## Physical Quantities Associated with a Complex Sound Field

Introduction:
A complex sound field $\tilde{S}(\vec{r}, t)$ at a point in space $\vec{r}=(x \hat{x}, y \hat{y}, z \hat{z})$ and instant in time $t$ can be completely/fully/uniquely characterized by measuring two physical quantities at the space-time point $(\vec{r}, t)$ : the instantaneous complex pressure $\tilde{p}(\vec{r}, t)$ (a scalar quantity) and the so-called instantaneous particle velocity $\overrightarrow{\tilde{u}}(\vec{r}, t)$ (a 3-D/ vector quantity) $\overrightarrow{\tilde{u}}(\vec{r}, t)=\tilde{u}_{x}(\vec{r}, t) \hat{x}+\tilde{u}_{y}(\vec{r}, t) \hat{y}+\tilde{u}_{z}(\vec{r}, t) \hat{z}$. The physical meaning of particle velocity is formally obtained from mathematics of vector calculus - the instantaneous particle velocity $\overrightarrow{\tilde{u}}(\vec{r}, t)$ is the 3-D/vector velocity associated with an infinitesimal volume element $d V(\vec{r})$ of the sound medium centered on the space-point $\vec{r}$. The instantaneous complex particle velocity $\overrightarrow{\tilde{u}}(\vec{r}, t)$ is the average over the instantaneous complex 3-D/vector velocities of individual air molecules contained within the infinitesimal volume element $d V(\vec{r})$ at the time $t$. On average, the thermal velocities of individual air molecules contained within $d V(\vec{r})$ at time $t$ associated with Brownian random walk motion cancel.

In the following, for simplicity's sake, we focus our attention on linear 1-D complex sound fields/1-D sound propagation only. By linear complex sound fields, specifically we mean loudness levels $L \ll 134 d B\{|\tilde{p}| \ll 100$ RMS Pascals $\}$ ). We additionally restrict the discussion throughout this document solely to that of monochromatic, harmonically-varying complex sound fields associated with a single frequency component at $\omega=2 \pi f$. For definiteness' sake, we choose the $\hat{z}$ direction for sound propagation in our 1-D system(s). It is a straightforward exercise to generalize the following 1-D vector results to 3-D. We leave this as an exercise for the interested reader.

A generic, instantaneous complex quantity $\tilde{a}(\vec{r}, t)$ associated with a monochromatic, harmonically-varying complex sound field can be written as:
$\tilde{a}(\vec{r}, t)=\left\{a_{r}(\vec{r})+i a_{i}(\vec{r})\right\} e^{+i \omega t}=|\tilde{a}(\vec{r})| e^{i \varphi_{a}} e^{+i \omega t}=\left\{|\tilde{a}(\vec{r})| \cos \varphi_{a}+i|\tilde{a}(\vec{r})| \sin \varphi_{a}\right\} e^{+i \omega t}$ where $i \equiv \sqrt{-1}$.
The time dependence associated with the generic complex quantity is of the form
$e^{+i \omega t}=\cos \omega t+i \sin \omega t$. Note that the $+\operatorname{sign}$ in this definition/convention is important/physically significant, because we use phase-sensitive lock-in amplifier techniques in various of the experiments in the 193POM/406POM lab, all of which are zero degree phase-referenced to a sine-wave function generator signal $V_{f g}(t)=V_{o} e^{+i \omega t}$ that also is used to create the complex sound field. Because of this fact, we are required to adopt the $e^{+i \omega t}$ convention in all of our formulae in order that the mathematical formulae used to describe the physics coincide with the actual functioning of the lock-in amplifiers used in the complex/phase-sensitive acoustic experiments in our lab.

Without getting bogged down in the gory details of how a lock-in amplifier actually works (which is amazing in itself!), the short/brief explanation is that by using the sine-wave function generator signal $\left|\tilde{V}_{f g}(t)=\left|\tilde{V}_{o}\right| e^{+i \omega t}\right.$ as a reference signal, the lock-in amplifier compares the complex signal $\tilde{a}(\vec{r}, t)=|\tilde{a}(\vec{r})| e^{+i \omega t} e^{i \varphi_{a}}$ to that of the reference signal and figures out how much of the complex signal $\tilde{a}(\vec{r}, t)$ is in phase (or $-180^{\circ}$ out of phase) with the reference signal $a_{r}(\vec{r})=|\tilde{a}(\vec{r})| \cos \varphi_{a}$ and how much of the complex signal $\tilde{a}(\vec{r}, t)$ is $\pm 90^{\circ}$ out of phase with the reference signal $a_{i}(\vec{r})=|\tilde{a}(\vec{r})| \sin \varphi_{a}$. Since $\tilde{V}_{f g}(t)=\left|\tilde{V}_{o}\right| e^{+i \omega t}$ and $\tilde{a}(\vec{r}, t)=|\tilde{a}(\vec{r})| e^{+i \omega t} e^{i \varphi_{a}}$ both have the same time dependence, the in-phase and $90^{\circ}$ out-of-phase components of $\tilde{a}(\vec{r}, t)$ output from the lock-in amplifier have no time dependence. The actual signals output from the lock-in amplifier are DC voltages representing the RMS (Root-Mean Square) in-phase and $90^{\circ}$ out-ofphase amplitudes of the complex signal $\tilde{a}(\vec{r}, t)$ being analyzed by the lock-in amplifier. The two DC voltages output from the lock-in amplifier can then be recorded e.g. using a pair of Analog-to-Digital Converters (ADCs) to digitize the two DC voltages, e.g. under computer control.

