial t^2.$$

Then since $F_y = ma_y \cong |T| \, \Delta x \, (\partial^2 y(x,t)/\partial x^2)$ and $F_y = ma_y = \mu \, \Delta x \, \partial^2 y(x,t)/\partial t^2$, we see that:

$$|T| \, \partial^2 y(x,t)/\partial x^2 = \mu \, \partial^2 y(x,t)/\partial t^2$$

Now, it turns out that the *ratio* of the magnitude of the tension, $|T|$ to the mass per unit length of the string, μ is actually the square of the *longitudinal* speed of propagation, $|v_x|^2$ of the waves on the string, *i.e.*
$$|v_x|^2 = |T|/\mu \quad \text{or} \quad |v_x| = (|T|/\mu)^{1/2}$$

Thus, the wave equation for propagation of transverse waves on a string is given by:

$$\partial^2 y(x,t)/\partial x^2 = 1/|v_x|^2 \, \partial^2 y(x,t)/\partial t^2$$

One can prove that right- and/or left-moving traveling waves, with transverse displacements $y(x, t) = f(x \mp v_x t)$ satisfy the above wave equation, by explicitly carrying out the differentiation on both sides of the wave equation, using the chain-rule of differentiation. We leave this as an exercise for the interested reader.

The Principle of Linear Superposition of Waves

The principle of linear superposition is a very powerful one. Mathematically, if $y_1(x,t)$ and $y_2(x,t)$ are both solutions to the wave equation, then it can be shown that the linear combination, $y_{tot}(x,t) = y_1(x,t) + y_2(x,t)$ is *also* a solution to the wave equation. Thus, if individual kinds of transverse traveling waves, $y_1(x,t)$ and $y_2(x,t)$ can propagate on a string, then it is also possible for the wave $y_{tot}(x,t) = y_1(x,t) + y_2(x,t)$ to propagate on this same string. We will see throughout this course, that the principle of linear superposition of waves manifests itself in many ways, and has many different consequences. In general, we could have:

$$y_{tot}(x,t) = y_1(x,t) + y_2(x,t) + y_3(x,t) + y_4(x,t) + ... = \sum_{n=1}^{n=\infty} y_n(x,t)$$

However, here we must remind the reader that one of our original constraints in deriving the wave equation was that the amplitudes of the wave(s) on the string were *small*. As long as the total/overall amplitude, $y_{tot}(x,t)$ resulting from superposing arbitrarily many individual waves is small, then the above mathematics is a good description of the physical system. If this is *not* true, then one needs to develop a more sophisticated theoretical description, valid for large-amplitude string vibrations. In this situation, use of the above formulae will be inaccurate.

Reflection of Waves at a Boundary

What happens to a transverse traveling wave when it reaches a boundary, such as the end of a taught string? The answer depends on whether the end of the string is rigidly fixed (i.e. immovable) or is able to freely undergo transverse vibrations, under the transitory influences (i.e. forces) associated with the transverse traveling wave when it reaches this end point.

A.) Reflection of Waves at a Fixed End

If the end of a taught string, *e.g.* at $x = L$ is rigidly/immovably fixed, then mathematically, this means that the transverse displacement, $y(x = L, t) = 0$ at the point $x = L$ for *any* time t. This mathematical statement of the (fixed) end condition, is generically known as a *boundary condition* associated with the transverse waves propagating on the string. An electric solid-body guitar or bass with a rigidly-mounted bridge attached to the body of the guitar is an example of fixed-end boundary condition for the reflection of transverse traveling waves on the strings of the guitar - or at least a good, first-order approximation of what actually happens.

If we have a right-moving transverse traveling wave in the shape e.g. of a gaussian pulse, described mathematically as $y(x,t) = y_o \exp\{-(x-v_xt)^2\}$, then as this waveform reaches the fixed endpoint at $x = L$, the boundary condition $y(x=L, t) = 0$ must be obeyed. As this pulse impinges on the fixed end support, the only way that $y(x=L,t) = 0$ can be obeyed is if this pulse is simultaneously reflected and polarity inverted. Reflection means that the right-moving gaussian-shaped transverse traveling wave, $y(x,t) = y_o \exp\{-(x-v_xt)^2\}$ is converted into a left-moving gaussian-shaped transverse traveling wave, $y(x,t) = y_o \exp\{-[(x-2L)+v_xt]^2\}$ at the point $x = L$; polarity inversion means that the left-moving

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gaussian-shaped transverse traveling wave is "flipped over" on the y-axis, *i.e.* y_o changes to $-y_o$. This is equivalent to a phase change, or phase shift of the reflected wave relative to the incident wave of 180°. Thus, the gaussian-shaped transverse traveling wave reflected from a fixed end located at $x = L$ is of the form $y(x,t) = -y_o \exp\{-[(x-2L)+v_xt)^2$, after reflection.

Another equivalent way to think about this process is to imagine a special kind of mirror at the fixed endpoint $x = L$. As the right-moving gaussian-shaped transverse traveling wave, $y(x,t) = y_o \exp\{-(x-v_xt)^2\}$ approaches the fixed end, an observer can see this wave directly, but looking in the special mirror, this observer *also* sees a *left*-moving, but *polarity-inverted* gaussian-shaped transverse traveling wave, $y(x,t) = -y_o$ $\exp\{-[(x-2L)+v_xt)^2$ approaching the endpoint at $x = L$, but from *behind* the mirror. This left-moving, polarity inverted pulse, approaching from *behind* the end point is known as an *image pulse*. This way of thinking about the problem arises from use of the principle of linear superposition. The fixed-end boundary condition, $y_{tot}(x=L,t) = 0$ for the incident, right-moving transverse traveling wave and its image pulse, independent of the detailed shape (transverse profiles) of these transverse traveling waves, is generically given by:

$$y_{tot}(x=L,t) = f(x-v_xt)|_{x=L} + f(x+v_xt)|_{x=L} = 0$$

where "$|_{x=L}$" means evaluating these functions at $x = L$. Thus, the fixed end boundary condition, $y_{tot}(x=L,t) = 0$ can only be satisfied if $f(x-v_xt)|_{x=L} = -f(x+v_xt)|_{x=L}$.

Note that the peak of the right-moving gaussian-shaped transverse traveling wave arrives at $x = L$ when $t = L/v_x$. The peak location, x of the right-moving gaussian pulse occurs at a given time, t when $(x-v_xt) = 0$. Thus, at $t = 0$, $x = 0$. Note also that the left-moving gaussian-shaped transverse traveling wave has had its origin shifted, such that when $t = 0$, its gaussian peak occurs at $x = 2L$, and when $t = L/v_x$, its gaussian peak also is at $x = L$. The peak, x of this left-moving gaussian pulse occurs at a given time, t when $[(x-2L)+v_xt] = 0$. Thus, at $t = 0$, $x = 2L$ for the left-moving gaussian pulse.

As time progresses, the two pulses move closer and closer to each other, *passing through* each other at $x = L$, and then keep on going. During the time that the two pulses overlap each other, the overall transverse displacement of the string at $x = L$ is just the *linear superposition* of the two waves, *i.e.*

$$y_{tot}(x=L, t) = y_{right}(x=L, t) + y_{left}(x=L, t) = y_o \exp\{-(L-v_xt)^2\} - y_o \exp\{-(-L+v_xt)^2\}$$

In general, for *any* point x, and time t, we will have, for reflection at a fixed end:

$$y_{tot}(x, t) = y_{right}(x, t) + y_{left}(x, t) = y_o \exp\{-(x-v_xt)^2\} - y_o \exp\{-[(x-2L)+v_xt]^2\}$$

Of course, what we see physically is only the portion of the string for the two waves with $x < L$. Mathematically, we can represent the wave that is reflected from a fixed end as an image wave, traveling in the opposite direction, and with inverted polarity, to the incoming traveling wave incident on the fixed end. Thus, because it takes a finite time for such a wave to undergo reflection from a fixed end, we can represent this process as a linear superposition of two waves, the original, right-moving transverse traveling wave,

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superposed with a polarity-inverted, left-moving transverse traveling wave, which is a "mirror" image of the original wave.

