Standing Wave – Simple Theory

In this example, we show plots of acoustic quantities associated with the superposition of two counter-propagating 1-D monochromatic plane traveling waves – *i.e.* a simple standing wave – in "free air". Please refer to Physics 406 Lecture Notes 12 p. 6-18 for details.

The individual right and left-traveling complex ***time-domain*** over-pressure amplitudes are: $\tilde{p}_A(x,t) = \tilde{A}e^{i(\omega t - kx)}$ and $\tilde{p}_B(x,t) = \tilde{B}e^{i(\omega t + kx)}$ with $\tilde{A} \neq \tilde{B}$ {necessarily}, where $\tilde{A} = \left|\tilde{A}\right| e^{i\varphi_A^o} \equiv Ae^{i\varphi_A^o}$ and $\tilde{B} = \left|\tilde{B}\right| e^{i\varphi_B^o} \equiv Be^{i\varphi_B^o}$. Thus: $\tilde{p}_{tot}(x,t) = \tilde{p}_A(x,t) + \tilde{p}_B(x,t) = \tilde{A}e^{i(\omega t - kx)} + \tilde{B}e^{i(\omega t + kx)}$ $(Pascals)$.

The individual right and left-traveling complex ***time-domain*** 1-D particle velocity amplitudes are: $\tilde{u}_A^{\parallel}(x,t) = \dfrac{\tilde{A}}{\rho_o c} e^{i(\omega t - kx)} \equiv \tilde{u}_{A_o}^{\parallel} e^{i(\omega t - kx)}$ and: $\tilde{u}_B^{\parallel}(x,t) = -\dfrac{\tilde{B}}{\rho_o c} e^{i(\omega t + kx)} \equiv -\tilde{u}_{B_o}^{\parallel} e^{i(\omega t + kx)}$ (using $c = \omega/k$).

Thus: $\tilde{u}_{tot}^{\parallel}(x,t) = \tilde{u}_A^{\parallel}(x,t) + \tilde{u}_B^{\parallel}(x,t) = \tilde{u}_{A_o}^{\parallel} e^{i(\omega t - kx)} + \tilde{u}_{B_o}^{\parallel} e^{i(\omega t + kx)} = \dfrac{\tilde{A}}{\rho_o c} e^{i(\omega t - kx)} - \dfrac{\tilde{B}}{\rho_o c} e^{i(\omega t + kx)} (m/s)$.

Defining: $\tilde{R} \equiv \dfrac{\tilde{B}}{\tilde{A}} = \dfrac{\left|\tilde{B}\right| e^{i\varphi_B^o}}{\left|\tilde{A}\right| e^{i\varphi_A^o}} = \dfrac{\left|\tilde{B}\right|}{\left|\tilde{A}\right|} e^{i\left(\varphi_B^o - \varphi_A^o\right)} = \left|\tilde{R}\right| e^{i\left(\varphi_B^o - \varphi_A^o\right)} = \left|\tilde{R}\right| e^{i\Delta\varphi_{BA}^o}$ where: $\Delta\varphi_{BA}^o \equiv \varphi_B^o - \varphi_A^o$, then:

$$\tilde{p}_{tot}(x,t) = \tilde{A}\left[e^{i(\omega t - kx)} + \left|\tilde{R}\right| e^{i(\omega t + kx)} \cdot e^{i\Delta\varphi_{BA}^o}\right] = \tilde{A}e^{i(\omega t - kx)}\left[1 + \left|\tilde{R}\right| e^{i\left(2kx + \Delta\varphi_{BA}^o\right)}\right]$$

and:

$$\tilde{u}_{tot}^{\parallel}(x,t) = \frac{\tilde{A}}{\rho_o c}\left[e^{i(\omega t - kx)} - \left|\tilde{R}\right| e^{i(\omega t + kx)} \cdot e^{i\Delta\varphi_{BA}^o}\right] = \frac{\tilde{A}}{\rho_o c} e^{i(\omega t - kx)}\left[1 - \left|\tilde{R}\right| e^{i\left(2kx + \Delta\varphi_{BA}^o\right)}\right]$$

