Two-Point Sound Field Relationships: $S(\vec{r}_1,t)$ vs. $S(\vec{r}_2,t)$

Introduction

At a given space-time point $(\vec{r},t)$ an arbitrary complex sound field $S(\vec{r},t)=\{p(\vec{r},t),\vec{u}(\vec{r},t)\}$ is completely/fully specified by making two physical measurements at that space-time point: the instantaneous complex acoustic pressure $p(\vec{r},t)$ {n.b. a <u>scalar</u> quantity} and the instantaneous complex three-dimensional acoustic particle velocity $\vec{u}(\vec{r},t)$ {n.b. a <u>vector</u> quantity}. At the common space-time point $(\vec{r},t)$ these two physical quantities are related to each other via the Euler equation for inviscid fluid flow (i.e. fluids – liquids and/or gases where <u>dissipative</u> forces are assumed to always be small in comparison to <u>inertial</u> forces):

$$\boxed{-\rho_o \frac{\partial \vec{u}(\vec{r},t)}{\partial t} = \vec{\nabla} p(\vec{r},t)}$$

The quantity ρ_o is the {equilibrium} mass volume density of the fluid, which for {bone-dry} air at NTP is $\rho_o = 1.204\, kg/m^3$. In *SI* units, the complex pressure $p(\vec{r},t)$ is measured in Pascals (1 *Pa* = 1 N/m^2 = 1 $kg\text{-}m/s^2$), the complex particle velocity $\vec{u}(\vec{r},t)$ is measured in units of *m*/*s*.

Usually, e.g. for a musical instrument such as the trumpet, we are interested in physical measurements of the complex acoustic sound field $S(\vec{r},t)$ at the input and output side of the trumpet associated with notes being played on the musical instrument, which manifestly involves the production (and propagation) of plane-type sound waves, where the amplitudes of the complex instantaneous pressure and particle velocity each have constant values everywhere on a given planar wavefront. Thus, the Euler equation describing the relationship between complex instantaneous pressure and particle velocity reduces to a 1-dimensional, rather than 3-dimensional problem for plane wave propagation, i.e.

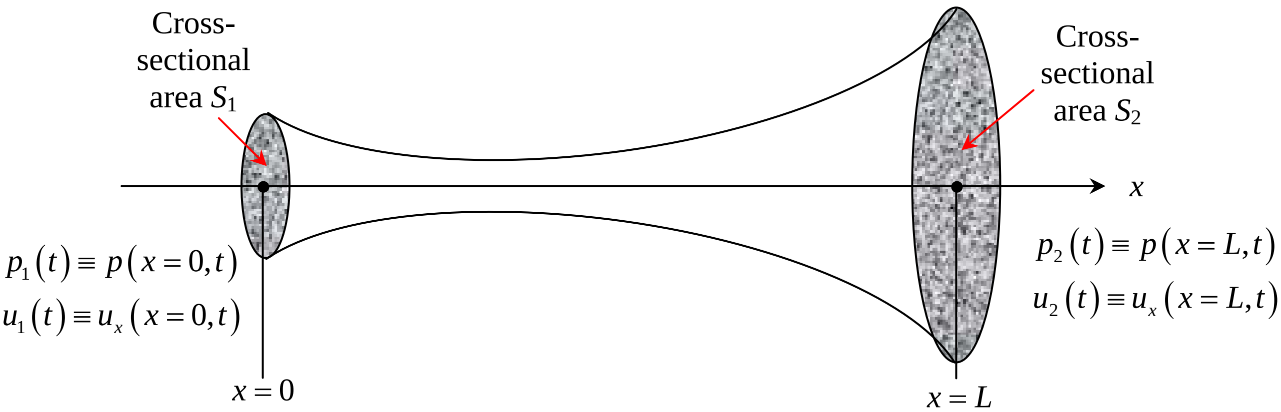
$$\boxed{-\rho_o \frac{\partial u_x(x,t)}{\partial t} = \frac{\partial p(x,t)}{\partial x}}$$

where $u_x(x,t)$ is the <u>longitudinal</u> component of the complex instantaneous particle velocity, i.e. parallel to/along the symmetry axis of the pipe/musical instrument.

<u>**Applications of the Four-Terminal Matrix Method in Complex Sound Field Analysis**</u>

A question that frequently arises is how do we relate the sound field $S(\vec{r}_1,t)=\{p(\vec{r}_1,t),\vec{u}(\vec{r}_1,t)\}$ at a given space-time point $(\vec{r}_1,t)$ to another sound field $S(\vec{r}_2,t)=\{p(\vec{r}_2,t),\vec{u}(\vec{r}_2,t)\}$ at the space-time point $(\vec{r}_2,t)$ that have a common sound source? There are various methods that can be used to accomplish this task. One method is to integrate the Euler equation both in space and (backwards) in time, which can be done correctly only if the detailed 3-D geometry is fully specified everywhere along the actual path taken by the propagation of sound waves from the point $\vec{r}_1$ to the point $\vec{r}_2$. Another approach is to use the methodology, appropriately modified, associated with the four-terminal method used in complex electrical network analysis, as discussed in Appendix 1.

For sound propagation in a confined, infinitely rigid structure such as a pipe, or a musical instrument such as a trumpet, the complex instantaneous pressure and particle velocity at one end of the pipe/musical instrument, $p(\vec{r}_1,t)$ and $\vec{u}(\vec{r}_1,t)$ with cross sectional area S_1 can be related to the complex instantaneous pressure and particle velocity $p(\vec{r}_2,t)$ and $\vec{u}(\vec{r}_2,t)$ with cross sectional area S_2 at the other end of the pipe/musical instrument using a mathematical technique developed for applications in analyzing electrical networks. The physical situation is shown in the figure below:



If the physical length L of the pipe/musical instrument is such that $kL \gg 1$, where the wavenumber $k = 2\pi/\lambda$ and $\lambda = v/f$, and where the speed of sound v in {bone-dry} air at NTP is $v \simeq 345\, m/s$, then use of the so-called four-terminal method *is* valid in this situation. The complex instantaneous pressure and particle velocity $p_1(t) \equiv p(x=0,t)$ and $\vec{u}_1(t) \equiv \vec{u}(x=0,t)$ at the "input" side of the structure at $x=0$ are different/decoupled from the complex instantaneous pressure and particle velocity $p_2(t) \equiv p(x=L,t)$ and $\vec{u}_2(t) \equiv \vec{u}(x=L,t)$ at the "output" side of the structure at $x=L$. For example, for a B$_b$ trumpet, with nominal tube length $L \simeq 1.47\, m$, the lowest note (other than the pedal note) is C4, i.e. $f_{C4} = 261.63\, Hz$ which corresponds to a {free-air} wavelength of $\lambda_{C4} = v/f_{C4} \simeq 345/261.63 = 1.32\, m$ {n.b. it is not an accident that λ_{C4} is comparable to the tube length L of the trumpet}. Thus we see that the criterion $k_{C4}L = (2\pi/\lambda_{C4})L \simeq (6.28/1.32)1.47 = 7.00 \gg 1$ is certainly satisfied for all normally-played notes of the trumpet. This is also true for other musical instruments in the "classic" brass/wind family.

Before launching into a discussion of applying the four-terminal network approach to this acoustics problem, we must first discuss a couple of issues. The complex *specific* acoustic impedance $z(x)$ at a given point, x is defined as the ratio of the complex pressure $p(x)$ to the complex *particle* velocity $u(x)$ at that point, i.e.:

Complex *specific* acoustic impedance: $$\boxed{z(x) \equiv \frac{p(x,t)}{u(x,t)} = \frac{p(x)e^{-i\omega t}}{u(x)e^{-i\omega t}} = \frac{p(x)}{u(x)}}$$

Note that the complex *specific* acoustic impedance $z(x)$ is a time-independent quantity.

The *SI* units of complex *specific* acoustic impedance $\boxed{z(x) = p(x)/u(x)}$ are $Pa\text{-}s/m = N\text{-}s/m^3 = kg\text{-}s/m^2$, or simply acoustic Ohms, i.e. $1\,Pa\text{-}s/m = 1\,\Omega_{ac}$, in direct analogy to the complex form of Ohms law for AC circuits, where the electrical AC impedance, $\boxed{Z(x) \equiv V(x)/I(x)}$, also a time-independent quantity.

The so-called complex acoustic impedance $Z(x)$ at a given point, x is defined as the ratio of the complex pressure $p(x)$ to the complex *volume* velocity $U(x)$ at that point, i.e.:

$$\text{Complex acoustic impedance: } \boxed{Z(x) \equiv \frac{p(x,t)}{U(x,t)} = \frac{p(x)e^{-i\omega t}}{U(x)e^{-i\omega t}} = \frac{p(x)}{U(x)}}$$

where the complex instantaneous *volume* velocity $U(x,t)$ is related to the complex instantaneous *particle* velocity $u(x,t)$ by $\boxed{U(x,t) = S(x)u(x,t)}$ where $S(x)$ is the cross sectional area (in m^2) of the pipe/musical instrument at the point x. Thus, the complex acoustic impedance $Z(x)$ at a given point, x can also be written as:

$$\text{Complex acoustic impedance: } \boxed{Z(x) \equiv \frac{p(x,t)}{U(x,t)} = \frac{p(x)e^{-i\omega t}}{U(x)e^{-i\omega t}} = \frac{p(x)}{U(x)} = \frac{p(x)}{S(x)u(x)} = \frac{1}{S(x)}z(x)}$$

Note that the complex acoustic impedance $Z(x)$ is also a time-independent quantity.

The *SI* units of complex acoustic impedance $\boxed{Z(x) = p(x)/U(x)}$ are $Pa\text{-}s/m^3 = N\text{-}s/m^5 = kg\text{-}s/m^4$, also known as *Rayls* (in honor of Lord Rayleigh, for his theoretical work in acoustical physics in the 19th century), i.e. $1\,Pa\text{-}s/m^3 = 1\,Rayl$, again in direct analogy to the complex form of Ohms law for AC circuits, where the complex AC electrical impedance, $\boxed{Z(x) \equiv V(x)/I(x)}$ (also a time-independent quantity)

Since the *SI* units of complex *particle* velocity $u(x)$ are m/s, the *SI* units of complex *volume* velocity $\boxed{U(x,t) = S(x)u(x,t)}$ are m^3/s, which is precisely why $U(x,t)$ *is* called the volume velocity. Multiplying the complex instantaneous volume velocity $U(x,t)$ by the equilibrium mass volume density $\rho_o = 1.204\,kg/m^3$ for {bone-dry} air at NTP gives the complex instantaneous *mass flow* $\boxed{M(x,t) = \rho_o U(x,t) = \rho_o S(x)u(x,t)}$ which has *SI* units of kg/s, i.e. the complex *mass flow* is in fact a complex *mass* *current*, and as such it is directly analogous to the complex instantaneous AC electrical current $I(x,t)$, which has *SI* units of $Coulombs/s = Amperes$. Note also that the complex instantaneous pressure $p(x,t)$ is directly analogous to the complex instantaneous AC electrical voltage (i.e. potential difference) $V(x,t)$.

Following the methodology described in Appendix 1 for the four-terminal network in complex AC electronic circuit analysis, the instantaneous complex pressure $p_1(t) \equiv p(x=0,t)$ and *volume* velocity $U_1(t) \equiv U(x=0,t) = S(x=0,t)u(x=0,t) = S_1 u(x=0,t) = S_1 u_1(t)$ at the "input" side of the acoustic structure at $x=0$ can be related to the complex instantaneous pressure $p_2(t) \equiv p(x=L,t)$ and *volume* velocity $U_2(t) \equiv U(x=L,t) = S(x=L,t)u(x=L,t) = S_2 u(x=L,t) = S_2 u_1(t)$ at the "output" side of the acoustic structure at $x=L$ via the complex matrix equation $\mathbb{P} = \mathbb{Z}\mathbb{U}$ which is of the form:

$$\begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} U_1(t) \\ U_2(t) \end{pmatrix}$$

Explicitly writing this relation out as two separate (but coupled) equations:

$$\begin{aligned} p_1(t) &= Z_{11}U_1(t) + Z_{12}U_2(t) \\ p_2(t) &= Z_{21}U_1(t) + Z_{22}U_2(t) \end{aligned}$$