The following plot shows a snapshot time sequence of a right-moving gaussian-shaped transverse traveling wave reflecting from a fixed end, located at $x = L = 10$ m. The amplitude of the pulse is $y_o = 0.5$ m, the longitudinal wave speed $v_x = 1.0$ m/sec. Note that the sequence of (11) time snapshots progress vertically downward in 1 second steps, starting at $t = 5$ sec, and ending at $t = 15$ sec. The left-moving image pulse is used to represent the reflection in the physical region ($x \leq L = 10$ m). Note, for the purposes of comparison with free-end reflection (see below), that the slope, $\partial y_{tot}(x=L,t)/\partial x$ is not zero at the fixed end, located at $x = L = 10$ m.

$Y_{tot}(x,t)$ vs. t
fixed end @ x = L= 10 m real + image pulses
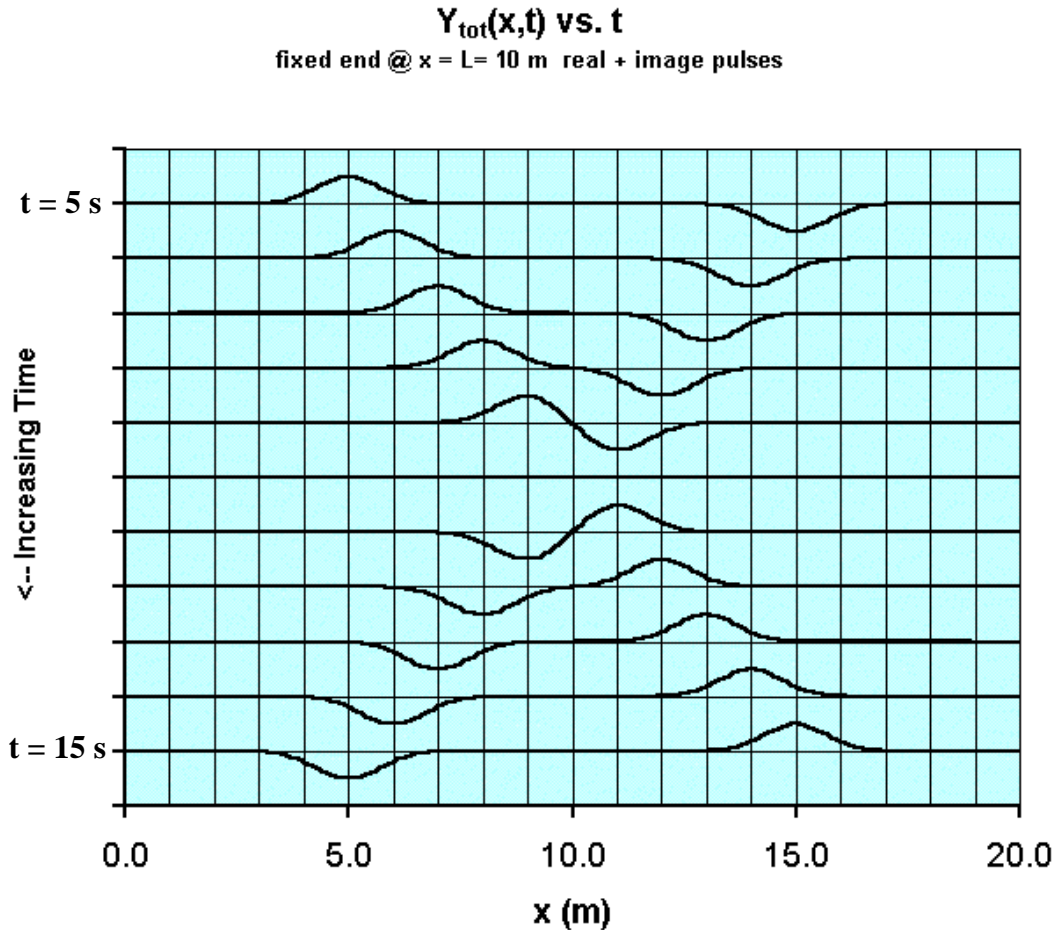
B.) Reflection of Waves at a Free End

If the end of a taught string, again at $x = L$ is held fixed *longitudinally*, but is freely able to vibrate *transversely*, for example, by attaching a horizontal string to the end support via a massless, frictionless ring, which can slide transversely (*e.g.* up and down) on a rigid, frictionless rod, the free-end *boundary condition* can be stated mathematically as the requirement that the *slope*, $\partial y/\partial x$ of the transverse displacement *at the point x = L* for *any* time *t must* be zero, *i.e.* $\partial y(x=L,t)/\partial x = 0$. Operationally, we first *compute* the

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partial derivative, $\partial y(x,t)/\partial x$, then *evaluate* it at $x = L$. Physically, the requirement that the slope, $\partial y/\partial x = 0$ for reflection of waves at a free-end of a taught string simply arises because of the fact that no *transverse* force, F_y can exist at this kind of end point.

If we have a right-moving transverse traveling wave, again in the shape of a gaussian pulse, $y(x,t) = y_o \exp\{-(x-v_xt)^2\}$, then as this waveform reaches the fixed endpoint at $x = L$, the free-end boundary condition $\partial y(x=L,t)/\partial x = 0$ must be obeyed. As the pulse impinges on the free end support, the only way that $\partial y(x=L,t)/\partial x = 0$ can be obeyed is if this pulse is simultaneously reflected, but with the *same* polarity as the incident wave. Again, reflection means that the right-moving gaussian-shaped transverse traveling wave, $y(x,t) = y_o \exp\{-(x-v_xt)^2\}$ is converted into a left-moving gaussian-shaped transverse traveling wave, $y(x,t) = y_o \exp\{-[(x-2L)+v_xt]^2\}$ at the point $x = L$; maintaining the original polarity means that the now left-moving gaussian-shaped transverse traveling wave has the same phase as the original wave on the y-axis, i.e. the amplitude y_o remains y_o. Thus, the gaussian-shaped transverse traveling wave reflected from a free end at $x = L$ is of the form $y(x,t) = y_o \exp\{-[(x-2L)+v_xt)^2$, after reflection.

Again, an equivalent way to think about this process is to imagine a special kind of mirror at the free endpoint $x = L$. As the right-moving gaussian-shaped transverse traveling wave, $y(x,t) = y_o \exp\{-(x-v_xt)^2\}$ approaches the free end, an observer can see this wave directly, but looking in the special mirror, this observer *also* sees a *left*-moving, gaussian-shaped transverse traveling wave, with the *same* polarity as the incident wave, $y(x,t) = y_o \exp\{-[(x-2L)+v_xt]^2\}$ approaching the endpoint at $x = L$, but from *behind* the mirror. Again, this left-moving, same-polarity pulse, approaching from *behind* the end point is also known as an *image pulse*. The free-end boundary condition, $\partial y_{tot}(x=L,t)/\partial x = 0$ for the incident, right-moving transverse traveling wave and its image pulse, independent of the detailed shape (transverse profiles) of these transverse traveling waves, is generically given by:

$$\partial y_{tot}(x=L,t)/\partial x = \partial f(x-v_xt)/\partial x|_{x=L} + \partial f(x+v_xt)/\partial x|_{x=L} = 0$$

Where "$|_{x=L}$" means evaluating the partial derivatives, $\partial/\partial x$ of these functions at $x = L$. Thus, the free-end boundary condition, $\partial y_{tot}(x=L,t)/\partial x = 0$ can only be satisfied if $\partial f(x-v_xt)/\partial x|_{x=L} = -\partial f(x+v_xt)/\partial x|_{x=L}$. If we *integrate* this relation, i.e.

$$\int \frac{\partial f(x-v_xt)}{\partial x}\Big|_{x=L} dx = -\int \frac{\partial f(x+v_xt)}{\partial x}\Big|_{x=L} dx$$

we obtain the result:

$$f(x-v_xt)|_{x=L} = + f(x+v_xt)|_{x=L}$$

The following plot shows a snapshot time sequence of a right-moving gaussian-shaped transverse traveling wave reflecting from a free end, located at $x = L = 10$ *m*. The amplitude of the pulse is $y_o = 0.5$ *m*, the longitudinal wave speed $v_x = 1.0$ *m/sec*. Note that the sequence of (11) time snapshots progress vertically downward in 1 second steps, starting at t = 5 *sec*, and ending at t = 15 *sec*. The left-moving image pulse is used to represent the reflection in the physical region ($x \leq L = 10$ *m*).

