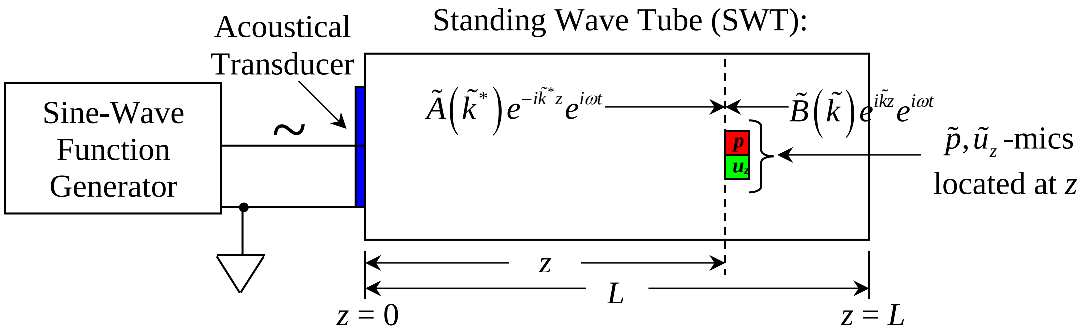
The Acoustical Physics of a Standing Wave Tube

A typical cylindrical-shaped standing wave tube (SWT) {*aka* impedance tube} of length L and diameter $D \ll L$ with infinitely rigid walls and closed ends is shown in the figure below:



Sound energy is input to the SWT at the position $z = 0$, *e.g.* using a sine-wave function generator connected to some kind of acoustical transducer, such as a wafer-thin piezo-electric transducer (or a loudspeaker). Ideally-speaking, the transducer should have no frequency-dependent phase-shift(s) relative to the driving sine-wave function generator. However, in the real world, such devices do not exist. At frequencies below the lowest cutoff frequency of the SWT ($f_c^{1,0} \simeq 1.84v/\pi D \sim 3300\ Hz$ for $v = 345\, m/s$ and $D = 6\, cm$) only 1-D type <u>plane</u> waves can propagate in the SWT.

Pressure (p) and differential/particle velocity (u_z) microphones are co-located at the "generic" position z along the symmetry axis of the SWT. They are used to record the complex instantaneous total pressure and the instantaneous complex 1-D longitudinal/z-component of the total particle velocity at that location associated with the presence of right- and left-moving acoustic traveling plane waves propagating in the SWT. The resultant instantaneous complex pressure standing wave at the point z is thus a linear superposition of these two traveling plane waves:

$$\boxed{\tilde{p}(z,t) = \tilde{A}(\tilde{k})e^{i(\omega t - \tilde{k}^* z)} + \tilde{B}(\tilde{k})e^{i(\omega t + \tilde{k}z)} = \tilde{A}(\tilde{k})e^{-i\tilde{k}^* z}e^{i\omega t} + \tilde{B}(\tilde{k})e^{i\tilde{k}z}e^{i\omega t} = \left[\tilde{A}(\tilde{k})e^{-i\tilde{k}^* z} + \tilde{B}(\tilde{k})e^{+i\tilde{k}z}\right]e^{+i\omega t}}$$

where the complex, frequency-dependent wavenumber $\tilde{k}(\omega) \equiv k(\omega) + i\kappa(\omega) = \omega/v(\omega) + i\kappa(\omega)$; the * denotes complex conjugation, *i.e.* $\tilde{k}^*(\omega) \equiv k(\omega) - i\kappa(\omega)$ and $i \equiv \sqrt{-1}$ and thus $-i \equiv -\sqrt{-1}$. The use of $\tilde{k}^*$ ensures that we are always appropriately mathematically describing <u>decaying</u> exponential attenuation phenomena, *i.e.* for $z > 0$, using $e^{-\kappa z}$ for both right- and left-traveling waves (as opposed to unphysical, exponentially <u>growing</u> phenomena, *i.e.* $e^{+\kappa z}$ with distance).