Mathematically speaking, if all four complex instantaneous pressures and *volume* velocities, $p_1(t)$, $p_2(t)$, $U_1(t)$ and $U_2(t)$ are measured/known, since we have two equations and *four* unknowns $(Z_{11}, Z_{12}, Z_{21}, Z_{22})$ associated with the two coupled complex relations

$$\begin{aligned} p_1(t) &= Z_{11}U_1(t) + Z_{12}U_2(t) \\ p_2(t) &= Z_{21}U_1(t) + Z_{22}U_2(t) \end{aligned}$$

then we do *not* have enough relations to completely/uniquely solve for the Z_{ij} elements of the complex $\mathbb{Z}$ matrix. However, due to the Reciprocity Theorem (in which the roles of the input and output of the acoustic structure are reversed), the off-diagonal elements of the complex $\mathbb{Z}$ matrix *must* be equal to each other, i.e. $\boxed{Z_{12} = Z_{21}}$, and thus we actually have only *three* unknowns $(Z_{11}, Z_{12} = Z_{21}, Z_{22})$. Note also that in general $\boxed{Z_{11} \neq Z_{22}}$, unless the geometry of the acoustic structure is such that it has manifest input-output reflection symmetry about its mid-point at $x = L/2$. Note further that the *microscopic* origin of the Reciprocity Theorem is intimately associated with the manifest time-reversal invariant nature of the electromagnetic interaction at the particle-physics level – i.e. the interaction between electrical charges via the exchange of virtual photons – the quanta/mediators of the electromagnetic force, which is intimately involved in the collisions/scatterings of gas molecules within the acoustic fluid – the air!

Thus, the relations for the complex $\mathbb{Z}$ matrix become (suppressing the common time dependence for notational clarity):

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \quad \text{and/or:} \quad \begin{aligned} p_1 &= Z_{11}U_1 + Z_{12}U_2 \\ p_2 &= Z_{12}U_1 + Z_{22}U_2 \end{aligned}$$

We also have the (auxiliary) relations for input and output impedances: $p_1 = Z_1 U_1$ and $p_2 = Z_2 U_2$.

Thus:

$$p_1 = Z_1 U_1 = Z_{11} U_1 + Z_{12} U_2$$
$$p_2 = Z_2 U_2 = Z_{12} U_1 + Z_{22} U_2$$

Rewriting these as:

$$\begin{aligned}(Z_1 - Z_{11})U_1 &= Z_{12}U_2 \\ (Z_2 - Z_{22})U_2 &= Z_{12}U_1\end{aligned} \quad \text{or:} \quad \begin{aligned}(Z_1 - Z_{11})U_1 &= Z_{12}U_2 \\ Z_{12}U_1 &= (Z_2 - Z_{22})U_2\end{aligned}$$

Dividing the two LHS equations (and eliminating Z_{12}) we see that: $\boxed{\dfrac{(Z_1 - Z_{11})}{(Z_2 - Z_{22})} = \left(\dfrac{U_2}{U_1}\right)^2}$.

Dividing the two RHS equations (eliminating the *U's*) we see that: $\boxed{\dfrac{(Z_1 - Z_{11})}{Z_{12}} = \dfrac{Z_{12}}{(Z_2 - Z_{22})}}$

i.e. $\boxed{Z_{12}^2 = (Z_1 - Z_{11})(Z_2 - Z_{22})}$, which can be rewritten either as $\boxed{Z_{11} = Z_1 - \dfrac{Z_{12}^2}{(Z_2 - Z_{22})}}$

or as $\boxed{Z_{22} = Z_2 - \dfrac{Z_{12}^2}{(Z_1 - Z_{11})}}$. The LHS equations can be rewritten again in the following form:

$$\boxed{\begin{aligned} Z_{11} &= Z_1 - Z_{12}(U_2/U_1) \\ Z_{22} &= Z_2 - Z_{12}(U_1/U_2) \end{aligned}}$$

From the relationship $\boxed{(Z_1 - Z_{11})/(Z_2 - Z_{22}) = (U_2/U_1)^2}$ we can also rewrite this as:

$$\boxed{\begin{aligned} Z_{11} &= Z_1 - (Z_2 - Z_{22})(U_2/U_1)^2 = Z_1 + (Z_{22} - Z_2)(U_2/U_1)^2 \\ Z_{22} &= Z_2 - (Z_1 - Z_{11})(U_1/U_2)^2 = Z_2 + (Z_{11} - Z_1)(U_1/U_2)^2 \end{aligned}}$$

This is about as far as we can go in terms of obtaining useful relations using <u>only</u> the two coupled equations of the complex $\mathbb{Z}$ matrix relation.

In order to uniquely determine/specify <u>all four</u> of the Z_{ij} elements of the complex $\mathbb{Z}$ matrix, one <u>additional</u> relation is needed. The relation we will use is associated with energy/energy density {n.b. which are both manifest <u>scalar</u> quantities}. The time-averaged energy density stored in the pressure and particle velocity of a complex sound field is:

$$\boxed{\langle e_{tot}(z)\rangle \equiv \langle e_{potl}(z)\rangle + \langle e_{kin}(z)\rangle = \frac{1}{4}\frac{|p(x,t)|^2}{\rho_o v^2} + \frac{1}{4}\rho_o |u(x,t)|^2 = \frac{1}{4}\frac{p(x,t)\,p^*(x,t)}{\rho_o v^2} + \frac{1}{4}\rho_o u(x,t)u^*(x,t)}$$

where * denotes complex conjugation, i.e. if $A = a + i\alpha$ then $A^* = a - i\alpha$ where $i \equiv \sqrt{-1}$ and $-i \equiv -\sqrt{-1}$. The *SI* units of energy density *e* are *Joules*/m^3.