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As time progresses, the two pulses move closer and closer to each other, *passing through* each other at $x = L$, and then keep on going. During the time that the two pulses overlap each other, the overall transverse displacement of the string at $x = L$ is just the *linear superposition* of the two waves, *i.e.*

$$y_{tot}(x{=}L, t) = y_{right}(x{=}L, t) + y_{left}(x{=}L, t) = y_o \exp\{-(L{-}v_xt)^2\} + y_o \exp\{-(-L{+}v_xt)^2\}$$

In general, for *any* point x, and time t, we will have, for reflection at a fixed end:

$$y_{tot}(x, t) = y_{right}(x, t) + y_{left}(x, t) = y_o \exp\{-(x{-}v_xt)^2\} + y_o \exp\{-[(x{-}2L){+}v_xt]^2\}$$

In the figure below, note that the slope, $\partial y_{tot}(x{=}L,t)/\partial x$ *is* always zero for reflection of a right-moving traveling wave from a free end, located at $x = L = 10$ *m*.

$Y_{tot}(x,t)$ vs. t
free end @ x = L= 10 m real + image pulses
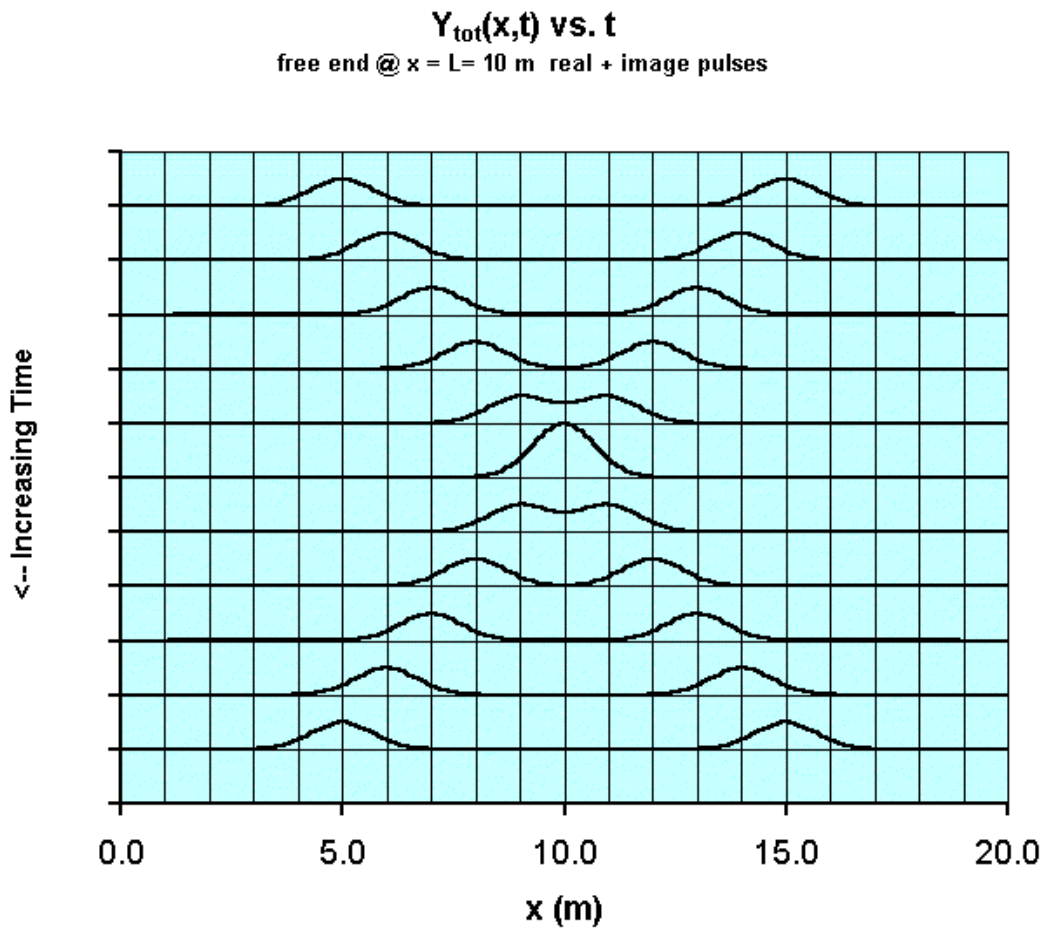
One comment here is that wave reflection of any kind, from an end support, or arising from a discontinuity in the propagation medium (*e.g*, a sudden "jump" in the mass per unit length, μ of a string, or *e.g.* the sudden change in sound waves propagating in the body and/or neck wood of a guitar, due to encountering the layer(s) of paint and/or lacquer on the guitar's outer surfaces), is a macroscopic form of a <u>*scattering process*</u>. The above snapshot time sequences really show this quite clearly, at least to a particle-physicist's eye! At the atomic level, a very great many atoms of the medium (typically ~

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10^{25}–10^{30} atoms, depending on the mass of the vibrating object) in which the wave is propagating are also vibrating (scattering), but with extremely small amplitudes. However, adding up all of the individual atomic contributions results in the macroscopic vibrations that we can see and hear with our own eyes and ears, respectively!

Standing Waves

We now consider wave propagation on a taught string, of length, L, tension T and mass per unit length, μ, with fixed end supports at $x = 0$ and $x = L$. If we consider sinusoidal-type transverse traveling wave solutions of the wave equation, mathematically, the most general possible solution of the wave equation for waves propagating on such a string consists of a linear combination of both right- and left-moving sine and cosine-type traveling waves - four total:

$$y_{tot}(x,t) = A\sin(kx - \omega t) + B\cos(kx - \omega t) + C\sin(kx + \omega t) + D\cos(kx + \omega t)$$

Here, A, B, C, D are the amplitudes (in *meters*, *m*) of the four types of sinusoidal waves, the wavenumber, $k = 2\pi/\lambda$ (in units of $1/m$, or m^{-1}), where λ is the wavelength (*meters*), associated with the frequency, f (*Hz*) and angular frequency $\omega = 2\pi f$ (*radians/sec*). The longitudinal wave speed on the string is $|v_x| = f\lambda = (T/\mu)^{1/2}$ (*m/sec*).

First, we note that the sine and cosine functions, $\sin(x)$ and $\cos(x)$ have some useful (and important) *reflection symmetry* properties - namely that $\sin(-x) = -\sin(x)$, and that $\cos(-x) = +\cos(x)$. When we change (*i.e.* reflect) $x \rightarrow -x$, the $\sin(x)$ function changes to $\sin(-x)$, which changes sign, to $-\sin(x)$ under this reflection operation (by the "*all-sin-tan-cos*" rule - the nemonic for which trigonometrical function is positive in each of the four quadrants associated with $0^o \leq x \leq 360^o$). Thus, we say that the $\sin(x)$ function has *odd* reflection symmetry (because of the sign change of $\sin(-x) = -\sin(x)$). Because the $\cos(x)$ function does not change sign under this reflection operation, we say that the $\cos(x)$ function has *even* reflection symmetry (because $\cos(-x) = +\cos(x)$).