The ***magnitudes*** of the complex total/resultant over-pressure $\left|\tilde{p}_{tot}(x)\right|$ and longitudinal particle velocity $\left|\tilde{u}_{tot}^{\parallel}(x)\right|$ are:

$$\left|\tilde{p}_{tot}(x)\right| = \left|\tilde{A}\right| \sqrt{1 + 2\left|\tilde{R}\right| \cos\left(2kx + \Delta\varphi_{BA}^o\right) + \left|\tilde{R}\right|^2} \text{ and: } \left|\tilde{u}_{tot}^{\parallel}(x)\right| = \frac{\left|\tilde{A}\right|}{\rho_o c} \sqrt{1 - 2\left|\tilde{R}\right| \cos\left(2kx + \Delta\varphi_{BA}^o\right) + \left|\tilde{R}\right|^2}$$

The ***phases*** associated with the complex total/resultant over-pressure $\varphi_{p_{tot}}(x)$ and longitudinal particle velocity $\varphi_{u_{tot}}$ are:

$$\varphi_{p_{tot}}(x) = \tan^{-1}\left[\frac{\sin kx\left(1 + \left|\tilde{R}\right| \cos\left(2kx + \Delta\varphi_{BA}^o\right)\right) + \left|\tilde{R}\right| \cos kx \cdot \sin\left(2kx + \Delta\varphi_{BA}^o\right)}{\cos kx\left(1 + \left|\tilde{R}\right| \cos\left(2kx + \Delta\varphi_{BA}^o\right)\right) - \left|\tilde{R}\right| \sin kx \cdot \sin\left(2kx + \Delta\varphi_{BA}^o\right)}\right]$$

$$\varphi_{u_{tot}}(x) = \tan^{-1}\left[\frac{\sin kx\left(1 - \left|\tilde{R}\right| \cos\left(2kx + \Delta\varphi_{BA}^o\right)\right) - \left|\tilde{R}\right| \cos kx \cdot \sin\left(2kx + \Delta\varphi_{BA}^o\right)}{\cos kx\left(1 - \left|\tilde{R}\right| \cos\left(2kx + \Delta\varphi_{BA}^o\right)\right) + \left|\tilde{R}\right| \sin kx \cdot \sin\left(2kx + \Delta\varphi_{BA}^o\right)}\right]$$

The complex ***frequency-domain*** longitudinal specific acoustic impedance, it's magnitude and phase are, noting {***here***} that $\omega = ck$:

$$\tilde{z}^{\parallel}_{a\,tot}(x,\omega) = z_o \frac{\left[\left\{1-\left|\tilde{R}\right|^2\right\} + 2i\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi^o_{BA}\right)\right]}{\left[\left\{1+\left|\tilde{R}\right|^2\right\} - 2\left|\tilde{R}\right|\cos\left(2kx+\Delta\varphi^o_{BA}\right)\right]}$$

$$\left|\tilde{z}^{\parallel}_{a\,tot}(x,\omega)\right| = z_o \frac{\sqrt{\left\{1-\left|\tilde{R}\right|^2\right\}^2 + 4\left|\tilde{R}\right|^2\sin^2\left(2kx+\Delta\varphi^o_{BA}\right)}}{\left[\left\{1+\left|\tilde{R}\right|^2\right\} - 2\left|\tilde{R}\right|\cos\left(2kx+\Delta\varphi^o_{BA}\right)\right]}$$

$$\varphi_{z_{a\,tot}}(x,\omega) = \tan^{-1}\left(\frac{2\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi^o_{BA}\right)}{\left\{1-\left|\tilde{R}\right|^2\right\}}\right) = \Delta\varphi_{p_{tot}-u^{\parallel}_{tot}}(x,\omega) = \varphi_{p_{tot}}(x,\omega) - \varphi_{u^{\parallel}_{tot}}(x,\omega)$$

The complex ***frequency-domain*** longitudinal acoustic intensity, it's magnitude and phase are:

$$\tilde{I}^{\parallel}_{a_{tot}}(x,\omega) = \frac{1}{2}\frac{\left|\tilde{A}\right|^2}{z_o}\left[\left\{1-\left|\tilde{R}\right|^2\right\} + 2i\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi^o_{BA}\right)\right]$$

$$\left|\tilde{I}^{\parallel}_{a_{tot}}(x,\omega)\right| = \frac{1}{2}\frac{\left|\tilde{A}\right|^2}{z_o}\sqrt{\left\{1-\left|\tilde{R}\right|^2\right\}^2 + 4\left|\tilde{R}\right|^2\sin^2\left(2kx+\Delta\varphi^o_{BA}\right)}$$

$$\varphi_{I_{a\,tot}}(x,\omega) = \tan^{-1}\left(\frac{2\left|\tilde{R}\right|\sin\left(2kx+\Delta\varphi^o_{BA}\right)}{\left\{1-\left|\tilde{R}\right|^2\right\}}\right) = \varphi_{z_{a\,tot}}(x,\omega) = \Delta\varphi_{p_{tot}-u^{\parallel}_{tot}}(x,\omega) = \varphi_{p_{tot}}(x,\omega) - \varphi_{u^{\parallel}_{tot}}(x,\omega)$$