From the above complex matrix relation $\mathbb{P} = \mathbb{Z}\mathbb{U}$, we can form quantities such as $\left|p(x,t)\right|^2$ by taking the complex transpose $\mathbb{P}^{*\mathrm{T}} = (\mathbb{Z}\mathbb{U})^{*\mathrm{T}} = \mathbb{U}^{*\mathrm{T}}\mathbb{Z}^{*\mathrm{T}}$ and then multiplying $\mathbb{P} = \mathbb{Z}\mathbb{U}$ on the left by its complex transpose, i.e.:

$$\mathbb{P}^{*\mathrm{T}}\mathbb{P} = \left(\mathbb{U}^{*\mathrm{T}}\mathbb{Z}^{*\mathrm{T}}\right)(\mathbb{Z}\mathbb{U}) = \mathbb{U}^{*\mathrm{T}}\left(\mathbb{Z}^{*\mathrm{T}}\mathbb{Z}\right)\mathbb{U}$$

In matrix form, this is explicitly (again suppressing the time-dependence for notational clarity):

$$\begin{pmatrix} p_1^* & p_2^* \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} U_1^* & U_2^* \end{pmatrix}\begin{pmatrix} Z_{11}^* & Z_{21}^* \\ Z_{12}^* & Z_{22}^* \end{pmatrix}\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} U_1^* & U_2^* \end{pmatrix}\begin{pmatrix} Z_{11}^* & Z_{12}^* \\ Z_{12}^* & Z_{22}^* \end{pmatrix}\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

Carrying out the matrix operations in stages we obtain:

$$\begin{pmatrix} p_1^* & p_2^* \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} Z_{11}^*U_1^* + Z_{12}^*U_2^* & Z_{12}^*U_1^* + Z_{22}^*U_2^* \end{pmatrix}\begin{pmatrix} Z_{11}U_1 + Z_{12}U_2 \\ Z_{12}U_1 + Z_{22}U_2 \end{pmatrix}$$

Or:

$$\begin{aligned} |p_1|^2 + |p_2|^2 &= \left(|Z_{11}|^2 + |Z_{12}|^2\right)|U_1|^2 + \left(|Z_{12}|^2 + |Z_{22}|^2\right)|U_2|^2 \\ &\quad + \left(Z_{11}Z_{12}^* + Z_{12}Z_{22}^*\right)U_1U_2^* + \left(Z_{11}^*Z_{12} + Z_{12}^*Z_{22}\right)U_1^*U_2 \end{aligned}$$

But $p_1 = Z_1U_1$ and $p_2 = Z_2U_2$, thus:

$$\begin{aligned} |Z_1|^2|U_1|^2 + |Z_2|^2|U_2|^2 &= \left(|Z_{11}|^2 + |Z_{12}|^2\right)|U_1|^2 + \left(|Z_{12}|^2 + |Z_{22}|^2\right)|U_2|^2 \\ &\quad + \left(Z_{11}Z_{12}^* + Z_{12}Z_{22}^*\right)U_1U_2^* + \left(Z_{11}^*Z_{12} + Z_{12}^*Z_{22}\right)U_1^*U_2 \end{aligned}$$

Before mindlessly/brute-force solving this relation, it is very useful to note that physically, energies and/or energy densities are ***additive*** scalar quantities, i.e. that $\langle e_{tot}\rangle = \langle e_1(x=0)\rangle + \langle e_2(x=L)\rangle + \ldots$ More specifically, the ***additive*** nature of energy/energy density means that ***no*** interference/***no*** cross-terms involving quantities such as $U_1(x=0,t)U_2^*(x=L,t)$ and/or $U_1^*(x=0,t)U_2(x=L,t)$ as in the above relation are allowed, for if they ***did*** exist, this would imply instantaneous/non-local/faster-than-light propagation/transfer of energy, e.g. from the point $x=0$ to the point $x=L$ (or vice-versa), which would be ***acausal***, a phenomenon which is ***not*** observed in our universe. Information ***cannot*** propagate faster than the speed of light, *c*. Hence any/all such cross terms in energy/energy density formulae are ***strictly*** forbidden. In our current situation, this ***explicitly*** means that:

$$\boxed{\left(Z_{11}Z_{12}^* + Z_{12}Z_{22}^*\right)U_1(\vec{r}_1,t)U_2^*(\vec{r}_2,t) + \left(Z_{11}^*Z_{12} + Z_{12}^*Z_{22}\right)U_1^*(\vec{r}_1,t)U_2(\vec{r}_2,t) = 0}$$

and thus:

$$\boxed{\left\{|Z_1|^2 - \left(|Z_{11}|^2 + |Z_{12}|^2\right)\right\}\left|U_1(\vec{r}_1,t)\right|^2 + \left\{|Z_2|^2 - \left(|Z_{12}|^2 + |Z_{22}|^2\right)\right\}\left|U_2(\vec{r}_2,t)\right|^2 = 0}$$

Solving either of these equations, in addition to using the two coupled equations of the complex $\mathbb{Z}$ matrix relation and the use of the Reciprocity Theorem, in combination with explicit measurements of at least _two_ of the four experimentally observable quantities $p_1(t)$, $p_2(t)$, $U_1(t)$ and $U_2(t)$ {n.b. at least one measurement _must_ be on the "input" side and at least one measurement _must_ be on the "output" side of the acoustic structure} and the use of the auxiliary relations $p_1 = Z_1U_1$ and $p_2 = Z_2U_2$ enables us to fully/uniquely determine _all four_ of the Z_{ij} elements of the complex $\mathbb{Z}$ matrix.

Note also from the immediately above equation, it ***might*** be tempting to ***believe*** that $|Z_1|^2 = |Z_{11}|^2 + |Z_{12}|^2$ and hence also that $|Z_2|^2 = |Z_{12}|^2 + |Z_{22}|^2$ (invoking the Reciprocity Theorem), which would certainly satisfy this equation; however, explicit/brute-force calculations show this is _**not**_ the case/is _**not**_ true. The reason is simple, because we have already previously obtained the following relations from the two coupled equations of the complex $\mathbb{Z}$ matrix equation:

$$\boxed{\begin{aligned} Z_{11} &= Z_1 - Z_{12}(U_2/U_1) \\ Z_{22} &= Z_2 - Z_{12}(U_1/U_2) \end{aligned}} \quad \text{or:} \quad \boxed{\begin{aligned} Z_1 &= Z_{11} + Z_{12}(U_2/U_1) \\ Z_2 &= Z_{22} + Z_{12}(U_1/U_2) \end{aligned}} \Rightarrow \boxed{\begin{aligned} |Z_1|^2 &= |Z_{11} + Z_{12}(U_2/U_1)|^2 \\ |Z_2|^2 &= |Z_{22} + Z_{12}(U_1/U_2)|^2 \end{aligned}}$$

Skipping much tedious complex algebra, we simply quote the result of using either of the above two "energy / energy density" relations and all the other relations, enabling us to express e.g. Z_{12} in terms of the four explicitly measured experimental quantities $p_1(t)$, $p_2(t)$, $U_1(t)$ and $U_2(t)$:

$$\boxed{Z_{12} = \frac{p_1U_2^* + p_2U_1^*}{|U_1|^2 + |U_2|^2} = Z_{21}}$$

Note the manifest symmetry in the above expression under the interchange of $1 \rightleftarrows 2$ indices, which arises as an explicit consequence of the Reciprocity Theorem, i.e. requiring $Z_{12} = Z_{21}$.