The boundary condition, $y_{tot}(x{=}0, t) = 0$ for the fixed end at $x = 0$, which *must* be obeyed for *any* time, t, can *only* be satisfied if

$$y_{tot}(x = 0,t) = -A\sin(\omega t) + B\cos(\omega t) + C\sin(\omega t) + D\cos(\omega t) = 0$$

We group like terms (sines and cosines) together:

$$y_{tot}(x = 0,t) = (C - A)\sin(\omega t) + (B + D)\cos(\omega t) = 0$$

The *only* way that the boundary condition, $y_{tot}(x{=}0, t) = 0$ for the fixed end at $x = 0$ can be satisfied for *any* time, t is if the amplitudes, $C = A$ *.and.* $B = -D$. Physically, this means that the amplitudes of the right- and left-moving traveling sine-waves *must* be equal, and be in phase with each other. Physically, the amplitudes of the right- and left-moving traveling cosine-waves *must* also be equal, but be out-of-phase with each other by 180° (use the angle-addition formula to show that $\cos(x + 180^o) = -\cos(x)$).

The boundary condition, $y_{tot}(x=L, t) = 0$ for the fixed end at $x = L$, which *must* be obeyed for *any* time, t, can *only* be satisfied if

$$y_{tot}(x = L, t) = A\sin(kL - \omega t) + B\cos(kL - \omega t) + C\sin(kL + \omega t) + D\cos(kL + \omega t) = 0$$

However, because of the requirement from the boundary condition $y_{tot}(x=0, t) = 0$, that the amplitudes $C = A$ *.and.* $B = -D$, this can be rewritten as:

$$y_{tot}(x = L) = A[\sin(kL - \omega t) + \sin(kL + \omega t)] + B[\cos(kL - \omega t) - \cos(kL + \omega t)]$$

Using the angle-addition formulae:

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

and

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

then:

$$y_{tot}(x = L, t) = A[\sin(kL)\cos(\omega t) - \cos(kL)\sin(\omega t) + \sin(kL)\cos(\omega t) + \cos(kL)\sin(\omega t)]$$
$$+ B[\cos(kL)\cos(\omega t) + \sin(kL)\sin(\omega t) - \cos(kL)\cos(\omega t) + \sin(kL)\sin(\omega t)]$$

which becomes:

$$y_{tot}(x = L, t) = 2A\sin(kL)\cos(\omega t) + 2B\sin(kL)\sin(\omega t)$$
$$= 2[A\cos(\omega t) + B\sin(\omega t)]\sin(kL)$$

The *only* way that the boundary condition, $y_{tot}(x=L, t) = 0$ for the fixed end at $x = L$ can be satisfied for *any* time, t is if $\sin(kL) = 0$. This happens when $kL = n\pi$, where n is a positive integer, *i.e.* $n = 1, 2, 3, 4, ... etc$. Then $\sin(n\pi) = 0$, and hence $y_{tot}(x=L, t) = 0$ is satisfied for any time t. Since $k = 2\pi/\lambda$, the case $n = 0$ is not allowed, since the distance between end supports, L is finite, and $n = 0$ corresponds to $\lambda = \infty$. Physically, the requirement that the transverse displacement at $x = L$ be zero is the same requirement as that for the transverse displacement at $x = 0$.

Note that the relation $kL = 2\pi L/\lambda = n\pi$ implies that $\lambda = 2L/n$ - *i.e.* that the allowed wavelengths associated with transverse waves propagating on this string of length L between the fixed endpoints $x = 0$ and $x = L$, can only be integer fractions of the total length, L of the string! We can denote these special wavelengths, λ and wavenumbers, k by the subscript n, i.e. $\lambda_n = 2L/n$ and $k_n = n\pi/L$, $n = 1, 2, 3, 4, ... etc$. Since the longitudinal wave speed, $|v_x| = f\lambda = \omega/k$, then $f = |v_x|/\lambda_n = n|v_x|/2L$; we can denote these special frequencies, f and angular frequencies, $\omega = 2\pi f$ as $f_n = n|v_x|/2L$ and $\omega_n = n\pi|v_x|/L$, $n = 1, 2, 3, 4, ... etc$.

Thus, we discover that only *certain* wavelengths, corresponding to *certain* frequencies of sinusoidal-type transverse waves are able to propagate on a taught string of length, L, with fixed ends! These frequencies, $f_n = n|v_x|/2L$ (wavelengths, $\lambda_n = 2L/n$) with $n = 1, 2, 3, 4, ... etc$. are integer multiples (integer fractions), or *harmonics*, of the *fundamental* frequency (*fundamental* wavelength), $f_1 = |v_x|/2L$ ($\lambda_1 = 2L$), respectively. Note that the

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so-called $n = 1$ *fundamental*, with frequency, $f_1 = |v_x| /2L$, and wavelength, $\lambda_1 = 2L$ is also known as the *first harmonic*. The 2nd harmonic, with $n = 2$, has frequency, $f_2 = |v_x| /L = 2f_1$ and wavelength, $\lambda_2 = L = \lambda_1/2$, and so on, for the higher harmonics, $n = 3, 4, 5, ... etc$.

Thus, we find that the taught string, of length L with fixed ends at $x = 0$ and $x = L$ has *normal modes of vibration*, which for *each* such normal mode, the transverse displacement for an arbitrary point, x on the string, at *any* time t, is given by:

$$y_n(x,t) = [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)]\sin(k_n x)$$

To connect this formula with the above formula for $y_{tot}(x, t)$, note that for a given normal mode of vibration, with harmonic number, $n = 1, 2, 3, 4, ... etc$. we have simply re-defined the amplitudes, such that $A_n = 2A$ and $B_n = 2B$.

Note also that the physical amplitude of the nth normal mode of vibration is $C_n = [A_n^2 + B_n^2]^{1/2}$ (*n.b.* this result follows from use of mathematics associated with *complex variables* - discussed in detail in the lecture notes on Fourier analysis (*aka* harmonic analysis) of waveforms). Thus, we can alternatively write the transverse displacement, $y_n(x, t)$ as:

$$y_n(x,t) = C_n \cos(\omega_n t + \delta_n)\sin(k_n x) = C_n \sin(\omega_n t + \delta_n{}')\sin(k_n x)$$

where C_n is the amplitude of the nth normal mode of vibration, and δ_n ($\delta_n{}' = \delta_n + 90°$) is its phase, respectively. Note that by a suitable re-definition of the zero of time, *e.g.* $\omega_n t' = \omega_n t + \delta_n$, or $t' = t + \delta_n/\omega_n$, or, instead, $\omega_n t^* = \omega_n t + \delta_n{}'$, or $t^* = t + \delta_n{}'/\omega_n$, the above expression for the transverse displacement, $y_n(x, t)$ for the nth normal mode of vibration can be rewritten either as:

$$y_n(x,t) = C_n \cos(\omega_n t')\sin(k_n x)$$

or as:

$$y_n(x,t) = C_n \sin(\omega_n t^*)\sin(k_n x)$$

The above mathematical expression(s) for the transverse displacement, $y_n(x, t)$ associated with the nth normal mode of vibration of a string of length, L with fixed end-points at $x = 0$ and $x = L$ describe a transverse *standing wave*. A standing wave is a wave that is *not* a *traveling* wave - it has no *longitudinal* motion, because, from the above mathematics, we see that a standing wave is in fact nothing more than a linear superposition of appropriately-phased right- and left-moving traveling waves of equal amplitude and frequency! The *envelope* of the standing wave as a function of the position x, over the interval $0 \leq x \leq L$ along the length of the string is given by $C_n sin(k_n x)$. The envelope of the standing wave is thus a snapshot of the standing wave, *e.g.* at $t' = 0$.

The following plot shows a snapshot time sequence of one half of a cycle of oscillation of a transverse standing wave - the $n = 1$ fundamental mode of vibration on a string of length, $L = 10$ m, with fixed ends located at $x = 0$ and $x = L = 10$ m. The amplitude of the pulse is $A_1 = 1.0$ m, the wavelength of the fundamental is $\lambda = 2L = 20$ m, the longitudinal wave speed $|v_x| = 1.0$ m/sec, thus the frequency is $f = |v_x|/\lambda = 0.05$ Hz.