For historical reasons, the in-phase amplitude component of the complex signal amplitude $\tilde{a}(\vec{r})$ is known as the so-called "real" component $a_{r}(\vec{r})=\operatorname{Re}\{\tilde{a}(\vec{r})\}=|\tilde{a}(\vec{r})| \cos \varphi_{a}$ of $\tilde{a}(\vec{r})$, the $90^{\circ}$ out-of-phase amplitude component of the signal amplitude $\tilde{a}(\vec{r})$ is known as the so-called "imaginary" component $a_{i}(\vec{r})=|\tilde{a}(\vec{r})| \sin \varphi_{a}$ of $\tilde{a}(\vec{r})$. The 2-D figure below $\{$ known as the complex plane\} shows the relation between the complex amplitude $\tilde{a}(\vec{r})$, its magnitude $|\tilde{a}(\vec{r})|$ and the in-phase/ "real" and $90^{\circ}$ out-of-phase components, $a_{r}(\vec{r})$ and $a_{i}(\vec{r})$ respectively. The horizontal axis is known as the Real axis (Re), the vertical axis is known as the Imaginary axis (Im). Thus, in the complex plane:


The magnitude of the generic complex amplitude $\tilde{a}(\vec{r})$ is defined as (suppressing the $\vec{r}$ dependence): $\| \tilde{a} \mid \equiv \sqrt{\tilde{a} \tilde{a}^{*}}=\sqrt{\left(a_{r}+i a_{i}\right)\left(a_{r}-i a_{i}\right)}=\sqrt{a_{r}^{2}+a_{r}^{2}}$ where the $*$ symbol denotes complex conjugation $i^{*} \equiv-i=-\sqrt{-1}$, thus $\tilde{a}^{*}=a_{r}-i a_{r}$. The complex phase of $\tilde{a}(\vec{r})$ is $\varphi_{a}=\tan ^{-1}\left(a_{i} / a_{r}\right)$
which is defined relative to the phase of the sine-wave signal output from the function generator (by definition $\varphi_{f g} \equiv 0$ ) because it is used as the reference signal by the lock-in amplifier(s).
In the table below, we summarize six complex quantities associated with a sound field $\tilde{S}(\vec{r}, t)$ :

## Complex Sound Field Quantity:

Magnitude:
Pressure:
1-D Part. Velocity: $\quad \tilde{u}_{z} \equiv u_{z r}+i u_{z i}=\left|\tilde{u}_{z}\right| e^{i \varphi_{u_{z}}}$

1-D Part. Displcmnt: | $\tilde{d}_{z} \equiv d_{z r}+i d_{z i}=\left\|\tilde{d}_{z}\right\| e^{i \varphi_{d_{z}}}$ | $\left\|\tilde{d}_{z}\right\| \equiv \sqrt{\tilde{d}_{z} \tilde{d}_{z}^{*}}=\sqrt{d_{z r}^{2}+d_{z i}^{2}}$ |
| :--- | :--- |

1-D Part. Accelrn: $\quad \tilde{a}_{z} \equiv a_{z r}+i a_{z i}=\left|\tilde{a}_{z}\right| e^{i \varphi_{a_{z}}}| |\left|\tilde{a}_{z}\right| \equiv \sqrt{\tilde{a}_{z} \tilde{a}_{z}^{*}}=\sqrt{a_{z r}^{2}+a_{z i}^{2}}$

1-D Spec.Ac.Impdnc: $:$| $\tilde{z}_{z} \equiv z_{z r}+i z_{z i}=\mid \tilde{z}_{z}$ | $e^{i \varphi_{z_{z}}}$ |
| :--- | :--- |
| $\tilde{z}_{z} \mid \equiv \sqrt{\tilde{z}_{z} \tilde{z}_{z}^{*}}=\sqrt{z_{z r}^{2}+z_{z i}^{2}}$ |  |

1-D Sound Intensity: | $\tilde{I}_{z} \equiv I_{z r}+i I_{z i}=\mid \tilde{I}_{z}$ | $e^{i \rho_{\varphi_{1}}}$ | $\left\|\tilde{I}_{z}\right\| \equiv \sqrt{\tilde{I}_{z} \tilde{I}_{z}^{*}}=\sqrt{I_{z r}^{2}+I_{z i}^{2}}$ |
| :---: | :---: | :---: | :---: |