Once Z_{12} has been determined, the other two elements of the complex $\mathbb{Z}$ matrix can be determined e.g. by using the relations:

$$\boxed{\begin{aligned} Z_{11} &= Z_1 - Z_{12}(U_2/U_1) \\ Z_{22} &= Z_2 - Z_{12}(U_1/U_2) \end{aligned}}$$

where the complex input and output acoustic impedances are $Z_1 = p_1/U_1$ and $Z_2 = p_2/U_2$, respectively

The Relationship Between the Four-Terminal Matrix Equation and Transfer Matrix Equation

The complex $\mathbb{Z}$ matrix equation $\mathbb{P} = \mathbb{Z}\mathbb{U}$ obtained via the four-terminal method individually relates complex instantaneous pressures at two different points in space to the acoustic volume velocities at those points via this equation. More explicitly:

$$\begin{pmatrix} p_1(x_1,t) \\ p_2(x_2,t) \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{pmatrix} \begin{pmatrix} U_1(x_1,t) \\ U_2(x_2,t) \end{pmatrix} \quad \text{or:} \quad \boxed{\begin{aligned} p_1(x_1,t) &= Z_{11}U_1(x_1,t) + Z_{12}U_2(x_2,t) \\ p_2(x_2,t) &= Z_{12}U_1(x_1,t) + Z_{22}U_2(x_2,t) \end{aligned}}$$

Perhaps physically more intuitive/meaningful is the equivalent relation, known as the <u>transfer matrix equation</u> $\mathbb{S}_2 = \mathbb{T}\mathbb{S}_1$ relating two complex sound fields $\mathbb{S}_1$ and $\mathbb{S}_2$ via the complex $\mathbb{T}$ matrix, where:

$$\begin{pmatrix} p_2(x_2,t) \\ U_2(x_2,t) \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} p_1(x_1,t) \\ U_1(x_1,t) \end{pmatrix} \quad \text{or:} \quad \boxed{\begin{aligned} p_2(x_1,t) &= T_{11}p_1(x_1,t) + T_{12}U_1(x_2,t) \\ U_2(x_2,t) &= T_{21}p_1(x_1,t) + T_{22}U_1(x_2,t) \end{aligned}}$$

The question arises, how is the complex transfer $(\mathbb{T})$ matrix equation $\mathbb{S}_2 = \mathbb{T}\mathbb{S}_1$ related to the complex $\mathbb{Z}$ matrix equation $\mathbb{P} = \mathbb{Z}\mathbb{U}$? Starting with the two $\mathbb{Z}$ matrix equations:

$$\boxed{\begin{aligned} p_1 &= Z_{11}U_1 + Z_{12}U_2 \\ p_2 &= Z_{12}U_1 + Z_{22}U_2 \end{aligned}}$$

Multiply the first equation by (Z_{22}/Z_{12}): $(Z_{22}/Z_{12})\,p_1 = (Z_{22}Z_{11}/Z_{12})U_1 + Z_{22}U_2$.
Subtract this equation from the 2nd equation above, eliminating U_2 to obtain:

$$p_2 - (Z_{22}/Z_{12})\,p_1 = \left[Z_{12} - (Z_{22}Z_{11}/Z_{12})\right]U_1 \quad \text{or:} \quad \boxed{p_2 = (Z_{22}/Z_{12})\,p_1 + \left[Z_{12} - (Z_{11}Z_{22}/Z_{12})\right]U_1}.$$

Next, simply rewrite the first equation above as: $Z_{12}U_2 = p_1 - Z_{11}U_1$ or: $\boxed{U_2 = (1/Z_{12})\,p_1 - (Z_{11}/Z_{12})U_1}$.

Thus, we have the following pair of coupled equations:

$$\begin{aligned} p_2 &= (Z_{22}/Z_{12})\,p_1 + \left(\left[Z_{12}^2 - Z_{11}Z_{22}\right]/Z_{12}\right)U_1 \\ U_2 &= \quad (1/Z_{12})\,p_1 \quad - \quad (Z_{11}/Z_{12})U_1 \end{aligned} \quad \text{or:} \quad \begin{pmatrix} p_2 \\ U_2 \end{pmatrix} = \begin{pmatrix} Z_{22}/Z_{12} & \left[Z_{12}^2 - Z_{11}Z_{22}\right]/Z_{12} \\ 1/Z_{12} & -Z_{11}/Z_{12} \end{pmatrix} \begin{pmatrix} p_1 \\ U_1 \end{pmatrix}$$

Thus, we see that the complex transfer matrix equation $\mathbb{S}_2 = \mathbb{T}\mathbb{S}_1$ relating two sound fields:

$$\begin{pmatrix} p_2 \\ U_2 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} p_1 \\ U_1 \end{pmatrix} \text{ or: } \begin{pmatrix} p_2 \\ U_2 \end{pmatrix} = \begin{pmatrix} Z_{22}/Z_{12} & \left[Z_{12}^2 - Z_{11}Z_{22}\right]/Z_{12} \\ 1/Z_{12} & -Z_{11}/Z_{12} \end{pmatrix} \begin{pmatrix} p_1 \\ U_1 \end{pmatrix} = \frac{1}{Z_{12}} \begin{pmatrix} Z_{22} & \left[Z_{12}^2 - Z_{11}Z_{22}\right] \\ 1 & -Z_{11} \end{pmatrix} \begin{pmatrix} p_1 \\ U_1 \end{pmatrix}$$

$$\boxed{\begin{aligned} p_2 &= T_{11}p_1 + T_{12}U_1 \\ U_2 &= T_{21}p_1 + T_{22}U_1 \end{aligned}} \quad \text{or:} \quad \boxed{\begin{aligned} p_2 &= (Z_{22}/Z_{12})\,p_1 + \left(\left[Z_{12}^2 - Z_{11}Z_{22}\right]/Z_{12}\right)U_1 \\ U_2 &= \quad (1/Z_{12})\,p_1 \quad - (Z_{11}/Z_{12})U_1 \end{aligned}}$$