Note that the sequence of (11) time snapshots progress vertically downward in 1 second steps, starting at $t = 0$ *sec*, and ending at $t = 10$ *sec*. A full cycle of oscillation of this string has a period, $\tau = 1/f = 20$ *sec*.

$Y_1(x, t)$ vs. x for Standing Wave
n = 1 Fundamental
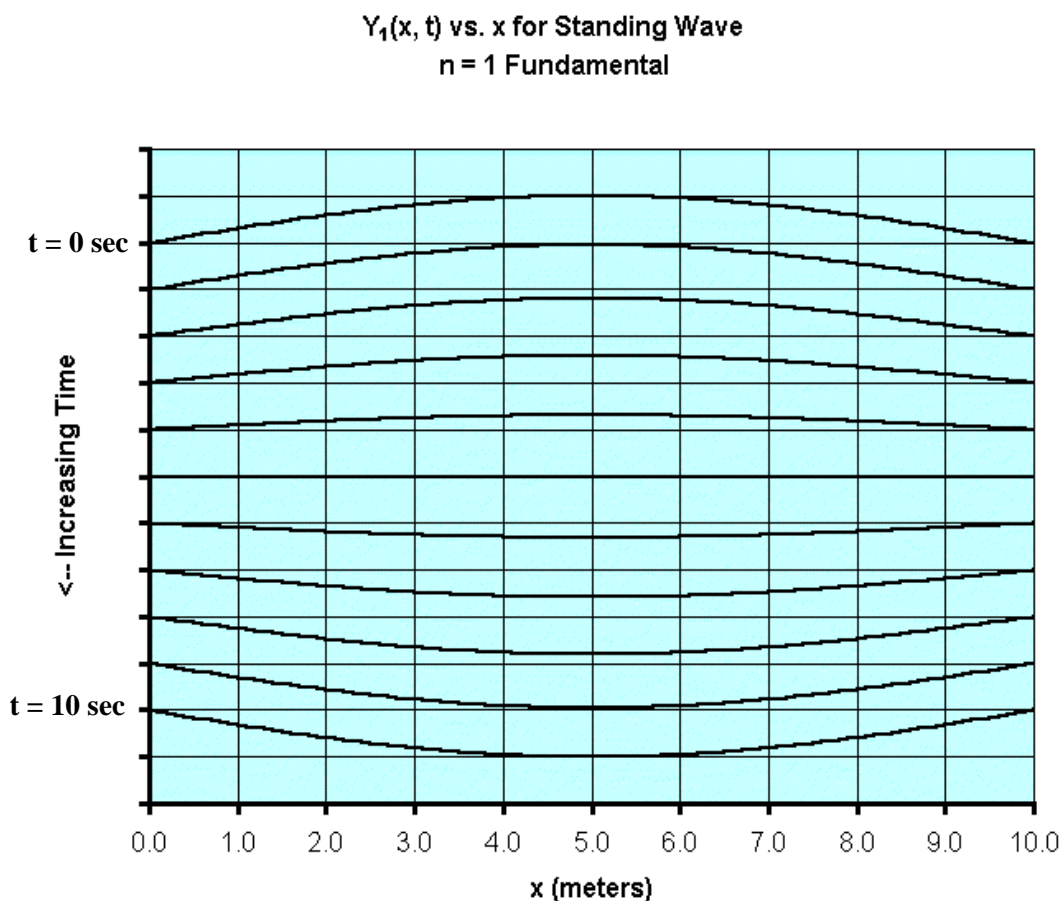
In general, it is entirely possible that *many* different normal modes of vibration could be simultaneously present on the string. In playing a guitar, this is in fact what actually happens! (We will discuss this in much greater detail in the lecture notes on Fourier analysis of waveforms.) The principle of linear superposition then tells us that the general solution of a vibrating string with fixed ends, with all possible normal modes of vibration present will have an overall transverse displacement that can be written as a sum (*i.e.* a linear superposition) of the individual transverse displacements, $y_n(x, t)$ of these normal modes:

$$y_{tot}(x,t) = \sum_{n=1}^{n=\infty} y_n(x,t) = \sum_{n=1}^{n=\infty} \left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right]\sin(k_n x)$$

Thus, we can alternatively write the general solution, $y_{tot}(x, t)$ as:

$$y_{tot}(x,t) = \sum_{n=1}^{n=\infty} y_n(x,t) = \sum_{n=1}^{n=\infty} C_n \cos(\omega_n t + \delta_n)\sin(k_n x) = \sum_{n=1}^{n=\infty} C_n \sin(\omega_n t + \delta_n')\sin(k_n x)$$

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where C_n is the amplitude of the nth normal mode of vibration, and δ_n ($\delta_n' = \delta_n + 90^o$) is its phase, respectively.

<div align="center"><i><u>Wave Energy</u></i></div>

As mentioned earlier, a wave e.g. on a string has associated with it energy, E and momentum, *P*. If we consider an infinitesimally small segment of string, of equilibrium length, Δx, then for small-amplitude vibrations, the mass associated with this string segment is $\Delta m = \mu\Delta x$. The transverse velocity of this string segment is $u_y(x,t)$. The *kinetic* energy (in *Joules*) associated with this string segment, located at the point *x* along the string and at time *t* is given by:

$$\Delta KE(x,t) = \tfrac{1}{2}\Delta p_y^2(x,t)\big/\Delta m = \tfrac{1}{2}\Delta m\, u_y^2(x,t) = \tfrac{1}{2}\,\mu\,\Delta x\,u_y^2(x,t)$$

Transverse momentum and transverse velocity of a given string segment are related to each other by $\Delta p_y(x,t) = \Delta m\, u_y(x,t)$. The kinetic energy, $\Delta KE(x,t)$ associated with a vibrating string segment is a maximum when the magnitude of the transverse velocity, $|u_y(x,t)|$ of the string segment is a maximum. For sinusoidal-type transverse waves on a string, this occurs when the transverse displacement of the string segment, $y(x,t) = 0$.

We can obtain the total kinetic energy associated with the entire vibrating string at a given instant in time, *t* by summing up (*i.e.* adding) all of the individual kinetic energy contributions associated with each infinitesimal string segment, over the entire length of the string:

$$KE(t) = \sum_{n=1}^{N}\Delta KE(x,t) = \tfrac{1}{2}\mu\sum_{n=1}^{N}\Delta x\,u_y^2(x,t)$$

In the limit that the lengths of individual string segments, Δx become truly infinitesimal, then $\Delta x \to dx$, and the above summation formally becomes an integration over the length of the string, say from $x = 0$ to $x = L$:

$$KE(t) = \tfrac{1}{2}\mu\int_{x=0}^{x=L}dx\,u_y^2(x,t) = \frac{p_y^2(t)}{2m}$$

The *potential* energy, $\Delta PE(x,t)$ (in *Joules*) of an infinitesimal string segment of length, Δx of the vibrating string is associated with the (elastic) *stretching* of the string, under tension, *T* from its equilibrium position, $y(x,t) = 0$, to its transversely displaced position, $y(x,t) \neq 0$. (At the atomic level, the macroscopic stretching of the string corresponds to atoms making up the string material stretching apart from each other by very small distances, a fraction of an Angstrom (1 Å = 10^{-10} m). Typical inter-atomic separation distances are on the order of ~ few Angstroms.) The displacement amplitude is again assumed to be very small, such that although the string stretches elastically, the string tension, *T* remains constant. Then the potential energy, $\Delta PE(x,t) = T\Delta L$, where ΔL is the *elongation* (*i.e.* stretching) of the infinitesimal string segment, as shown in the figure below:
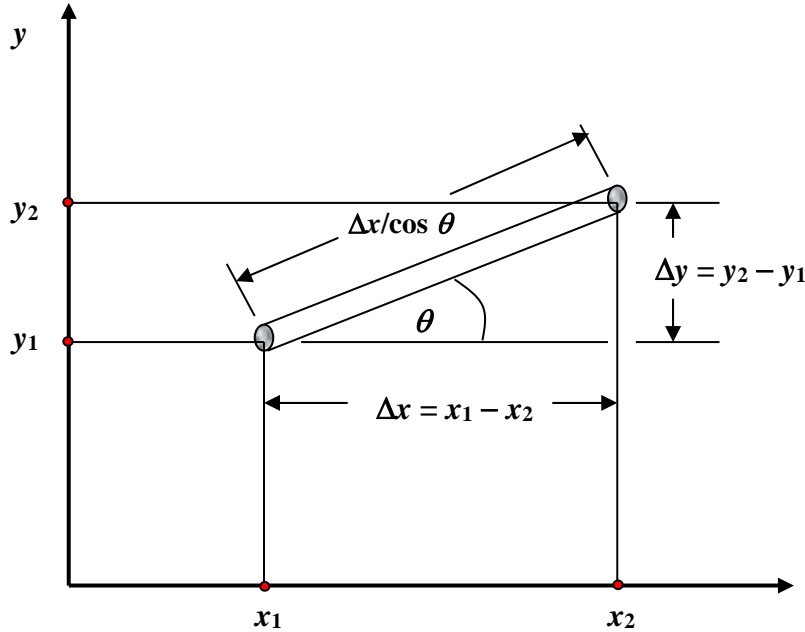
The elongation, ΔL of this infinitesimal string segment in changing from its equilibrium position, $y(x,t) = 0$ to its transversely displaced position, $y(x,t) \neq 0$ is given by

$$\Delta L(x,t) = \frac{\Delta x}{\cos\theta(x,t)} - \Delta x = \Delta x\left(\frac{1}{\cos\theta(x,t)} - 1\right)$$

Note that the mass, $\Delta m = \mu\Delta x$ associated with the (now stretched out) infinitesimal string segment does *not* change in the stretching process.

For small amplitude vibrations the angle, $\theta(x,t)$ is very small. Because of this, we can approximate cos θ by the first two leading terms of its Taylor series expansion:

$$\cos\theta = \sum_{n=0}^{n=\infty}\frac{(-1)^n}{2n!}\theta^{2n} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ... \approx 1 - \frac{\theta^2}{2!} = 1 - \frac{\theta^2}{2}$$

where $n! = n(n-1)(n-2)...1$, and $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 12$, *etc*. Now because the angle, θ is small, then tan $\theta \approx \theta$, but tan $\theta = \partial y/\partial x$ = slope of the infinitesimal string segment. (The small-angle relation(s): tan $\theta \approx \theta$, and also sin $\theta \approx \theta$ are again consequences of truncating the Taylor series expansions for tan θ and sin θ to the first terms in these expansions.) Thus, for small amplitudes, $\theta(x,t) \approx \partial y(x,t)/\partial x$ = slope of the infinitesimal string segment.

Thus, for small amplitudes, the elongation, $\Delta L(x,t)$ of the string segment is given by:

$$\Delta L(x,t) = \frac{\Delta x}{1 - \frac{1}{2}\left(\frac{\partial y(x,t)}{\partial x}\right)^2 + ...} - \Delta x \approx \tfrac{1}{2}\Delta x\left(\frac{\partial y(x,t)}{\partial x}\right)^2$$

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where we have *also* used the approximation to the Taylor series expansion, $1/(1-\varepsilon) \approx \varepsilon$ for $\varepsilon \approx 0$, in the above formula.

Putting this all together, the potential energy, $\Delta PE(x,t)$ of an infinitesimal string segment of length, Δx for small amplitude string vibrations, is given by:

$$\Delta PE(x,t) = \tfrac{1}{2}T\Delta x \left(\frac{\partial y(x,t)}{\partial x} \right)^2$$

The potential energy, $\Delta KE(x,t)$ associated with a vibrating string segment is a maximum when the magnitude of the slope, $|slope| = |\partial y(x,t)/\partial x|$ of the string segment is a maximum. For sinusoidal-type transverse waves on a string, this occurs when the transverse displacement, $y(x,t)$ is a maximum.

We can again obtain the total potential energy associated with the entire vibrating string at a given instant in time, t by summing up (i.e. adding) all of the individual potential energy contributions associated with each infinitesimal string segment, over the entire length of the string:

$$PE(t) = \sum_{n=1}^{N} \Delta PE(x,t) = \tfrac{1}{2}T\sum_{n=1}^{N} \Delta x \left(\frac{\partial y(x,t)}{\partial x} \right)^2$$

In the limit that the lengths of individual string segments, Δx become truly infinitesimal, then $\Delta x \to dx$, and the above summation formally becomes an integration over the length of the string, say from $x = 0$ to $x = L$:

$$PE(t) = \tfrac{1}{2}T\int_{x=0}^{x=L} dx \left(\frac{\partial y(x,t)}{\partial x} \right)^2$$

As we have seen above, a standing wave in the nth normal mode of vibration has a transverse displacement as a function of position, x and time, t of:

$$y_n(x,t) = C_n \cos(\omega_n t)\sin(k_n x)$$

The transverse velocity, $u_{yn}(x,t)$ of the nth normal mode of vibration as a function of position, x and time, t is given by:

$$u_{yn}(x,t) = \frac{dy(x,t)}{dt} = \frac{d}{dt}\left[C_n \cos(\omega_n t)\sin(k_n x)\right] = C_n \frac{d}{dt}\left[\cos(\omega_n t)\right]\sin(k_n x) = -\omega_n C_n \sin(\omega_n t)\sin(k_n x)$$

The kinetic energy of the standing wave associated with the nth normal mode of vibration of the entire string, of length, L as a function of time, t is given by:

$$KE_n(t) = \tfrac{1}{2}\mu\int_{x=0}^{x=L} dx \ u_{yn}^2(x,t) = \tfrac{1}{2}\mu\omega_n^2 C_n^2 \sin^2(\omega_n t)\int_{x=0}^{x=L} dx \sin^2(k_n x)$$

Now the integral $\int_{x=0}^{x=L} dx \sin^2(k_n x) = \left[\frac{x}{2} - \frac{\sin(2k_n x)}{4k_n} \right]_{x=0}^{x=L} = \frac{L}{2} - \frac{\sin(2k_n L)}{4k_n}$

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However, recall that for the nth normal mode of vibration, the wavenumber, $k_n = 2\pi/\lambda_n$ and $\lambda_n = 2L/n$, $n = 1, 2, 3, 4, ...$ etc. Thus, $(2k_nL) = (k_n2L) = [(2\pi/\lambda_n)*2L] = [(2\pi n/2L)*2L] = 2\pi n$. But $\sin(2\pi n) = 0$ for $n = 1, 2, 3, 4, ...$ etc. Thus, this integral $= L/2$. Hence, the kinetic energy of the standing wave associated with the nth normal mode of vibration of the entire string, of length, L as a function of time, t is

$$KE_n(t) = \tfrac{1}{4}\mu L\omega_n^2 C_n^2 \sin^2(\omega_n t) = \tfrac{1}{4}m\omega_n^2 C_n^2 \sin^2(\omega_n t)$$

Note that the last term on the right hand side of the above equation used the relation for the mass per unit length of the string, $\mu = m/L$.