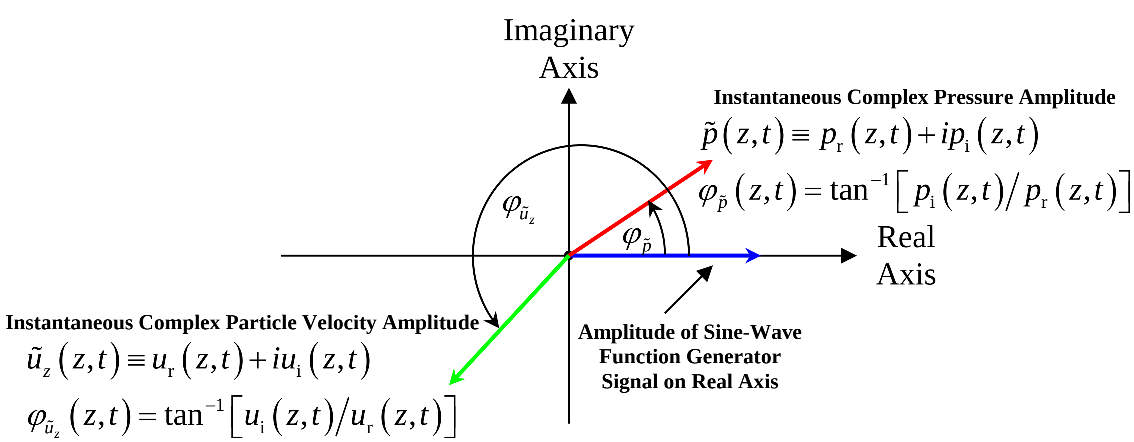
An {extremely} important micro-detail here is that in order to be able to correctly compare theoretical prediction(s) to experimental data, the choice of using $e^{i\omega t}$ vs. $e^{-i\omega t}$ in the theory is in fact <u>*not*</u> arbitrary. In the UIUC Physics 193POM/406POM SWT experiment, in order to obtain the necessary phase-sensitive information on the complex nature of pressure (p) and 1-D particle velocity (u_z) as a function of frequency, the electrical signals output from the pressure and particle velocity microphone preamplifiers are each input to separate lock-in amplifiers (SRS model # DSP-830) which also use the signal output from the sine-wave function generator as the <u>reference</u> signal for each of the lock-in amplifiers. In our SWT experiment we have explicitly selected <u>0^o referencing</u>

of the two lock-in amplifiers to the function generator's sine-wave signal, and thus because of the way a lock-in amplifier works, also implicitly means that we have selected the $e^{+i\omega t}$ sign convention. Had we instead selected *e.g.* <u>180° referencing</u> of the lock-in amplifier to the sine-wave reference signal, then we would have implicitly instead selected the $e^{-i\omega t}$ sign convention. Because of our $e^{+i\omega t}$ choice in referencing of the lock-in amplifiers to the function generator's sine-wave signal, both the instantaneous complex pressure and instantaneous complex particle velocity precess (*i.e.* rotate) <u>counter-clockwise</u> (CCW) in the complex plane as time <u>increases</u>, since $e^{+i\omega t} = \cos\omega t + i\sin\omega t$ whereas $e^{-i\omega t} = \cos\omega t - i\sin\omega t$.

Generically, the instantaneous complex pressure (SI units: Pascals) and instantaneous complex particle velocity (SI units: *m*/*s*) can be written as:

$$\boxed{\tilde{p}(z,t) \equiv p_{\mathrm{r}}(z,t) + ip_{\mathrm{i}}(z,t) = |\tilde{p}(z)|e^{i\varphi_{\tilde{p}}}e^{+i\omega t}} \quad \text{and} \quad \boxed{\tilde{u}_z(z,t) \equiv u_{\mathrm{r}}(z,t) + iu_{\mathrm{i}}(z,t) = |\tilde{u}(z)|e^{i\varphi_{\tilde{u}}}e^{+i\omega t}}$$

The <u>real</u> parts of the complex instantaneous pressure $\tilde{p}(z,t)$ and/or complex instantaneous 1-D particle velocity $\tilde{u}_z(z,t)$ are in-phase (if +*ve*) or 180° out-of phase (if –*ve*) relative to the reference signal output from the sine-wave function generator; the <u>imaginary</u> parts of the complex instantaneous pressure $\tilde{p}(z,t)$ and/or complex instantaneous 1-D particle velocity $\tilde{u}_z(z,t)$ are $+90^{\circ}$ out-of-phase (if +*ve*) or -90° out-of phase (if –*ve*) relative to the reference signal output from the sine-wave function generator, as shown below {for a general case/generic situation} in the so-called phase diagram – *i.e.* the complex plane, at time t = 0:



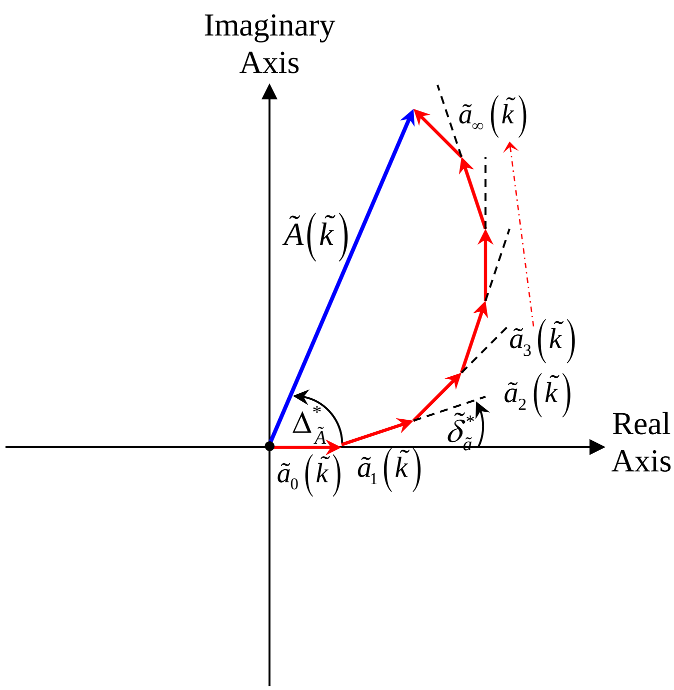
For *small* amplitudes, the instantaneous complex pressure and instantaneous 3-D complex vector particle velocity are related to each other via Euler's equation for compressible, inviscid fluid flow {inviscid fluid flow means that any/all viscous/dissipative forces $\ll$ inertial forces}:

$\boxed{-\rho_o\,\partial\vec{\tilde{u}}(\vec{r},t)/\partial t = \vec{\nabla}\tilde{p}(\vec{r},t)}$ where ρ_o = mass volume density of the fluid and $\boxed{\vec{\tilde{u}}(\vec{r},t)\bullet\vec{\nabla}\left\{\vec{\tilde{u}}(\vec{r},t)\right\} = 0}$ is assumed. For {bone-dry} air at NTP, $\rho_o = 1.204\,kg/m^3$. For the SWT with 1-D longitudinal particle velocity measurement, the corresponding 1-D Euler's equation for plane waves propagating in the SWT reduces to $\boxed{-\rho_o\,\partial\tilde{u}_z(z,t)/\partial t = \partial\tilde{p}(z,t)/\partial z}$.

The total instantaneous complex pressure at the position *z* associated with the presence of a standing ***plane*** wave in the SWT is the instantaneous linear superposition of an (overall) right-propagating complex traveling plane wave and an (overall) left-propagating traveling plane wave:

$$\boxed{\tilde{p}(z,t) = \left[\tilde{A}(\tilde{k})e^{-i\tilde{k}^* z} + \tilde{B}(\tilde{k})e^{+i\tilde{k}z}\right]e^{+i\omega t}}$$

The overall instantaneous complex pressure amplitude is in fact a linear superposition of an infinite number of individual right- and left-moving complex traveling pressure waves with complex amplitudes $\tilde{a}_n(\tilde{k})$ and $\tilde{b}_n(\tilde{k})$, $n = 0,1,2,3...\infty$ respectively, each of which are associated with the sine-wave signal output from the acoustical transducer (located at *z* = 0) at times earlier than *t* = 0. Thus, mathematically the complex amplitudes associated with right- and left-moving complex amplitudes can each be represented by the infinite series $\tilde{A}(\tilde{k}) = \sum_{n=0}^{\infty}\tilde{a}_n(\tilde{k})$ and $\tilde{B}(\tilde{k}) = \sum_{n=0}^{\infty}\tilde{b}_n(\tilde{k})$. Both of these series representations can be represented graphically via phasor diagrams in the complex plane, *e.g*. for $\tilde{A}(\tilde{k}) = \sum_{n=0}^{\infty}\tilde{a}_n(\tilde{k})$ as shown in the figure below for *t* = 0:



Precisely on a resonance of the SWT, the individual complex amplitudes $\tilde{a}_n(\tilde{k})$ and $\tilde{b}_n(\tilde{k})$ associated with the individual right- and left-moving traveling waves respectively, are perfectly in phase with each other, *i.e*. all of the individual relative phases $\delta^*_{\tilde{a}_n} = 2n\pi = 0$, $\delta^*_{\tilde{b}_n} = 2n\pi = 0$ and thus the overall phases $\Delta^*_{\tilde{A}} = \sum_{n=0}^{\infty}\delta^*_{\tilde{a}_n} = 0$ and $\Delta^*_{\tilde{B}} = \sum_{n=0}^{\infty}\delta^*_{\tilde{b}_n} = 0$, graphically corresponding to "straight-line" phasor diagrams for $\tilde{A}(\tilde{k}) = \sum_{n=0}^{\infty}\tilde{a}_n(\tilde{k})$ and $\tilde{B}(\tilde{k}) = \sum_{n=0}^{\infty}\tilde{b}_n(\tilde{k})$.