Thus, we see that the T_{ij} elements of the complex transfer matrix $\mathbb{T}$ are:

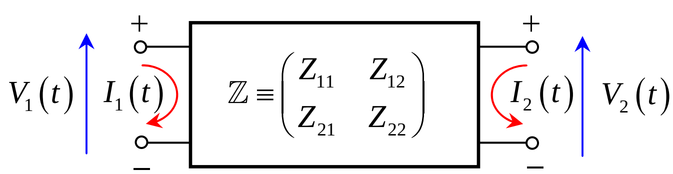
$$\boxed{\begin{array}{ll} T_{11} = \left(Z_{22}/Z_{12}\right) & T_{12} = \left[Z_{12}^2 - Z_{11}Z_{22}\right]/Z_{12} \\ T_{21} = \left(1/Z_{12}\right) & T_{22} = -\left(Z_{11}/Z_{12}\right) \end{array}} \text{ where: } \boxed{Z_{12} = \frac{p_1U_2^* + p_2U_1^*}{\left|U_1\right|^2 + \left|U_2\right|^2} = Z_{21}} \text{ and: } \boxed{\begin{array}{l} Z_{11} = Z_1 - Z_{12}\left(U_2/U_1\right) \\ Z_{22} = Z_2 - Z_{12}\left(U_1/U_2\right) \end{array}}.$$

Note that while the complex $\mathbb{Z}$ matrix has $Z_{12} = Z_{21}$, in contrast, the complex transfer matrix $\mathbb{T}$ manifestly has $T_{12} \neq T_{21}$ since the off-diagonal elements do not/can not have the same physical *SI* units, i.e. $T_{12} = \left[Z_{12}^2 - Z_{11}Z_{22}\right]/Z_{12}$ whereas $T_{21} = \left(1/Z_{12}\right)$. The diagonal elements T_{11} and T_{22} are dimensionless.

Appendix 1:

The Four-Terminal Method As Used In Complex AC Electrical Network Analysis

In electrical network theory, complex instantaneous voltages and currents at the "input" and "output" sides of an arbitrary, i.e. "black-box" four-terminal network circuit as shown in the figure below



are related to each other via the complex matrix equation $\mathbb{V} = \mathbb{Z}\mathbb{I}$ which is of the form:

$$\begin{pmatrix} V_1(t) \\ V_2(t) \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I_1(t) \\ I_2(t) \end{pmatrix}$$

Explicitly writing this relation out as two separate (but coupled) equations:

$$\begin{aligned} V_1(t) &= Z_{11}I_1(t) + Z_{12}I_2(t) \\ V_2(t) &= Z_{21}I_1(t) + Z_{22}I_2(t) \end{aligned}$$

Note that the four Z_{ij} elements of the complex $\mathbb{Z}$ matrix have *SI* units of that for impedance, Z i.e. Ohms (= Volts/Ampere), since the complex form of Ohm's law is $V(t) = Z\,I(t)$. Note also that for a given ***load*** impedance $Z_2 = V_2(t)/I_2(t)$ attached to the "output" side of the four-terminal network circuit, the ***input*** impedance $Z_1 = V_1(t)/I_1(t)$, as viewed from the "input" side of the four-terminal network circuit can also be obtained using the above complex matrix relation.

The detailed nature of the electrical network contained inside the "black box" may be explicitly known *a priori*, such as e.g. a passive AC filter network (consisting e.g. of resistors with resistances R_i, capacitors with capacitances C_j and inductors with inductances L_k); the "black box" could also e.g. be a transformer, with a primary winding of N_1 turns on the input side and a secondary winding of N_2 turns on the output side. However, the electrical components contained within the "black box" may be completely unknown, and thus in this case, it would truly be a black box. In this situation, if we physically measure each of the four complex instantaneous voltages and currents, $V_1(t)$, $V_2(t)$, $I_1(t)$ and $I_2(t)$ then from the above complex matrix relation, we should be able to explicitly / quantitatively determine all four of the elements Z_{ij} of the complex $\mathbb{Z}$ matrix. Note however, that if the "black box" contains <u>*any*</u> reactive electrical components (i.e. capacitors and/or inductors), this analysis method <u>*only*</u> works for finite frequencies (i.e. $f > 0$); formally it fails in situations where $f = 0$ (when capacitive/inductive reactances are infinite/zero, respectively); thus since $\lambda = v/f$, if $f = 0$ then $\lambda = \infty$ and also the wavenumber $k = 2\pi/\lambda = 2\pi/\infty = 0$ for $f = 0$.

Mathematically speaking, if all four complex instantaneous voltages and currents, $V_1(t)$, $V_2(t)$, $I_1(t)$ and $I_2(t)$ are measured/known, since we have <u>two</u> equations and <u>*four*</u> unknowns $(Z_{11}, Z_{12}, Z_{21}, Z_{22})$ associated with the two coupled complex relations

$$V_1(t) = Z_{11}I_1(t) + Z_{12}I_2(t)$$
$$V_2(t) = Z_{21}I_1(t) + Z_{22}I_2(t)$$

then we do <u>*not*</u> have enough relations to completely/uniquely solve for the Z_{ij} elements of the complex $\mathbb{Z}$ matrix. However, due to the <u>Reciprocity Theorem</u> (in which the roles of the input and output of the 4-terminal network are reversed), the off-diagonal elements of the complex $\mathbb{Z}$ matrix <u>*must*</u> be equal to each other, i.e. $\boxed{Z_{12} = Z_{21}}$, and thus we actually have only <u>*three*</u> unknowns $(Z_{11}, Z_{12} = Z_{21}, Z_{22})$. Note also that in general $\boxed{Z_{11} \neq Z_{22}}$, unless the innards of the four-terminal "black-box" network is such that it has manifest input-output reflection symmetry. Note further that the <u>*microscopic*</u> origin of the Reciprocity Theorem is intimately associated with the manifest time-reversal invariant nature of the electromagnetic interaction at the particle-physics level – i.e. the interaction between electrical charges via the exchange of virtual photons – the quanta/mediators of the electromagnetic force.