Phase: RMS SI Units:
$\varphi_{p} \equiv \tan ^{-1}\left(p_{i} / p_{r}\right)$ Pascals
$\varphi_{u_{z}} \equiv \tan ^{-1}\left(u_{z i} / u_{z r}\right) \mathrm{m} / \mathrm{sec}$
$\varphi_{d_{z}} \equiv \tan ^{-1}\left(d_{z i} / d_{z r}\right)$ meters
$\varphi_{a_{z}} \equiv \tan ^{-1}\left(a_{z i} / a_{z r}\right) \mathrm{m} / \mathrm{sec}^{2}$
$\varphi_{z_{z}} \equiv \tan ^{-1}\left(z_{z i} / z_{z r}\right)$ Ac.Ohms
$\varphi_{I_{z}} \equiv \tan ^{-1}\left(I_{z i} / I_{z r}\right)$ Watts $/ \mathrm{m}^{2}$

## 1-D/Longitudinal Complex Particle Displacement:

The 1-D/longitudinal complex particle displacement $\tilde{d}_{z}(\vec{r}, t)$ is obtained from the 1-D/ longitudinal complex particle velocity $\tilde{u}_{z}(\vec{r}, t)$ since they are related to each other via: $\tilde{u}_{z}(\vec{r}, t) \equiv \partial \tilde{d}_{z}(\vec{r}, t) / \partial t$. For a harmonically-varying/pure-tone/single frequency complex sound field $\tilde{d}_{z}(\vec{r}, t)=\left|\tilde{d}_{z}(\vec{r})\right| e^{+i \omega t} e^{i \phi_{d_{2}}}$ and $\tilde{u}_{z}(\vec{r}, t)=\left|\tilde{u}_{z}(\vec{r})\right| e^{+i \omega t} e^{i \phi_{u_{z}}}$, thus:
$\tilde{u}_{z}(\vec{r}, t)=\left|\tilde{u}_{z}(\vec{r})\right| e^{+i \omega t} e^{i \varphi_{u_{z}}} \equiv \partial \tilde{d}_{z}(\vec{r}, t) / \partial t=+i \omega\left|\tilde{d}_{z}(\vec{r})\right| e^{+i \omega t} e^{i \varphi_{d_{z}}}=+i \omega \tilde{d}_{z}(\vec{r}, t)$. Note that this relation tells us that the complex phases are related to each other via: $\varphi_{u_{z}}=\pi / 2+\varphi_{d_{2}}$ or: $\varphi_{d_{z}}=\varphi_{u_{z}}-\pi / 2$ since: $+i=e^{+i \pi / 2}=\cos \pi / 2+i \sin \pi / 2$. Expanding the LHS and RHS of this relation: $\tilde{u}_{z}=u_{z r}+i u_{z i}=i \omega\left(d_{z r}+i d_{z i}\right)=i \omega d_{z r}-\omega d_{z i}=i \omega \tilde{d}_{z}$ and equating real and imaginary parts on the LHS to those on the RHS we see that: $u_{z r}(\vec{r}, t)=-\omega d_{z i}(\vec{r}, t)$ and $u_{z i}(\vec{r}, t)=\omega d_{z r}(\vec{r}, t)$, where $\omega=2 \pi f$. Thus: $d_{z r}(\vec{r}, t)=u_{z i}(\vec{r}, t) / \omega$ and $d_{z i}(\vec{r}, t)=-u_{z r}(\vec{r}, t) / \omega$.

## 1-D/Longitudinal Complex Particle Acceleration:

Likewise, the 1-D/longitudinal complex particle acceleration $\tilde{a}_{z}(\vec{r}, t)$ is obtained from the 1-D/ longitudinal complex particle velocity $\tilde{u}_{z}(\vec{r}, t)$ since they are related to each other via:
$\tilde{a}_{z}(\vec{r}, t) \equiv \partial \tilde{u}_{z}(\vec{r}, t) / \partial t$. For a harmonically-varying/pure-tone/single frequency complex sound field, $\tilde{a}_{z}(\vec{r}, t)=\left|\tilde{a}_{z}(\vec{r})\right| e^{+i \omega t} e^{i \varphi_{a_{z}}}$ and $\tilde{u}_{z}(\vec{r}, t)=\left|\tilde{u}_{z}(\vec{r})\right| e^{+i \omega t} e^{i \varphi_{u_{z}}}$, thus:
$\tilde{a}_{z}(\vec{r}, t)=\left|\tilde{a}_{z}(\vec{r})\right| e^{+i \omega t} e^{i \varphi_{a_{z}}} \equiv \partial \tilde{u}_{z}(\vec{r}, t) / \partial t=+i \omega\left|\tilde{u}_{z}(\vec{r})\right| e^{+i \omega t} e^{i \varphi_{u_{z}}}=+i \omega \tilde{u}_{z}(\vec{r}, t)$. the complex phases are related to each other via: $\varphi_{a_{z}}=\varphi_{u_{z}}+\pi / 2$. Expanding the LHS and RHS of
this relation $\tilde{a}_{z}=a_{z r}+i a_{z i}=i \omega\left(u_{z r}+i u_{z i}\right)=i \omega u_{z r}-\omega u_{z i}=i \omega \tilde{u}_{z}$ and again equating real and imaginary parts on the LHS to those on the RHS, we see that: $a_{z r}(\vec{r}, t)=-\omega u_{z i}(\vec{r}, t)$ and $a_{z i}(\vec{r}, t)=\omega u_{z r}(\vec{r}, t)$.