Explicitly writing out the complex pressure amplitudes $\tilde{A}(\tilde{k}) = \sum_{n=0}^{\infty} \tilde{a}_n(\tilde{k})$ and $\tilde{B}(\tilde{k}) = \sum_{n=0}^{\infty} \tilde{b}_n(\tilde{k})$ associated with the overall right- and left-moving complex plane waves:

$$\boxed{\tilde{A}(\tilde{k}) = A_o e^{-i\tilde{\delta}_0^*} + A_o e^{-i\tilde{\delta}_1^*} + A_o e^{-i\tilde{\delta}_2^*} + A_o e^{-i\tilde{\delta}_3^*} + \ldots = A_o\left[e^{-i\tilde{\delta}_0^*} + e^{-i\tilde{\delta}_1^*} + e^{-i\tilde{\delta}_2^*} + e^{-i\tilde{\delta}_3^*} \ldots\right] = A_o \sum_{n=0}^{\infty} e^{-i\tilde{\delta}_n^*}}$$

And:

$$\boxed{\tilde{B}(\tilde{k}) = A_o e^{-i\tilde{\delta}_0^*} + A_o e^{-i\tilde{\delta}_1^*} + A_o e^{-i\tilde{\delta}_2^*} + A_o e^{-i\tilde{\delta}_3^*} + \ldots = A_o\left[e^{-i\tilde{\delta}_0^*} + e^{-i\tilde{\delta}_1^*} + e^{-i\tilde{\delta}_2^*} + e^{-i\tilde{\delta}_3^*} \ldots\right] = A_o \sum_{n=0}^{\infty} e^{-i\tilde{\delta}_n^*} = \tilde{A}(\tilde{k})}$$

The multiplicative phase factor $e^{-i\tilde{\delta}_n}$ associated with the n^{th} term in each of the two infinite series arises from the fact that each such contributing wave had to originate at an earlier time, $t_n < 0$ in order for all such waves to arrive simultaneously at the $z = z$ position at the time $t = t$. Note that since $e^{i\omega t}$ rotates complex quantities CCW in the complex plane as the time t increases, the sign in the argument of the $e^{-i\tilde{\delta}_n^*}$ phase factor associated with waves arriving at the $z = z$ position at the time $t = t$ from the earlier time $t_n < 0$ must be negative. Since the *elapsed* time for n round trips of right- or left-moving waves propagating in the SWT is $\Delta t_n \equiv t - t_n = n(2L)/v = 2nL/v$, then the complex phase shift associated with n round trips of right- or left-moving waves propagating in the SWT is $\boxed{\tilde{\delta}_n^* = \omega\Delta t_n = 2nL\omega/\tilde{v} = 2n\tilde{k}^* L}$ and thus $\boxed{e^{-i\tilde{\delta}_n^*} = e^{-2in\tilde{k}^* L} = e^{-2in(k-i\kappa)L} = e^{-2\kappa L}e^{-2inkL}}$.

Thus: $$\boxed{\tilde{A}(\tilde{k}) = A_o + A_o e^{-2\kappa L}e^{-2ikL} + A_o e^{-4\kappa L}e^{-4ikL} + A_o e^{-6\kappa L}e^{-6ikL} + \ldots = A_o \sum_{n=0}^{\infty} e^{-2n\kappa L}e^{-2inkL} = \tilde{B}(\tilde{k})}$$

Note that since the end walls of the SWT (located at $z = 0$ and $z = L$ respectively) are assumed to be infinitely rigid, we have tacitly/implicitly assumed that no additional phase shift(s) of the right-/left-moving traveling waves occurs upon reflection at the end walls. If such reflection-induced phase shifts were to occur, then additional phase factors $e^{-in\varphi_o}$ and $e^{-in\varphi_L}$ would need to be included in the above expressions in order to explicitly take into account/properly mathematically describe general/generic phase shifts associated with reflection of the individual right- and left-moving plane waves at {non-perfectly rigid} end-walls of the SWT.

Is it possible to obtain an ***analytic***, closed-form expression for the infinite series associated with the complex amplitudes $\tilde{A}(\tilde{k})$ and $\tilde{B}(\tilde{k})$? The answer is a most definite ***yes***!