The slope, $\partial y_n(x,t)/\partial x$ of the the nth normal mode of vibration as a function of position, x and time, t is given by:

$$\frac{\partial y_n(x,t)}{\partial x} = \frac{\partial}{\partial x}\big[C_n\cos(\omega_n t)\sin(k_n x)\big] = C_n\cos(\omega_n t)\frac{\partial}{\partial x}\big[\sin(k_n x)\big] = k_n C_n\cos(\omega_n t)\cos(k_n x)$$

The potential energy of the standing wave associated with the nth normal mode of vibration of the entire string, of length, L as a function of time, t is given by

$$PE_n(t) = \tfrac{1}{2}T\int_{x=0}^{x=L} dx\left(\frac{\partial y(x,t)}{\partial x}\right)^2 = \tfrac{1}{2}T\ k_n^2\cos^2(\omega_n t)\int_{x=0}^{x=L} dx\cos(k_n x)$$

Now the integral

$$\int_{x=0}^{x=L} dx\cos^2(k_n x) = \left[\frac{x}{2} + \frac{\sin(2k_n x)}{4k_n}\right]_{x=0}^{x=L} = \frac{L}{2} + \frac{\sin(2k_n L)}{4k_n}$$

Thus, from the above, we know that this integral $= L/2$. Hence the potential energy of the standing wave associated with the the nth normal mode of vibration of the entire string, of length L, as a function of time, t is

$$PE_n(t) = \tfrac{1}{4}T\ L\ k_n^2 C_n^2\cos^2(\omega_n t)$$

Now, since the longitudinal wave speed, $|v_x| = \omega_n/k_n$, or $k_n = \omega_n/|v_x|$, then

$$PE_n(t) = \tfrac{1}{4}T\ L\ k_n^2 C_n^2\cos^2(\omega_n t) = \tfrac{1}{4}T\ L\frac{\omega_n^2}{v_x^2}C_n^2\cos^2(\omega_n t)$$

However, we also know that $v_x^2 = T/\mu$. Thus, the potential energy of the standing wave associated with the nth normal mode of vibration of the entire string, of length, L as a function of time, t is

$$PE_n(t) = \tfrac{1}{4}\mu L\ \omega_n^2 C_n^2\cos^2(\omega_n t) = \tfrac{1}{4}m\omega_n^2 C_n^2\cos^2(\omega_n t)$$

The *total* energy, $E_n(t)$ (in *Joules*) of the standing wave associated with the nth normal mode of vibration of the entire string, of length, L as a function of time, t is

$$E_n(t) = KE_n(t) + PE_n(t) = \tfrac{1}{4}m\omega_n^2 C_n^2\sin^2(\omega_n t) + \tfrac{1}{4}m\omega_n^2 C_n^2\cos^2(\omega_n t) = \tfrac{1}{4}m\omega_n^2 C_n^2$$

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Note that the last term on the right hand side of the above equation used the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$. Thus, we discover that the total energy, $E_n(t)$ of the standing wave associated with the nth normal mode of vibration of the entire string, of length, L as a function of time, t is a ***constant*** - the total energy, $E_n(t) = E_n$, *i.e.* it has no time dependence! Note also that the potential energy, kinetic energy, and total energy of the standing wave for the nth normal mode of vibration all vary as the square of the frequency, since $\omega_n = 2\pi f_n$, and as the square of the amplitude, C_n, and depend linearly on the total mass, m of the string of length, L.

The potential energy, $PE_n(t)$ kinetic energy $KE_n(t)$ and total energy $E_n(t)$ associated with the nth normal mode of vibration of the entire string, of length, L as a function of time, t are shown in the figure below. The values used for making this plot were $m = 1.0$ kg ($L = 10$ m), $n = 1$, $\tau_1 = 10.0$ sec, $f_1 = 1/\tau_1 = 1/20$ Hz, $|v_x| = 1.0$ m/s ($\lambda_1 = 2L = 20$ m), and $C_1 = 1.0$ m.

KE(t), PE(t) and Etot(t) vs. Time, t
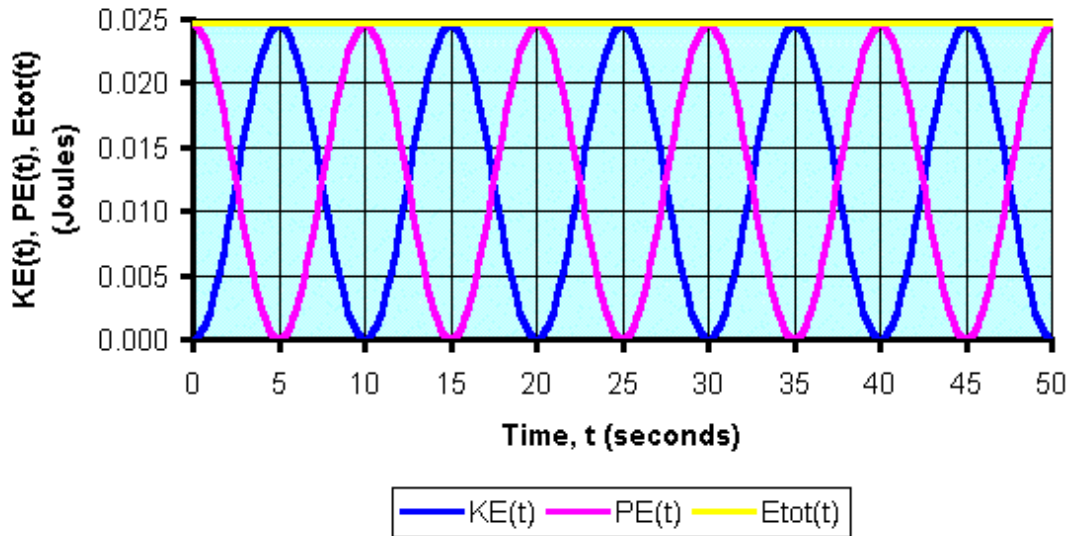
At one instant in time, say at $t = 0$ sec in the above figure, when the transverse displacement of the string is a maximum, with zero transverse velocity - the total energy of the string is all in the form of potential energy. One-quarter of a cycle later at $t = 5$ *sec*, when the string has maximum transverse velocity, the transverse displacement, $y_n(x,t=5)$ $= 0 =$ the equilibrium position of the string, and here the total energy of the string is all in the form of kinetic energy. Another one-quarter of a cycle later, at $t = 10$ sec, the transverse displacement of the string is again at a maximum (but is now on the *opposite* side of the equilibrium position, $y_n(x,t=5) = 0$), with zero transverse velocity, the total energy of the string is again all in the form of potential energy. One further quarter of a cycle later, at $t = 15$ *sec*, the string has again zero transverse displacement, $y_n(x,t=15) = 0$, but again has maximal transverse velocity, and again the total energy of the string is all in the form of kinetic energy. A further quarter of a cycle, at $t = 20$ sec, the string is back where it originally started, with maximum transverse displacement, again with zero

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transverse velocity - the total energy of the string is all in the form of potential energy. And on and on, through each successive cycle.

The time-average values of the kinetic energy and/or potential energy of the standing wave associated with the nth normal mode of vibration of the entire string, of length, L are

$$\langle KE_n(t) \rangle = \tfrac{1}{2} KE_n(t) = \tfrac{1}{8} m\omega_n^2 C_n^2$$

and

$$\langle PE_n(t) \rangle = \tfrac{1}{2} PE_n(t) = \tfrac{1}{8} m\omega_n^2 C_n^2 = \langle KE_n(t) \rangle$$

This is because the time-averaged values of $\sin^2(\omega_n t)$ and $\cos^2(\omega_n t)$ are (averaging over one cycle, of period $\tau_n = 1/f_n$):

$$\langle \sin^2(\omega_n t) \rangle = \tfrac{1}{\tau_n} \int_{t=0}^{t=\tau_n} dt\, \sin^2(\omega_n t) = \tfrac{1}{2}$$

and

$$\langle \cos^2(\omega_n t) \rangle = \tfrac{1}{\tau_n} \int_{t=0}^{t=\tau_n} dt\, \cos^2(\omega_n t) = \tfrac{1}{2} = \langle \sin^2(\omega_n t) \rangle$$

If there are many normal modes of oscillation present as standing waves on a string, then the total energy, E_{tot} summing over all modes is given by:

$$E_{tot} = \sum_{n=1}^{n=\infty} E_n = \sum_{n=1}^{n=\infty} E_n(t) = \sum_{n=1}^{n=\infty} KE_n(t) + PE_n(t) = \tfrac{1}{4} m \sum_{n=1}^{n=\infty} \omega_n^2 C_n^2$$

The time-averaged total energy, <E_{tot}> summing over all modes is given by:

$$\langle E_{tot} \rangle = \sum_{n=1}^{n=\infty} \langle E_n \rangle = \sum_{n=1}^{n=\infty} \langle E_n(t) \rangle \equiv \sum_{n=1}^{n=\infty} \langle KE_n(t) \rangle + \langle PE_n(t) \rangle = \tfrac{1}{8} m \sum_{n=1}^{n=\infty} \omega_n^2 C_n^2 = \tfrac{1}{2} E_{tot}$$

Note that the time-averaged total energy, <E_{tot}> is *not* the same as the instantaneous energy, E_{tot} because of how it is defined - the time averages of the kinetic and potential energy for each mode, <$KE_n(t)$> and <$PE_n(t)$> are carried out *separately*. Thus, E_{tot} is also known as the *peak* total energy, while <E_{tot}> = ½ E_{tot} is known as the so-called *root-mean-square*, or *rms* total energy.