## 1-D/Longitudinal Complex Specific Acoustic Impedance:

The 1-D/longitudinal complex specific acoustic impedance $\tilde{z}_{z}(\vec{r}, t)$ is a physical property of the medium in which sound propagation occurs and is defined as the ratio of complex pressure to 1-D complex particle velocity: $\tilde{z}_{z}(\vec{r}, t) \equiv \tilde{p}(\vec{r}, t) / \tilde{u}_{z}(\vec{r}, t)$ (the acoustic analog of the complex form of Ohm's law: $\tilde{Z} \equiv \tilde{V} / \tilde{I}$ ). For a harmonically-varying/single frequency complex sound field, since $\tilde{p}(\vec{r}, t)=|\tilde{p}(\vec{r})| e^{+i \omega t} e^{i \varphi_{p}}$ and $\tilde{u}_{z}(\vec{r}, t)=\left|\tilde{u}_{z}(\vec{r})\right| e^{+i \omega t} e^{i \varphi_{u_{z}}}$ then we see that in general, the 1-D/longitudinal complex specific acoustic impedance is also a time-independent quantity, i.e. $\tilde{z}_{z}(\vec{r}) \equiv \tilde{p}(\vec{r}) / \tilde{u}_{z}(\vec{r})$ (for loudness levels $\ll 134 d B$ ). Suppressing the $(\vec{r})$-dependence:

$$
\begin{aligned}
\tilde{z}_{z}=z_{z r}+i z_{z i}=\frac{\tilde{p}}{\tilde{u}_{z}}=\frac{p_{r}+i p_{i}}{u_{z r}+i u_{z i}}=\left(\frac{p_{r}+i p_{i}}{u_{z r}+i u_{z i}}\right)\left(\frac{u_{z r}-i u_{z i}}{u_{z r}-i u_{z i}}\right) & =\left(\frac{p_{r} u_{z r}+p_{i} u_{z i}}{u_{z r}^{2}+u_{z i}^{2}}\right)+i\left(\frac{p_{i} u_{z r}-p_{r} u_{z i}}{u_{z r}^{2}+u_{z i}^{2}}\right) \\
& =\left(\frac{p_{r} u_{z r}+p_{i} u_{z i}}{\mid \tilde{u}_{z}^{2}}\right)+i\left(\frac{p_{i} u_{z r}-p_{r} u_{z i}}{\left|\tilde{u}_{z}\right|^{2}}\right)
\end{aligned}
$$

$\therefore$ We see that: $z_{z r} \equiv \frac{p_{r} u_{z r}+p_{i} u_{z i}}{u_{z r}^{2}+u_{z i}^{2}}=\frac{p_{r} u_{z r}+p_{i} u_{z i}}{\left|\tilde{u}_{z}\right|^{2}}$ and: $z_{z i} \equiv \frac{p_{i} u_{z r}-p_{r} u_{z i}}{u_{z r}^{2}+u_{z i}^{2}}=\frac{p_{i} u_{z r}-p_{r} u_{z i}}{\left|\tilde{u}_{z}\right|^{2}}$
Since the 1-D/longitudinal complex specific acoustic impedance is the ratio of complex pressure to complex particle velocity $\tilde{z}_{z} \equiv \tilde{p} / \tilde{u}_{z}$, the SI units of complex specific acoustic impedance are:

$$
\frac{R M S \text { Pascals }}{R M S \mathrm{~m} / \mathrm{s}}=\frac{R M S \mathrm{~N} / \mathrm{m}^{2}}{R M S \mathrm{~m} / \mathrm{s}}=\frac{R M S\left(\mathrm{~kg}-\mathrm{m} / \mathrm{s}^{2}\right) / \mathrm{m}^{2}}{R M S \mathrm{~m} / \mathrm{s}}=\frac{\mathrm{kg}-\mathrm{m}}{\mathrm{~m}^{3}-\mathrm{s}^{2} / \mathrm{s}}=\mathrm{kg} / \mathrm{s}-\mathrm{m}^{2} \equiv \text { Acoustic Ohms }
$$

## 1-D Complex Sound Intensity:

The 1-D/longituding complex sound intensity is defined as the product of complex pressure and the complex conjugate of the 1-D/longitudinal particle velocity: $\tilde{I}_{z}(\vec{r}, t) \equiv \tilde{p}(\vec{r}, t) \tilde{u}_{z}^{*}(\vec{r}, t)$ (Again, the acoustic analog of complex electrical power: $\tilde{P} \equiv \tilde{I}^{*}$ ). For a harmonically-varying/ single frequency complex sound field $\tilde{p}(\vec{r}, t)=|\tilde{p}(\vec{r})| e^{+i \omega t} e^{i \rho_{p}}$ and $\tilde{u}_{z}^{*}(\vec{r}, t)=\left|\tilde{u}_{z}(\vec{r})\right| e^{-i \omega t} e^{-\varphi_{u_{z}}}$; the 1-D/longitudinal complex intensity is also a time-independent quantity, i.e. $\tilde{I}_{z}(\vec{r}) \equiv \tilde{p}(\vec{r}) \tilde{u}_{z}^{*}(\vec{r})$ (for loudness levels $\ll 140 \mathrm{~dB}$ ). Suppressing the $(\vec{r})$-dependence:

$$
\begin{gathered}
\tilde{I}_{z}=I_{z r}+i I_{z i}=\tilde{p} \tilde{u}_{z}^{*}=\left(p_{r}+i p_{i}\right)\left(u_{z r}-i u_{z i}\right)=\left(p_{r} u_{z r}+p_{i} u_{z i}\right)+i\left(p_{i} u_{z r}-p_{r} u_{z i}\right) \\
I_{z r} \equiv \operatorname{Re}\left\{I_{z}\right\}=\operatorname{Re}\left\{\tilde{p} \tilde{u}_{z}^{*}\right\}=\operatorname{Re}\left\{\left(p_{r}+i p_{i}\right)\left(u_{z r}-i u_{z i}\right)\right\}=\operatorname{Re}\left\{\left(p_{r} u_{z r}+p_{i} u_{z i}\right)+i\left(p_{i} u_{z r}-p_{r} u_{z i}\right)\right\}=p_{r} u_{z r}+p_{i} u_{z i}
\end{gathered}
$$

$I_{z i} \equiv \operatorname{Im}\left\{I_{z}\right\}=\operatorname{Im}\left\{\tilde{p} \tilde{u}_{z}^{*}\right\}=\operatorname{Im}\left\{\left(p_{r}+i p_{i}\right)\left(u_{z r}-i u_{z i}\right)\right\}=\operatorname{Im}\left\{\left(p_{r} u_{z r}+p_{i} u_{z i}\right)+i\left(p_{i} u_{z r}-p_{r} u_{z i}\right)\right\}=p_{i} u_{z r}-p_{r} u_{z i}$
Since the 1-D/longitudinal complex intensity is the product of complex pressure with the complex conjugate of particle velocity $\tilde{I}_{z} \equiv \tilde{p} \tilde{u}_{z}^{*}$, the SI units of complex intensity are:

$$
R M S \text { Pascals } \cdot R M S \mathrm{~m} / \mathrm{s}=\frac{R M S \mathrm{~N}}{\mathrm{~m}^{2}} \cdot \frac{R M S \mathrm{~m}}{\mathrm{~s}}=\frac{R M S \mathrm{~J} / \mathrm{s}}{\mathrm{~m}^{2}}=R M S \mathrm{Watts} / \mathrm{m}^{2}
$$

Since: $\tilde{I}_{z} \equiv \tilde{p} \tilde{u}_{z}^{*}$ and $\tilde{z}_{z} \equiv \tilde{p} / \tilde{u}_{z}$ then $\tilde{p}=\tilde{u}_{z} \tilde{z}_{z}$ thus we also see that: $\tilde{I}_{z} \equiv \tilde{p} \tilde{u}_{z}^{*}=\tilde{u}_{z} \tilde{u}_{z}^{*} \tilde{z}_{z}=\left|\tilde{u}_{z}\right|^{2} \tilde{z}_{z}$.
Since: $\tilde{I}_{z} \equiv I_{z r}+i I_{z i}=\left|\tilde{u}_{z}\right|^{2}\left(z_{z r}+i z_{z i}\right)=\left|\tilde{u}_{z}\right|^{2} \tilde{z}_{z}$ we also see that $I_{z r}=\left|\tilde{u}_{z}\right|^{2} z_{z r}$ and $I_{z i}=\left|\tilde{u}_{z}\right|^{2} z_{z i}$.

## 1-D Sound Intensity Factor (SIF) for Harmonic (i.e. Sine-Wave) Complex Sound Fields:

Since $I_{z r} \equiv\left|\tilde{I}_{z}\right| \cos \varphi_{I_{z}}=|\tilde{p}|\left|\tilde{u}_{z}\right| \cos \varphi_{I_{z}}$ and $I_{i} \equiv|\tilde{I}| \sin \varphi_{I}=|\tilde{p}|\left|\tilde{u}^{*}\right| \sin \varphi_{I}$ for harmonically-varying (i.e. sine-wave/single-frequency) complex sound fields, the 1-D Sound Intensity Factor, SIF is defined as: $\operatorname{SIF} \equiv 100 \times \cos \varphi_{I_{z}}=100 \times\left(\frac{I_{z r}}{\left|\tilde{I}_{z}\right|}\right)=100 \times \frac{I_{z r}}{|\tilde{p}|\left|\tilde{u}_{z}^{*}\right|}(\%)$.

The definition of the 1-D/longitudinal Sound Intensity Factor, SIF is again in complete analogy to the definition of the so-called Power Factor, PF associated with complex electrical power:
$P_{r} \equiv|\tilde{P}| \cos \varphi_{P}=|\tilde{V}|\left|\tilde{I}^{*}\right| \cos \varphi_{P}$ and: $P F \equiv 100 \times \cos \varphi_{P}=100 \times\left(\frac{P_{r}}{|\tilde{P}|}\right)=100 \times \frac{P_{r}}{|\tilde{V}|\left|\tilde{I}^{*}\right|}(\%)$.