Defining $\boxed{t \equiv 2\kappa L > 0}$ and $\boxed{x \equiv 2kL}$, and noting that $\boxed{\sum_{n=0}^{\infty} e^{-nt}e^{-inx} = \sum_{n=0}^{\infty} e^{-nt}\cos nx - i\sum_{n=0}^{\infty} e^{-nt}\sin nx}$ the analytic/closed-form expressions for the two ∞ series on the RHS of this relation, for $t > 0$ are **[1]**:

$$\boxed{\sum_{n=0}^{\infty} e^{-nt}\sin nx = \frac{1}{2}\left(\frac{\sin x}{\cosh t - |\cos x|}\right)} \text{ and: } \boxed{\sum_{n=0}^{\infty} e^{-nt}\cos nx = \frac{1}{2}\left(\frac{\sinh t}{\cosh t - \cos x} + 1\right)}.$$

Thus: $$\boxed{\sum_{n=0}^{\infty} e^{-nt}\cos nx - i\sum_{n=0}^{\infty} e^{-nt}\sin nx = \sum_{n=0}^{\infty} e^{-nt}e^{-inx} = \frac{1}{2}\left[\left(\frac{\sinh t}{\cosh t - \cos x}+1\right) - i\left(\frac{\sin x}{\cosh t - |\cos x|}\right)\right]}$$

Hence the analytic/closed-form expression for complex $\tilde{A}(\tilde{k}) = \tilde{B}(\tilde{k})$ is:

$$\boxed{\tilde{A}(\tilde{k}) = \tilde{B}(\tilde{k}) = A_o\left[\sum_{n=0}^{\infty} e^{-2n\kappa L}e^{-2inkL}\right] = \frac{1}{2}A_o\left[\left(1+\frac{\sinh(2\kappa L)}{\cosh(2\kappa L)-\cos(2kL)}\right) - i\left(\frac{\sin(2kL)}{\cosh(2\kappa L)-|\cos(2kL)|}\right)\right]}$$

Thus, the overall instantaneous complex pressure amplitude $\tilde{p}(z,t)$ is:

$$\boxed{\begin{aligned}\tilde{p}(z,t) &= \left[\tilde{A}(\tilde{k})e^{-i\tilde{k}^* z} + \tilde{B}(\tilde{k})e^{+i\tilde{k}z}\right]e^{+i\omega t} = \tilde{A}(\tilde{k})\left[e^{-i\tilde{k}^* z} + e^{+i\tilde{k}z}\right]e^{+i\omega t} = 2\tilde{A}(\tilde{k})e^{-\kappa z}\cos(kz)\,e^{+i\omega t} \\ &= A_o e^{-\kappa z}\cos(kz)\left[\left(1+\frac{\sinh(2\kappa L)}{\cosh(2\kappa L)-\cos(2kL)}\right) - i\left(\frac{\sin(2kL)}{\cosh(2\kappa L)-|\cos(2kL)|}\right)\right]e^{+i\omega t}\end{aligned}}$$

Note that the physics associated with standing plane acoustic waves inside a SWT is similar to that associated with standing plane electromagnetic/visible light waves inside a Fabry-Perot etalon with semi-transparent/partially-silvered and/or dielectric-coated plane-parallel mirrors!

Defining: $$\boxed{\tilde{\mathcal{F}}(\tilde{k}) \equiv \left[\left(1+\frac{\sinh(2\kappa L)}{\cosh(2\kappa L)-\cos(2kL)}\right) - i\left(\frac{\sin(2kL)}{\cosh(2\kappa L)-|\cos(2kL)|}\right)\right]}$$

Then $\tilde{p}(z,t)$ can be written as: $\boxed{\tilde{p}(z,t) = A_o\tilde{\mathcal{F}}(\tilde{k})e^{-\kappa z}\cos(kz)\,e^{+i\omega t}}$

The 1-D complex particle velocity is of the general form $\tilde{u}_z(z,t) = \tilde{u}_{o_z}(z)e^{+i\omega t}$ and is related to the complex pressure $\tilde{p}(z,t)$ via the 1-D Euler equation:

$$\boxed{-\rho_o\frac{\partial\tilde{u}_z(z,t)}{\partial t} = \frac{\partial\tilde{p}(z,t)}{\partial z}}$$

Thus: $$\boxed{\tilde{u}_z(z,t) = -i\frac{1}{\omega\rho_o}A_o\tilde{\mathcal{F}}(\tilde{k})e^{-\kappa z}\left[\kappa\cos kz + k\sin kz\right]e^{+i\omega t}}$$

Since $\boxed{v(\omega) = \omega/k(\omega)}$ this relation can also be written as:

$$\boxed{\tilde{u}_z(z,t) = -i\frac{1}{\rho_o v}A_o\tilde{\mathcal{F}}(\tilde{k})e^{-\kappa z}\left[\left(\frac{\kappa}{k}\right)\cos kz + \sin kz\right]e^{+i\omega t}}$$

For inviscid fluid flow, note that $(\kappa/k) \ll 1$ or equivalently, that $\kappa \ll k$.