The complex ***frequency-domain*** potential, kinetic and total energy densities associated with counter-propagating 1-D monochromatic traveling plane waves (*n.b.* all ***purely real*** quantities):

$$w_{potl}(x,\omega) \equiv \frac{1}{4}\frac{1}{\rho_o c^2}\left|\tilde{p}_{tot}(x,\omega)\right|^2 = \frac{1}{4}\frac{\left|\tilde{A}\right|^2}{\rho_o c^2}\left[1+\left|\tilde{R}\right|^2 + 2\left|\tilde{R}\right|\cos\left(2kx+\Delta\varphi^o_{BA}\right)\right]$$

$$w_{kin}(x,\omega) \equiv \frac{1}{4}\rho_o\left|\tilde{u}^{\parallel}_{tot}(x,\omega)\right|^2 = \frac{1}{4}\frac{\left|\tilde{A}\right|^2}{\rho_o c^2}\left[1+\left|\tilde{R}\right|^2 - 2\left|\tilde{R}\right|\cos\left(2kx+\Delta\varphi^o_{BA}\right)\right]$$

$$w_{tot}(x,\omega) \equiv w_{potl}(x,\omega) + w_{kin}(x,\omega) = \frac{1}{2}\frac{\left|\tilde{A}\right|^2}{\rho_o c^2}\left[1+\left|\tilde{R}\right|^2\right] = \frac{1}{2}\frac{\left|\tilde{A}\right|^2}{z_o c}\left[1+\left|\tilde{R}\right|^2\right]$$

We coded up the above formulas using MATLAB and show plots in the figures below of the magnitudes and phases of complex total over-pressure, particle velocity, longitudinal specific acoustic impedance and longitudinal acoustic intensity $\left|\tilde{p}_{tot}(\theta)\right|$, $\left|\tilde{u}^{\parallel}_{tot}(\theta)\right|$, $\left|\tilde{z}^{\parallel}_{a\,tot}(\theta)\right|$ and $\left|\tilde{I}^{\parallel}_{a_{tot}}(\theta)\right|$ *vs.* $\theta \equiv kx$ for $0 \le \left|\tilde{R}\right| \le 1$ in steps of 0.1, with $\Delta\varphi^o_{BA} = 0.0$. Please note the suppressed zeroes in some of the following plots!