## The Relationship Between the Complex Phases $\varphi_{I_{I_{z}}}$ and $\varphi_{z_{7}}$ :

We explicitly note that: $\varphi_{I_{z}} \equiv \tan ^{-1}\left(\frac{I_{z i}}{I_{z r}}\right)=\varphi_{z_{z}} \equiv \tan ^{-1}\left(\frac{z_{z i}}{z_{z r}}\right)$ because: $\frac{I_{z i}}{I_{z r}}=\frac{p_{i} u_{z r}-p_{r} u_{z i}}{p_{r} u_{z r}+p_{i} u_{z i}}$
whereas: $\frac{z_{z i}}{z_{z r}}=\left(\frac{p_{i} u_{z r}-p_{r} u_{z i}}{\left|\tilde{y}_{z}\right|^{2}}\right) /\left(\frac{p_{r} u_{z r}+p_{i} u_{z i}}{\left|\tilde{y}_{z}\right|^{2}}\right)=\frac{p_{i} u_{z r}-p_{r} u_{z i}}{p_{r} u_{z r}+p_{i} u_{z i}}=\frac{I_{z i}}{I_{z r}}$.
$\underline{\text { The Relation Between } \varphi_{I_{z}}=\varphi_{z_{z}} \text { and } \varphi_{p}, \varphi_{u_{z}} \text { : }}$
Since $\tilde{z}_{z} \equiv \tilde{p} / \tilde{u}_{z}$ and $\tilde{I}_{z} \equiv \tilde{p} \tilde{u}_{z}^{*}$ then we see that:
$\tilde{z}_{z}=\left|\tilde{z}_{z}\right| e^{i \varphi_{z_{z}}} \equiv|\tilde{p}| e^{i \varphi_{p}} /\left|\tilde{u}_{z}\right| e^{i \varphi_{u_{z}}}=\left(|\tilde{p}| /\left|\tilde{u}_{z}\right|\right) e^{i\left(\varphi_{p}-\varphi_{u_{z}}\right)}=\left|\tilde{z}_{z}\right| e^{i\left(\varphi_{p}-\varphi_{u_{z}}\right)}$ and
$\tilde{I}_{z}=\left|\tilde{I}_{z}\right| e^{i \varphi_{\varphi_{z}}} \equiv|\tilde{p}| e^{i \varphi_{p}}\left|\tilde{u}_{z}^{*}\right| e^{-i \varphi_{u_{z}}}=\left(|\tilde{p}|\left|\tilde{u}_{z}^{*}\right|\right) e^{i\left(\varphi_{p}-\varphi_{u_{z}}\right)}=\left|\tilde{I}_{z}\right| e^{i\left(\varphi_{p}-\varphi_{u_{z}}\right)}$ since $\left|\tilde{z}_{z}\right| \equiv|\tilde{p}| /\left|\tilde{u}_{z}\right|$ and $\left|\tilde{I}_{z}\right| \equiv|\tilde{p}|\left|\tilde{u}_{z}^{*}\right|$
thus we see that $\varphi_{z_{z}}=\varphi_{I_{z}}=\Delta \varphi_{p-u_{z}}=\varphi_{p}-\varphi_{u_{z}}$, i.e. physically the complex phases $\varphi_{z_{z}} \equiv \varphi_{I_{z}}$ are the
difference between the phase of the complex pressure $\varphi_{p}$ and the phase of the complex longitudinal particle velocity $\varphi_{u}$ as shown in the two complex plane figures below:



## There are several limiting cases of special interest:

When $\varphi_{z_{z}}=\varphi_{I_{z}}=\Delta \varphi_{p-u_{z}}=\varphi_{p}-\varphi_{u_{z}}=0$ the complex pressure $\tilde{p}(\vec{r}, t)$ and 1-D/longitudinal complex particle velocity $\tilde{u}_{z}(\vec{r}, t)$ are in phase with each other; the Sound Intensity Factor $S I F \equiv 100 \times \cos \varphi_{I_{z}}=100 \times \cos \varphi_{z_{z}}=100 \%$.

When $\varphi_{z_{z}}=\varphi_{I_{z}}=\Delta \varphi_{p-u_{z}}=\varphi_{p}-\varphi_{u_{z}}= \pm 90^{\circ}$ the complex pressure $\tilde{p}(\vec{r}, t)$ leads/lags the 1-D/ longitudinal complex particle velocity $\tilde{u}_{z}(\vec{r}, t)$ by $90^{\circ}$ in phase; the Sound Intensity Factor $S I F \equiv 100 \times \cos \varphi_{I_{z}}=100 \times \cos \varphi_{z_{z}}=0 \%$.

When $\varphi_{z_{z}}=\varphi_{I_{z}}=\Delta \varphi_{p-u_{z}}=\varphi_{p}-\varphi_{u_{z}}= \pm 180^{\circ}$ the complex pressure $\tilde{p}(\vec{r}, t)$ leads/lags the 1-D/ longitudinal complex particle velocity $\tilde{u}_{z}(\vec{r}, t)$ by $180^{\circ}$ in phase; the Sound Intensity Factor $S I F \equiv 100 \times \cos \varphi_{I_{z}}=100 \times \cos \varphi_{Z_{z}}=-100 \%$.