The complex longitudinal particle displacement {from its nominal equilibrium position} $\tilde{\xi}(z,t)$ is related to the complex longitudinal particle velocity $\tilde{u}_z(z,t)$ via $\boxed{\tilde{u}_z(z,t) = \partial\tilde{\xi}(z,t)/\partial t}$.

Thus: $\boxed{\tilde{\xi}_z(z,t) = -\frac{1}{\omega\rho_o v} A_o \tilde{\mathcal{F}}(\tilde{k}) e^{-\kappa z}\left[\left(\frac{\kappa}{k}\right)\cos kz + \sin kz\right] e^{+i\omega t}}$

The complex *specific* acoustic longitudinal impedance $\tilde{z}(z)$ of the SWT tube at the position z is defined as the ratio of complex pressure to complex longitudinal particle velocity:

$$\boxed{\tilde{z}(z) \equiv \frac{\tilde{p}(z,t)}{\tilde{u}_z(z,t)} = \frac{\tilde{p}(z)e^{+i\omega t}}{\tilde{u}_z(z)e^{+i\omega t}} = \frac{\tilde{p}(z)}{\tilde{u}_z(z)} = +i\rho_o v \frac{\cos kz}{\left[\left(\frac{\kappa}{k}\right)\cos kz + \sin kz\right]}}$$

Note that the complex *specific* acoustic impedance $\tilde{z}(z)$ of the SWT is purely imaginary, and is also a time-independent quantity, since $\tilde{p}(z,t)$ and $\tilde{u}_z(z,t)$ have the same time-dependence factor $e^{+i\omega t}$. The SI units of complex *specific* acoustic longitudinal impedance $\tilde{z}(z)$ are $Pa\text{-}s/m = kg/s\text{-}m^2$, also known simply as acoustic ohms, also known as Rayls (in honor of Lord Rayleigh).

The 1-D complex *longitudinal* acoustic intensity $\tilde{I}_z(z)$ at the position z is defined as:

$$\boxed{\tilde{I}_z(z) \equiv \tfrac{1}{2}\tilde{p}(z,t)\tilde{u}_z^*(z,t) = \frac{1}{2}\tilde{p}(z)\tilde{u}_z^*(z) = -i\frac{A_o^2}{2\rho_o v}\left|\tilde{\mathcal{F}}(\tilde{k})\right|^2 e^{-2\kappa z}\cos(kz)\left[\left(\frac{\kappa}{k}\right)\cos kz + \sin kz\right]}$$

The complex *longitudinal* acoustic intensity $\tilde{I}_z(z)$ in the SWT is purely imaginary and is also a time-independent quantity. The SI units of complex acoustic intensity $\tilde{I}_z(z)$ are $Watts/m^2$.

The time-averaged total acoustic energy density $\langle w_a^{tot}(z)\rangle$ at the position z is the additive sum of the individual time-averaged acoustic potential energy density $\langle w_a^{potl}(z)\rangle$ and the time-averaged acoustic kinetic energy density $\langle w_a^{kin}(z)\rangle$:

$$\boxed{\langle w_a^{tot}(z)\rangle = \langle w_a^{potl}(z)\rangle + \langle w_a^{kin}(z)\rangle \equiv \frac{1}{4}\frac{|\tilde{p}(z,t)|^2}{\rho_o v^2} + \frac{1}{4}\rho_o|\tilde{u}_z(z,t)|^2 = \frac{1}{4}\frac{|\tilde{p}(z)|^2}{\rho_o v^2} + \frac{1}{4}\rho_o|\tilde{u}_z(z)|^2}$$

The time-averaged energy densities are purely real, time-independent quantities. The SI units of energy density are $Joules/m^3$.

Precisely at one of the resonant frequencies of the SWT $\boxed{f_n = v/\lambda_n = nv/2L}$, with both of the *p* and u_z mics located *e.g.* at *z* = *L*, then $\boxed{k_n L = 2\pi L/\lambda_n = n\pi}$, and hence $\boxed{\cos(k_n L) = \cos(n\pi) = (-1)^n}$, $\boxed{\cos(2k_n L) = \cos(2n\pi) = 1}$, $\boxed{\sin(k_n L) = \sin(n\pi) = 0}$ and $\boxed{\sin(2k_n L) = \sin(2n\pi) = 0}$ and thus the instantaneous overall complex pressure amplitude at *z* = *L* on the n^{th} resonance of the SWT becomes:

$$\boxed{\tilde{p}_n(z = L, t) = (-1)^n A_o e^{-\kappa L}\left(\frac{\sinh(2\kappa L)}{\cosh(2\kappa L) - 1} + 1\right) e^{+i\omega_n t}}$$