Figure 1. $\left|\tilde{R}\right| = 0.0$

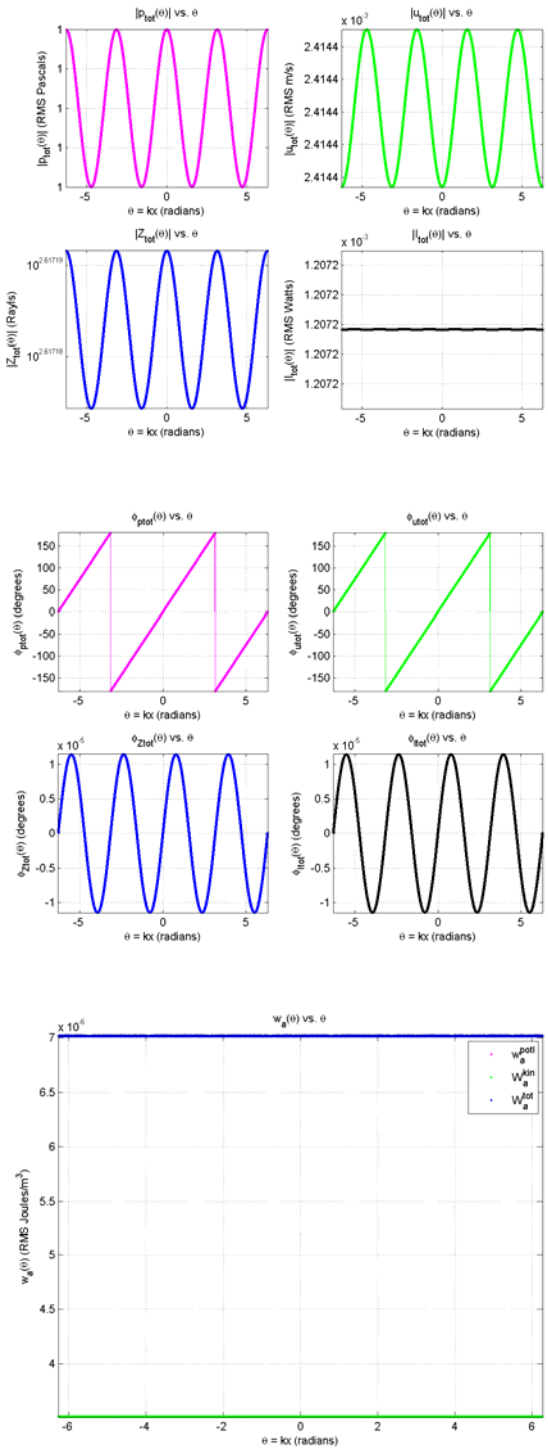


Figure 2. $\left|\tilde{R}\right| = 0.1$

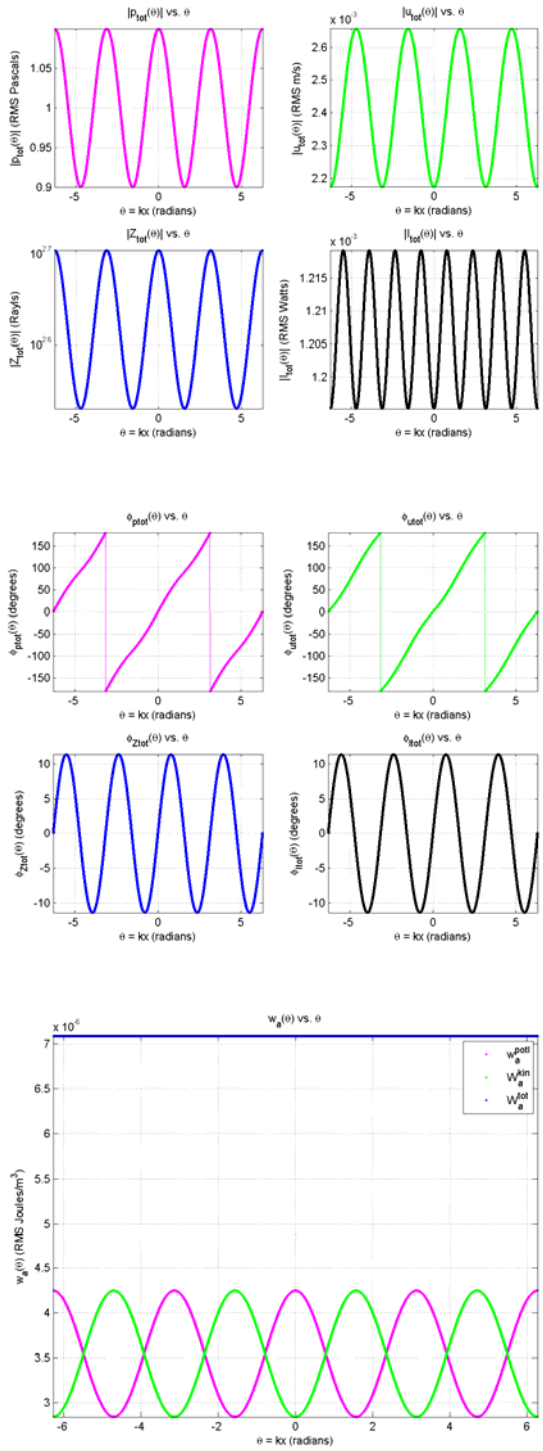