Thus, the relations for the complex $\mathbb{Z}$ matrix become (suppressing the common time dependence for notational clarity):

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \quad \text{and/or:} \quad \begin{aligned} V_1 &= Z_{11}I_1 + Z_{12}I_2 \\ V_2 &= Z_{12}I_1 + Z_{22}I_2 \end{aligned}$$

We also have the (auxiliary) relations for input and output impedances: $V_1 = Z_1 I_1$ and $V_2 = Z_2 I_2$.

Thus:

$$V_1 = Z_1 I_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_2 I_2 = Z_{12}I_1 + Z_{22}I_2$$

Rewriting these as:

$$\begin{aligned}(Z_1 - Z_{11}) I_1 &= Z_{12} I_2 \\ (Z_2 - Z_{22}) I_2 &= Z_{12} I_1\end{aligned} \quad \text{or:} \quad \begin{aligned}(Z_1 - Z_{11}) I_1 &= Z_{12} I_2 \\ Z_{12} I_1 &= (Z_2 - Z_{22}) I_2\end{aligned}$$

Dividing the two LHS equations (and eliminating Z_{12}) we see that: $\boxed{\dfrac{(Z_1 - Z_{11})}{(Z_2 - Z_{22})} = \left(\dfrac{I_2}{I_1}\right)^2}$.

Dividing the two RHS equations (eliminating the currents) we see that: $\boxed{\dfrac{(Z_1 - Z_{11})}{Z_{12}} = \dfrac{Z_{12}}{(Z_2 - Z_{22})}}$

i.e. $\boxed{Z_{12}^2 = (Z_1 - Z_{11})(Z_2 - Z_{22})}$, which can be rewritten either as $\boxed{Z_{11} = Z_1 - \dfrac{Z_{12}^2}{(Z_2 - Z_{22})}}$

or as $\boxed{Z_{22} = Z_2 - \dfrac{Z_{12}^2}{(Z_1 - Z_{11})}}$. The LHS equations can be rewritten again in the following form:

$$\boxed{\begin{aligned}Z_{11} &= Z_1 - Z_{12}(I_2/I_1) \\ Z_{22} &= Z_2 - Z_{12}(I_1/I_2)\end{aligned}}$$

From the relationship $\boxed{(Z_1 - Z_{11})/(Z_2 - Z_{22}) = (I_2/I_1)^2}$ we can also rewrite this as:

$$\boxed{\begin{aligned}Z_{11} &= Z_1 - (Z_2 - Z_{22})(I_2/I_1)^2 = Z_1 + (Z_{22} - Z_2)(I_2/I_1)^2 \\ Z_{22} &= Z_2 - (Z_1 - Z_{11})(I_1/I_2)^2 = Z_2 + (Z_{11} - Z_1)(I_1/I_2)^2\end{aligned}}$$

This is about as far as we can go in terms of obtaining useful relations using <u>only</u> the two coupled equations of the complex $\mathbb{Z}$ matrix relation.

In order to uniquely determine/specify <u>all four</u> of the Z_{ij} elements of the complex $\mathbb{Z}$ matrix, one <u>additional</u> relation is needed. The relation we will use is associated with energy/energy density {n.b. which are both manifest <u>scalar</u> quantities}. For electrical circuits with reactive components, the instantaneous energy stored in capacitors and inductors is $U_C(t) = \tfrac{1}{2} C |V(t)|^2 = \tfrac{1}{2} C V(t) V^*(t)$ and $U_L(t) = \tfrac{1}{2} L |I(t)|^2 = \tfrac{1}{2} C I(t) I^*(t)$ respectively, where * denotes complex conjugation, i.e. if $A = a + i\alpha$ then $A^* = a - i\alpha$ where $i \equiv \sqrt{-1}$ and $-i \equiv -\sqrt{-1}$.

From the above complex matrix relation $\mathbb{V} = \mathbb{Z}\mathbb{I}$, we can form quantities like $|V(t)|^2$ by taking the complex transpose $\mathbb{V}^{*\mathrm{T}} = (\mathbb{Z}\mathbb{I})^{*\mathrm{T}} = \mathbb{I}^{*\mathrm{T}} \mathbb{Z}^{*\mathrm{T}}$ and then multiplying $\mathbb{V} = \mathbb{Z}\mathbb{I}$ on the left by its complex transpose, i.e.:

$$\mathbb{V}^{*\mathrm{T}} \mathbb{V} = \left(\mathbb{I}^{*\mathrm{T}} \mathbb{Z}^{*\mathrm{T}}\right)(\mathbb{Z}\mathbb{I}) = \mathbb{I}^{*\mathrm{T}} \left(\mathbb{Z}^{*\mathrm{T}} \mathbb{Z}\right) \mathbb{I}$$

In matrix form, this is explicitly (again suppressing the time-dependence for notational clarity):

$$\begin{pmatrix} V_1^* & V_2^* \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} I_1^* & I_2^* \end{pmatrix}\begin{pmatrix} Z_{11}^* & Z_{21}^* \\ Z_{12}^* & Z_{22}^* \end{pmatrix}\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} I_1^* & I_2^* \end{pmatrix}\begin{pmatrix} Z_{11}^* & Z_{12}^* \\ Z_{12}^* & Z_{22}^* \end{pmatrix}\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

Carrying out the matrix operations in stages we obtain:

$$\begin{pmatrix} V_1^* & V_2^* \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} Z_{11}^* I_1^* + Z_{12}^* I_2^* & Z_{12}^* I_1^* + Z_{22}^* I_2^* \end{pmatrix}\begin{pmatrix} Z_{11} I_1 + Z_{12} I_2 \\ Z_{12} I_1 + Z_{22} I_2 \end{pmatrix}$$