Now since $\boxed{\coth x = \dfrac{1}{\tanh x} = \dfrac{\cosh x}{\sinh x}}$ and using the fact that[2] $\boxed{\tanh\left(\tfrac{1}{2}x\right) = \dfrac{\cosh x - 1}{\sinh x}}$, then we see that $\boxed{\coth\left(\tfrac{1}{2}x\right) = \dfrac{1}{\tanh\left(\tfrac{1}{2}x\right)} = \dfrac{\sinh x}{\cosh x - 1}}$ and thus if $\boxed{x = 2\kappa L}$ we see that the above expression for $\tilde{p}_n(z = L, t)$ on the SWT resonances can equivalently be written as:

$$\boxed{\tilde{p}_n(z = L, t) = (-1)^n A_o e^{-\kappa L}\left[\coth(\kappa L) + 1\right] e^{+i\omega_n t}}$$

Thus, we see that there are pressure <u>anti</u>-<u>nodes</u> at both *z* = 0 and *z* = *L* on the resonances of the SWT for infinitely rigid/closed end walls.

The instantaneous 1-D longitudinal particle <u>velocity</u> at *z* = *L* on the resonances of the SWT is:

$$\boxed{\tilde{u}_{z_n}(z = L, t) = -i\frac{(-1)^n}{\rho_o v} A_o e^{-\kappa L}\left(\frac{\kappa}{k_n}\right)\left(\frac{\sinh(2\kappa L)}{\cosh(2\kappa L) - 1} + 1\right) e^{+i\omega_n t}}$$

Again, using the relation $\boxed{\coth\left(\tfrac{1}{2}x\right) = \dfrac{1}{\tanh\left(\tfrac{1}{2}x\right)} = \dfrac{\sinh x}{\cosh x - 1}}$ the longitudinal particle <u>velocity</u> at *z* = *L* on one of the resonances of the SWT can be rewritten as:

$$\boxed{\tilde{u}_{z_n}(z = L, t) = -i\frac{(-1)^n}{\rho_o v} A_o e^{-\kappa L}\left(\frac{\kappa}{k_n}\right)\left[\coth(\kappa L) + 1\right] e^{+i\omega_n t}}$$

Note that on a resonance of the SWT, $\tilde{u}_{z_n}(z = L, t)$ is -90° out-of-phase relative to $\tilde{p}_n(z = L, t)$.

The instantaneous 1-D longitudinal particle <u>displacement</u> at *z* = *L* on a resonance of the SWT is:

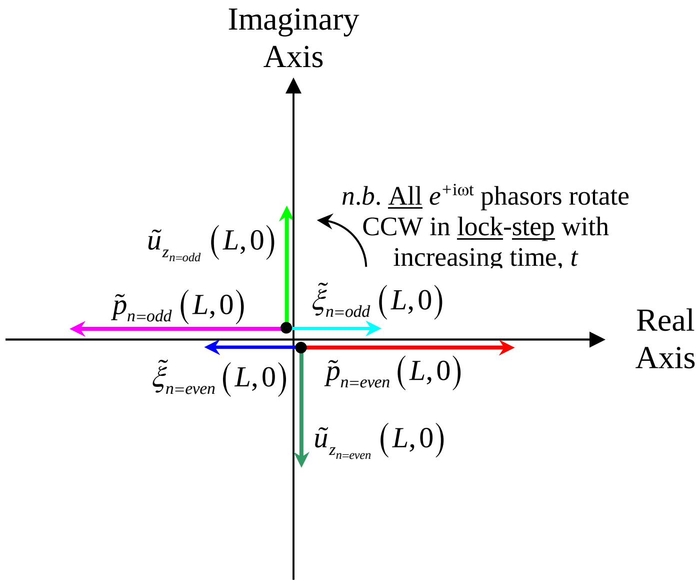
$$\boxed{\tilde{\xi}_{z_n}(z = L, t) = -\frac{(-1)^n}{\omega_n \rho_o v} A_o e^{-\kappa L}\left(\frac{\kappa}{k_n}\right)\left\{\left(\frac{\sinh(2\kappa L)}{\cosh(2\kappa L) - 1} + 1\right)\right\} e^{+i\omega_n t}}$$

Again, using the relation $\boxed{\coth\left(\tfrac{1}{2}x\right)=\dfrac{1}{\tanh\left(\tfrac{1}{2}x\right)}=\dfrac{\sinh x}{\cosh x-1}}$ the longitudinal particle displacement at $z = L$ on one of the resonances of the SWT can be rewritten as:

$$\boxed{\tilde{\xi}_{z_n}(z=L,t)=-\frac{(-1)^n}{\omega_n\rho_o}A_o e^{-\kappa L}\left(\frac{\kappa}{k_n}\right)\left(\coth(\kappa L)+1\right)e^{+i\omega_n t}}$$

Thus we see that on the resonances of the SWT, $\tilde{\xi}_{z_n}(z=L,t)$ is $-90°$ out of phase relative to $\tilde{u}_{z_n}(z=L,t)$ and is $-180°$ out of phase relative to $\tilde{p}_n(z=L,t)$.