Wave Power

The instantaneous power, $P_n(t)$ (in *Watts*, or *Joules/sec*) associated with the nth normal mode of vibration of the entire string, of length, L at a given time, t in a standing wave is given by

$$P_n(t) = \frac{1}{\tau_n}\left[KE_n(t) + PE_n(t)\right] = \frac{1}{\tau_n} E_n(t) = \frac{1}{\tau_n} E_n = \frac{1}{\tau_n}\left[\tfrac{1}{4} m\omega_n^2 C_n^2\right] = P_n$$

Note that the instantaneous power has *no* time dependence - it is a constant, independent of time. The instantaneous power is also known as the *peak* power.

The time-averaged power, $<P(t)>$ associated with the nth normal mode of vibration of the entire string, of length, L in a standing wave is given by

$$\langle P_n(t)\rangle \equiv \frac{1}{\tau_n}\left[\langle KE_n(t)\rangle + \langle PE_n(t)\rangle\right] = \frac{1}{\tau_n}\langle E_n(t)\rangle = \frac{1}{\tau_n}\left[\tfrac{1}{8}m\omega_n^2 C_n^2\right] = \tfrac{1}{2}P_n$$

Here again, the time-averaged power, $<P_n>$) associated with the nth normal mode of vibration of the entire string, of length, L is *not* the same as the instantaneous power, P_n because of how it is defined - the time averages of the kinetic and potential energy for each mode, $<KE_n(t)>$ and $<PE_n(t)>$ are carried out *separately*. Thus, P_n is also known as the *peak* power, while $<P_n> = \frac{1}{2}P_n$ is known as the *root-mean-square*, or *rms* power.

If there are many normal modes of oscillation present as standing waves on a string, then the total power, P_{tot} summing over all modes is given by:

$$P_{tot} = \sum_{n=1}^{n=\infty} P_n = \sum_{n=1}^{n=\infty} P_n(t) = \sum_{n=1}^{n=\infty}\left(\frac{1}{\tau_n}\right)\left[KE_n(t) + PE_n(t)\right] = \sum_{n=1}^{n=\infty}\left(\frac{1}{\tau_n}\right)\left[\tfrac{1}{4}m\omega_n^2 C_n^2\right] = \sum_{n=1}^{n=\infty}\left(\frac{1}{\tau_n}\right)E_n$$

The time-averaged total power, $<P_{tot}>$ summing over all modes is given by:

$$<P_{tot}> = \sum_{n=1}^{n=\infty} <P_n(t)> \equiv \sum_{n=1}^{n=\infty}\left(\frac{1}{\tau_n}\right)\left(<KE_n(t)> + <PE_n(t)>\right) = \sum_{n=1}^{n=\infty}\left(\frac{1}{\tau_n}\right)\left[\tfrac{1}{8}m\omega_n^2 C_n^2\right] = \sum_{n=1}^{n=\infty}\left(\frac{1}{\tau_n}\right)<E_n>$$

Note that the time-averaged total power, $<P_{tot}>$ is *not* the same as the instantaneous power, P_{tot} because of how it is defined - the time averages of the kinetic and potential energy for each mode, $<KE_n(t)>$ and $<PE_n(t)>$ are carried out *separately*. Thus, P_{tot} is also known as the *peak* total power, while $<P_{tot}> = \frac{1}{2}P_{tot}$ is known as the *root-mean-square*, or *rms* total power.

Exercises:

1. On a sheet of graph paper, draw an arbitrarily-shaped waveform, $y(x, t) = f(x - v_x t)$ representing the transverse displacement, $y(x, t=0)$ of a right-moving traveling wave along the x-axis at time $t = 0$ - *e.g.* a triangular-shaped pulse. Having chosen the amplitude, y_o and longitudinal velocity, v_x of this traveling wave, draw where this waveform is *e.g.* when $t = 1$ second, $t = 5$ seconds, and/or $t = 10$ seconds. Verify that $(x - v_x t) = $ constant $= K$ for a given point on the waveform for each choice of time, t.

2. On a sheet of graph paper, draw an arbitrarily-shaped waveform, $y(x, t) = f(x + v_x t)$ representing the transverse displacement, $y(x, t=0)$ of a left-moving traveling wave along the x-axis at time $t = 0$ - *e.g.* a triangular-shaped pulse. Having chosen the amplitude, y_o and longitudinal velocity, $-v_x$ of this traveling wave, draw where this waveform is *e.g.* when $t = 1$ second, $t = 5$ seconds, and/or $t = 10$ seconds. Verify that $(x + v_x t) = $ constant $= K'$ for a given point on the waveform for each choice of time, t.

3. For a right-moving gaussian traveling wave, $y(x, t) = y_o \exp\{-(x - v_x t)^2\}$, with amplitude, $y_o = 1\ m$ (= transverse displacement of the string from its equilibrium position), and *longitudinal* wave velocity, $v_x = +1\ m/sec$, show that the partial derivative of $y(x, t)$ with respect to x, $\partial y(x, t)/\partial x$ (which physically is the local slope of the string at the point x, at a given time t), is given by the expression $\partial y(x, t)/\partial x = -2(x - v_x t)\ y_o \exp\{-(x - v_x t)^2\} = -2(x - v_x t)\ y(x, t)$. Then, using *e.g.* a spread-sheet program, such as Excel, or other mathematical software, make plot(s) of the transverse velocity of the right-moving gaussian traveling wave on the string, $u_y(x, t) = -(\partial y(x, t)/\partial x)\ v_x$, as a function of position, x, for times $t = -5$, 0 and $+5$ seconds.

4. Show that a traveling wave (either right- or left-moving), $y(x, t) = f(x \mp v_x t)$ satisfies the wave equation, $\partial^2 y(x,t)/\partial x^2 = 1/|v_x|^2\ \partial^2 y(x,t)/\partial t^2$ by explicitly carrying out the differentiation on both sides of the equation, using the chain-rule of differentiation.

5. After you have read through, learned and understood the contents of the lecture notes on Fourier analysis of waveforms, then come back to these lecture notes and derive the relation(s):

$$y_n(x,t) = C_n \cos(\omega_n t + \delta_n)\sin(k_n x) = C_n \sin(\omega_n t + \delta_n')$$

$$y_n(x,t) = C_n \cos(\omega_n t')\sin(k_n x)$$

$$y_n(x,t) = C_n \sin(\omega_n t\,*)\sin(k_n x)$$

from the relation:

$$y_n(x,t) = [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)]\sin(k_n x)$$

References for Waves and Further Reading:

1. The Physics of Musical Instruments, 2nd Edition, Neville H. Fletcher and Thomas D. Rossing, Springer, 1997.

2. The Acoustical Foundations of Music, John Backus, W.W. Norton and Company, 1969.

3. Mathematical Methods of Physics, 2nd Edition, Jon Matthews and R.L. Walker, W.A. Benjamin, Inc., 1964.

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