Or:

$$\begin{aligned} |V_1|^2 + |V_2|^2 &= \left(|Z_{11}|^2 + |Z_{12}|^2\right)|I_1|^2 + \left(|Z_{12}|^2 + |Z_{22}|^2\right)|I_2|^2 \\ &\quad + \left(Z_{11}Z_{12}^* + Z_{12}Z_{22}^*\right)I_1 I_2^* + \left(Z_{11}^* Z_{12} + Z_{12}^* Z_{22}\right)I_1^* I_2 \end{aligned}$$

But $V_1 = Z_1 I_1$ and $V_2 = Z_2 I_2$, thus:

$$\begin{aligned} |Z_1|^2 |I_1|^2 + |Z_2|^2 |I_2|^2 &= \left(|Z_{11}|^2 + |Z_{12}|^2\right)|I_1|^2 + \left(|Z_{12}|^2 + |Z_{22}|^2\right)|I_2|^2 \\ &\quad + \left(Z_{11}Z_{12}^* + Z_{12}Z_{22}^*\right)I_1 I_2^* + \left(Z_{11}^* Z_{12} + Z_{12}^* Z_{22}\right)I_1^* I_2 \end{aligned}$$

Before mindlessly/brute-force solving this relation, it is very useful to note that physically, electromagnetic energies and/or energy densities are <u>*additive*</u> scalar quantities, i.e. that $U_{tot}(t) = U_1(\vec{r}_1,t) + U_2(\vec{r}_2,t) + \ldots$ More specifically, the additive nature of *EM* energy/energy density means that <u>*no*</u> interference/<u>*no*</u> cross-terms involving quantities such as $I_1(\vec{r}_1,t)\,I_2^*(\vec{r}_2,t)$ and/or $I_1^*(\vec{r}_1,t)\,I_2(\vec{r}_2,t)$ as in the above relation are allowed, for if they <u>*did*</u> exist, this would imply instantaneous/non-local/faster-than-light propagation/transfer of energy, e.g. from the point $\vec{r}_1$ to the point $\vec{r}_2$ (or vice-versa), which would be <u>*acausal*</u>, a phenomenon which is <u>*not*</u> observed in our universe. Information <u>*cannot*</u> propagate faster than the speed of light, *c*. Hence any/all such cross terms in energy/energy density formulae are <u>*strictly*</u> forbidden. In our current situation, this <u>*explicitly*</u> means that:

$$\boxed{\left(Z_{11}Z_{12}^* + Z_{12}Z_{22}^*\right)I_1(\vec{r}_1,t)\,I_2^*(\vec{r}_2,t) + \left(Z_{11}^* Z_{12} + Z_{12}^* Z_{22}\right)I_1^*(\vec{r}_1,t)\,I_2(\vec{r}_2,t) = 0}$$

and thus:

$$\boxed{\left\{|Z_1|^2 - \left(|Z_{11}|^2 + |Z_{12}|^2\right)\right\}\left|I_1(\vec{r}_1,t)\right|^2 + \left\{|Z_2|^2 - \left(|Z_{12}|^2 + |Z_{22}|^2\right)\right\}\left|I_2(\vec{r}_2,t)\right|^2 = 0}$$

Solving either of these equations, in addition to using the two coupled equations of the complex $\mathbb{Z}$ matrix relation and the use of the Reciprocity Theorem, in combination with explicit measurements of at least <u>*two*</u> of the four experimentally observable quantities $V_1(t)$, $V_2(t)$, $I_1(t)$ and $I_2(t)$ {n.b. at least one measurement <u>must</u> be on the input side and at least one measurement <u>must</u> be on the output side of the four-terminal network} and the use of the auxiliary relations $V_1 = Z_1 I_1$ and $V_2 = Z_2 I_2$ enables us to fully/uniquely determine <u>all four</u> of the Z_{ij} elements of the complex $\mathbb{Z}$ matrix.

Note also from the immediately above equation, it ***might*** be tempting to ***believe*** that $|Z_1|^2 = |Z_{11}|^2 + |Z_{12}|^2$ and hence also that $|Z_2|^2 = |Z_{12}|^2 + |Z_{22}|^2$ (invoking the Reciprocity Theorem), which would certainly satisfy this equation; however, explicit/brute-force calculations show this is ***not*** the case/is ***not*** true. The reason is simple, because we have already previously obtained the following relations from the two coupled equations of the complex $\mathbb{Z}$ matrix equation:

$$\boxed{\begin{aligned} Z_{11} &= Z_1 - Z_{12}\left(I_2/I_1\right) \\ Z_{22} &= Z_2 - Z_{12}\left(I_1/I_2\right) \end{aligned}} \text{ or: } \boxed{\begin{aligned} Z_1 &= Z_{11} + Z_{12}\left(I_2/I_1\right) \\ Z_2 &= Z_{22} + Z_{12}\left(I_1/I_2\right) \end{aligned}} \Rightarrow \boxed{\begin{aligned} |Z_1|^2 &= \left|Z_{11} + Z_{12}\left(I_2/I_1\right)\right|^2 \\ |Z_2|^2 &= \left|Z_{22} + Z_{12}\left(I_1/I_2\right)\right|^2 \end{aligned}}$$

Skipping much tedious complex algebra, we simply quote the result of using either of the above two "energy / energy density" relations and all the other relations, enabling us to express e.g. Z_{12} in terms of the four explicitly measured experimental quantities $V_1(t)$, $V_2(t)$, $I_1(t)$ and $I_2(t)$:

$$\boxed{Z_{12} = \frac{V_1 I_2^* + V_2 I_1^*}{|I_1|^2 + |I_2|^2} = Z_{21}}$$

Note the manifest symmetry in the above expression under the interchange of $1 \rightleftarrows 2$ indices, which arises as an explicit consequence of the Reciprocity Theorem, i.e. requiring $Z_{12} = Z_{21}$.

Once Z_{12} has been determined, the other two elements of the complex $\mathbb{Z}$ matrix can be determined e.g. by using the relations:

$$\boxed{\begin{aligned} Z_{11} &= Z_1 - Z_{12}\left(I_2/I_1\right) \\ Z_{22} &= Z_2 - Z_{12}\left(I_1/I_2\right) \end{aligned}}$$

where the complex input and output impedances are $Z_1 = V_1/I_1$ and $Z_2 = V_2/I_2$, respectively.