In terms of a phasor diagram, the complex pressure $\tilde{p}_n(z=L,t)$, complex longitudinal particle velocity $\tilde{u}_{z_n}(z=L,t)$ and complex longitudinal displacement $\tilde{\xi}_n(z=L,t)$ on the resonances of the SWT as observed at $z = L$ and at time $t = 0$ are oriented as shown in the figure below:



The complex specific longitudinal acoustic impedance at $z = L$ on the resonances of the SWT is purely imaginary and time-independent:

$$\boxed{z_n(z=L)\equiv\frac{\tilde{p}_n(z=L,t)}{\tilde{u}_{z_n}(z=L,t)}=+i\,\rho_o v\left(\frac{k_n}{\kappa}\right)}$$

The complex longitudinal intensity at $z = L$ on the resonances of the SWT is also purely imaginary and time-independent:

$$\boxed{\tilde{I}_{z_n}(z=L)\equiv\frac{1}{2}\tilde{p}_n(z=L)\tilde{u}^*_{z_n}(z=L)=+i\frac{A_o^2}{2\rho_o v}e^{-2\kappa L}\left(\frac{\kappa}{k_n}\right)\left[\coth(\kappa L)+1\right]^2}$$

Note that since there are acoustic ***standing*** waves present in the SWT, the complex longitudinal sound intensity $\tilde{I}_z(z)$ *must* be purely reactive (*i.e*. purely imaginary). A non-zero value associated with the real component of $\tilde{I}_z(z)$ is due to an actual {time-averaged} flux, or flow of energy down the SWT – this cannot be due to a *standing* wave – only a *traveling* sound wave can have this!

The *time-averaged* total energy density at the position $z = L$ on a resonance of the SWT is:

$$\boxed{\begin{aligned}\left\langle w_{a_n}^{tot}(z=L)\right\rangle &\equiv \left\langle w_{a_n}^{potl}(z=L)\right\rangle + \left\langle w_{a_n}^{kin}(z=L)\right\rangle = \frac{1}{4}\frac{\left|\tilde{p}_n(z=L,t)\right|^2}{\rho_o v^2} + \frac{1}{4}\rho_o\left|\tilde{u}_{z_n}(z=L,t)\right|^2 \\ &= \frac{1}{4}\frac{A_o^2}{\rho_o v^2}e^{-2\kappa L}\left[\coth(\kappa L)+1\right]^2 + \frac{1}{4}\frac{A_o^2}{\rho_o v^2}e^{-2\kappa L}\left(\frac{\kappa}{k_n}\right)^2\left[\coth(\kappa L)+1\right]^2\end{aligned}}$$

Or:
$$\boxed{\left\langle w_{a_n}^{tot}(z=L)\right\rangle \equiv \left\langle w_{a_n}^{potl}(z=L)\right\rangle + \left\langle w_{a_n}^{kin}(z=L)\right\rangle = \frac{1}{4}\frac{A_o^2}{\rho_o v^2}e^{-2\kappa L}\left[1+\left(\frac{\kappa}{k_n}\right)^2\right]\left[\coth(\kappa L)+1\right]^2}$$

Note that since $(\kappa/k_n) \ll 1$ for inviscid fluid flow, on the resonances of the SWT we see that $\left\langle w_{a_n}^{potl}(z=L)\right\rangle \gg \left\langle w_{a_n}^{kin}(z=L)\right\rangle$.

References:

[1] *Table of Integrals, Series and Products*, I.S. Gradshteyn and I.M. Ryzhik, page 42, Academic Press, Inc. 1980.

In this reference, please note that the expression $\boxed{\sum_{n=0}^{\infty} e^{-nt}\sin nx = \left(\frac{\sin x}{\cosh t - \cos x}\right)}$ is factually in error in the 2nd and 3rd quadrants (where $\cos x < 0$). The correct expression, valid in all four quadrants is:

$$\boxed{\sum_{n=0}^{\infty} e^{-nt}\sin nx = \left(\frac{\sin x}{\cosh t - \left|\cos x\right|}\right)}$$

[2] *ibid*, page 25.