## Other Useful Relations:

Using $\tilde{z}_{z} \equiv \tilde{p} / \tilde{u}_{z}$ and $\tilde{I}_{z} \equiv \tilde{p} \tilde{u}_{z}^{*}$, eliminating the complex particle velocity $\tilde{u}_{z}$ we can see that $|\tilde{p}|^{2}=\tilde{p} \tilde{p}^{*}=\tilde{I}_{z} \tilde{z}_{z}^{*}$, which is a purely real quantity. If we instead eliminate the complex pressure $\tilde{p}$ we obtain $\left|\tilde{u}_{z}\right|^{2}=\tilde{u}_{z} \tilde{u}_{z}^{*}=\tilde{I}_{z} / \tilde{z}_{z}$, which is also a purely real quantity. Equivalently: $\tilde{I}_{z}=\left|\tilde{u}_{z}\right|^{2} \tilde{z}_{z}$.

Also: $\left|\tilde{z}_{z}\right|^{2}=\tilde{z}_{z} \tilde{z}_{z}^{*}=\frac{\tilde{p}}{\tilde{u}_{z}} \frac{\tilde{p}_{z}^{*}}{\tilde{u}_{z}^{*}}=\frac{\tilde{p} \tilde{p}^{*}}{\tilde{u}_{z} \tilde{u}_{z}^{*}}=\frac{|\tilde{p}|^{2}}{\left|\tilde{u}_{z}\right|^{2}}$ and: $\left|\tilde{I}_{z}\right|^{2}=\tilde{I}_{z} \tilde{I}_{z}^{*}=\left(\tilde{p} \tilde{u}_{z}^{*}\right)\left(\tilde{p}^{*} \tilde{u}_{z}\right)=\left(\tilde{p} \tilde{p}^{*}\right)\left(\tilde{u}_{z} \tilde{u}_{z}^{*}\right)=|\tilde{p}|^{2}\left|\tilde{u}_{z}\right|^{2}$.
Acoustic potential, kinetic and total energy densities: (n.b. all energy densities are purely real quantities!)
Acoustic potential energy density: $w_{p}=\frac{1}{2}\left(|\tilde{p}|^{2} / \rho_{o} c^{2}\right)\left(R M S\right.$ Joules $\left./ m^{2}\right), ~ \rho_{o}=$ ambient density of air
Acoustic kinetic energy density: $\quad w_{u}=\frac{1}{2} \rho_{o}|\tilde{u}|^{2}\left(R M S\right.$ Joules $\left./ m^{2}\right) \quad c=$ speed of sound in air

Total acoustic energy density:

$$
w_{\text {tot }}=w_{p}+w_{u}=\frac{1}{2}\left(|\tilde{p}|^{2} / \rho_{o} c^{2}\right)+\frac{1}{2} \rho_{o}|\tilde{u}|^{2}\left(\text { RMS Joules } / m^{2}\right)
$$

Time-averaged acoustic energy densities:
Since $w_{p} \propto|\tilde{p}|^{2}$ and $w_{u} \propto|\tilde{u}|^{2}$, for harmonic (i.e. sine-wave) dependence, we see that:
$\left\langle w_{p}\right\rangle=\frac{1}{2} w_{p}$ and $\left\langle w_{u}\right\rangle=\frac{1}{2} w_{u}$, thus: $\left\langle w_{\text {tot }}\right\rangle=\left\langle w_{p}+w_{u}\right\rangle=\left\langle w_{p}\right\rangle+\left\langle w_{u}\right\rangle=\frac{1}{2} w_{p}+\frac{1}{2} w_{u}=\frac{1}{2} w_{\text {tot }}$.

## Ratio of PE/KE densities:

Using $|\tilde{p}|^{2}=\tilde{I}_{z} \tilde{Z}_{z}^{*}$ and $|\tilde{u}|^{2}=\tilde{I}_{z} / \tilde{z}_{z}$ we see that: $w_{p}=\frac{1}{2}\left(|\tilde{p}|^{2} / \rho_{o} c^{2}\right)=\frac{1}{2}\left(\tilde{I}_{z} \tilde{Z}_{z}^{*} / \rho_{o} c^{2}\right)$ and:
$w_{u}=\frac{1}{2} \rho_{o}|\tilde{u}|^{2}=\frac{1}{2} \rho_{o}\left(\tilde{I}_{z} / \tilde{z}_{z}\right)$. Taking the ratio of potential to kinetic energy density:

$$
\frac{P E}{K E}=\frac{w_{p}}{w_{u}}=\frac{\frac{1}{2}\left(|\tilde{p}|^{2} / \rho_{o} c^{2}\right)}{\frac{1}{2} \rho_{o}|\tilde{u}|^{2}}=\frac{\frac{1}{2}\left(\tilde{I}_{z} \tilde{z}_{z}^{*} / \rho_{o} c^{2}\right)}{\frac{1}{2} \rho_{o}\left(\tilde{I}_{z} / \tilde{z}_{z}\right)}=\frac{\tilde{z}_{z} \tilde{z}_{z}^{*}}{\rho_{o}^{2} c^{2}}=\frac{\left|\tilde{z}_{z}\right|^{2}}{\rho_{o}^{2} c^{2}}
$$

However, the characteristic specific acoustic longitudinal impedance of associated with 1-D plane waves propagating in open/free air is: $\tilde{z}_{\text {fieed }}^{\text {ree }} \equiv \rho_{o} c=z_{o} \simeq 415 \Omega$ (n.b. a purely real quantity - since $p$ and $u$ are in phase with each other for 1-D plane waves propagating in the free-air sound field).

Thus: $\frac{P E}{K E}=\frac{w_{p}}{w_{u}}=\frac{\frac{1}{2}\left(|\tilde{p}|^{2} / \rho_{o} c^{2}\right)}{\frac{1}{2} \rho_{o}|\tilde{u}|^{2}}=\frac{\frac{1}{2}\left(\tilde{I}_{z} \tilde{z}_{z}^{*} / \rho_{o} c^{2}\right)}{\frac{1}{2} \rho_{o}\left(\tilde{I}_{z} / \tilde{z}_{z}\right)}=\frac{\tilde{z}_{z} \tilde{Z}_{z}^{*}}{\rho_{o}^{2} c^{2}}=\frac{\left|\tilde{z}_{z}\right|^{2}}{\rho_{o}^{2} c^{2}}=\frac{\left|\tilde{z}_{z}\right|^{2}}{z_{o}^{2}}$. In a free-air sound field we see that $P E / K E=w_{p} / w_{u}=1$, i.e. the acoustic energy density associated with a free-air sound field has equal amounts of PE and KE density - again, analogous to the result for monochromatic plane $E M$ waves propagating in free space.

Since $w_{\text {tot }}=w_{p}+w_{u}=\frac{1}{2}\left(|\tilde{p}|^{2} / \rho_{o} c^{2}\right)+\frac{1}{2} \rho_{o}|\tilde{u}|^{2}$, then again using $|\tilde{p}|^{2}=\tilde{I}_{z} \tilde{z}_{z}^{*}, \quad|\tilde{u}|^{2}=\tilde{I}_{z} / \tilde{z}_{z}$ and $\tilde{z}_{\text {field }}^{\text {free }} \equiv \rho_{o} c=z_{o} \simeq 415 \Omega$ we see that:
$w_{\text {tot }}=w_{p}+w_{u}=\frac{1}{2}\left(\frac{\tilde{I}_{z} \tilde{Z}_{z}^{*}}{\rho_{o} c^{2}}\right)+\frac{1}{2} \rho_{o}\left(\frac{\tilde{I}_{z}}{\tilde{z}_{z}}\right)=\frac{1}{2} \rho_{o}\left(\frac{\tilde{I}_{z} \tilde{Z}_{z}^{*}}{\rho_{o}^{2} c^{2}}\right)+\frac{1}{2} \rho_{o}\left(\frac{\tilde{I}_{z} \tilde{Z}_{z}^{*}}{\tilde{z}_{z} \tilde{Z}_{z}^{*}}\right)=\frac{1}{2} \rho_{o}\left[\frac{1}{z_{o}^{2}}+\frac{1}{\left|\tilde{Z}_{z}\right|^{2}}\right] \tilde{I}_{z} \tilde{z}_{z}^{*}$
which (again) is a purely real quantity.
We can turn this relation around such that the complex 1-D sound intensity $\tilde{I}_{z}$ can be written in terms of $w_{\text {tot }}$ and complex $\tilde{Z}^{*}$ as:

$$
\tilde{I}_{z}=\frac{w_{\text {tot }}}{\frac{1}{2} \rho_{o}\left[\frac{1}{z_{o}^{2}}+\frac{1}{\left|\tilde{z}_{z}\right|^{2}}\right] \tilde{z}_{z}^{*}}=\frac{c w_{\text {tot }}}{\frac{1}{2} \rho_{o} c\left[\frac{1}{z_{o}^{2}}+\frac{1}{\left|\tilde{z}_{z}\right|^{2}}\right] \tilde{z}_{z}^{*}}=\frac{c w_{\text {tot }}}{\frac{1}{2}\left[\frac{1}{z_{o}^{2}}+\frac{1}{\left|\tilde{z}_{z}\right|^{2}}\right] Z_{o} \tilde{Z}^{*}}
$$

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For a free-air/purely "active" sound field associated with propagation of monochromatic plane waves, the complex pressure and 1-D particle velocity are in phase with each other, thus the complex specific acoustic impedance is a purely real quantity, i.e. $\tilde{z}_{z}^{*}=\tilde{z}_{z}=\tilde{z}_{z}^{\text {free }}$ fied $=z_{o}=\rho_{o} c$, and thus we also see that $\tilde{I}_{z}^{\text {free }}=c w_{\text {tot }}$, is also a purely real quantity, analogous to the result e.g. for propagation of monochromatic $E M$ plane waves in free space: $\tilde{I}_{E M}^{\text {free }}=c w_{\text {tot }}^{E M}$.