Figure 3. $\left|\tilde{R}\right| = 0.2$

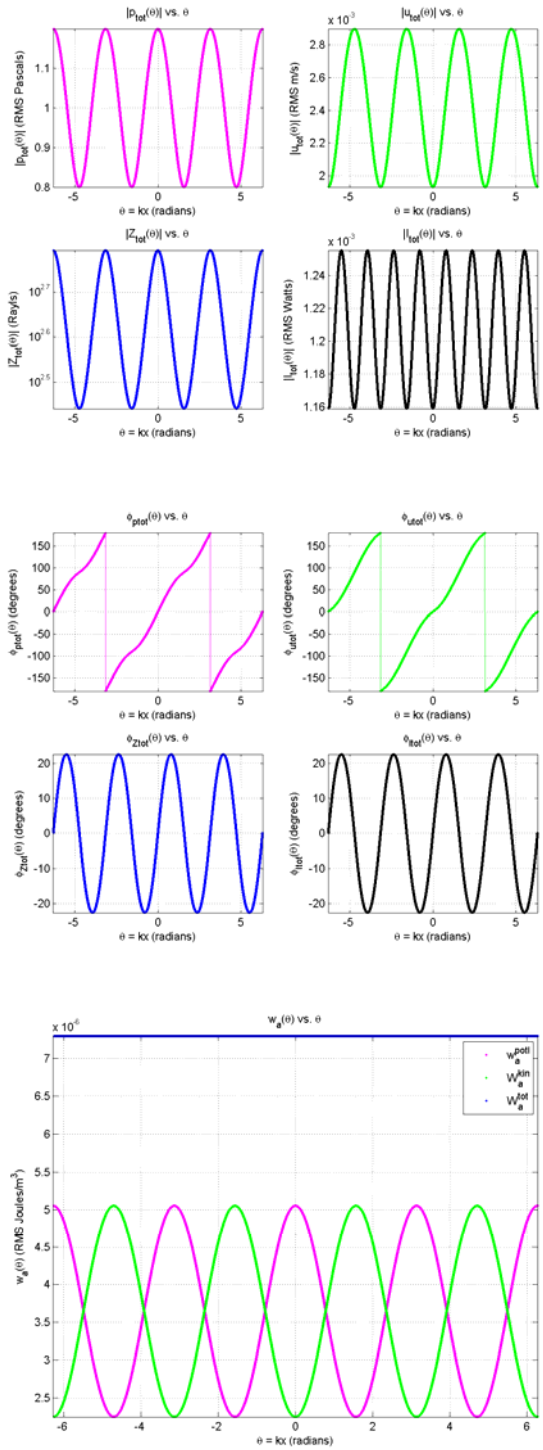


Figure 4. $\left|\tilde{R}\right| = 0.3$

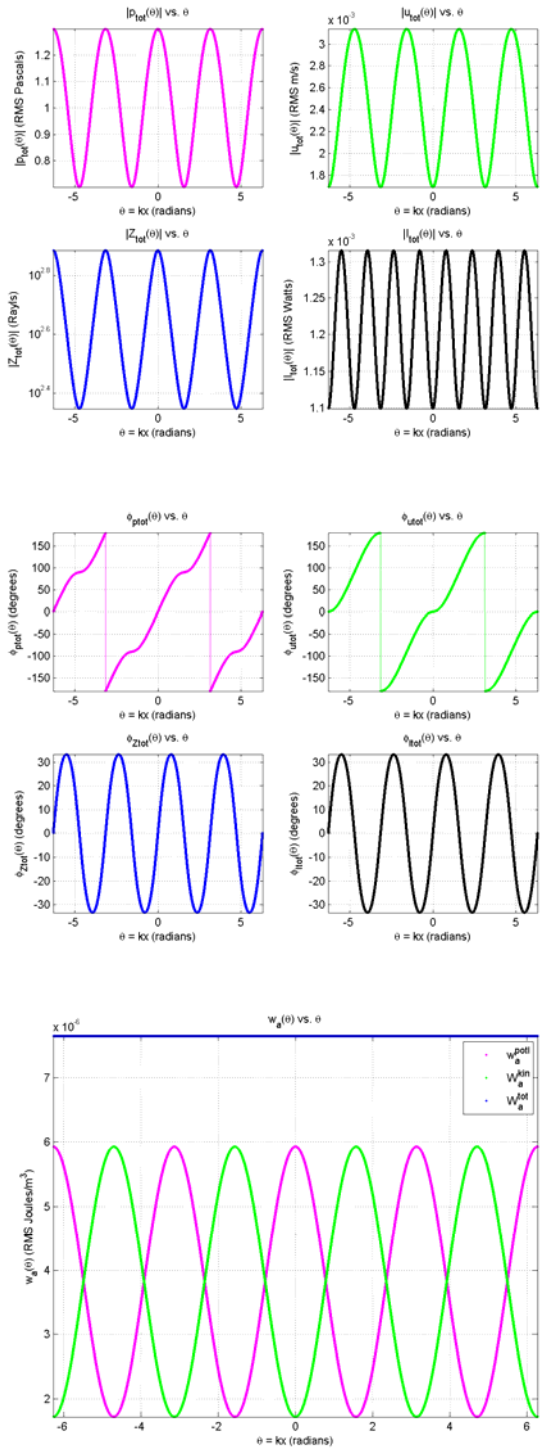


Figure 5. $\left|\tilde{R}\right| = 0.4$

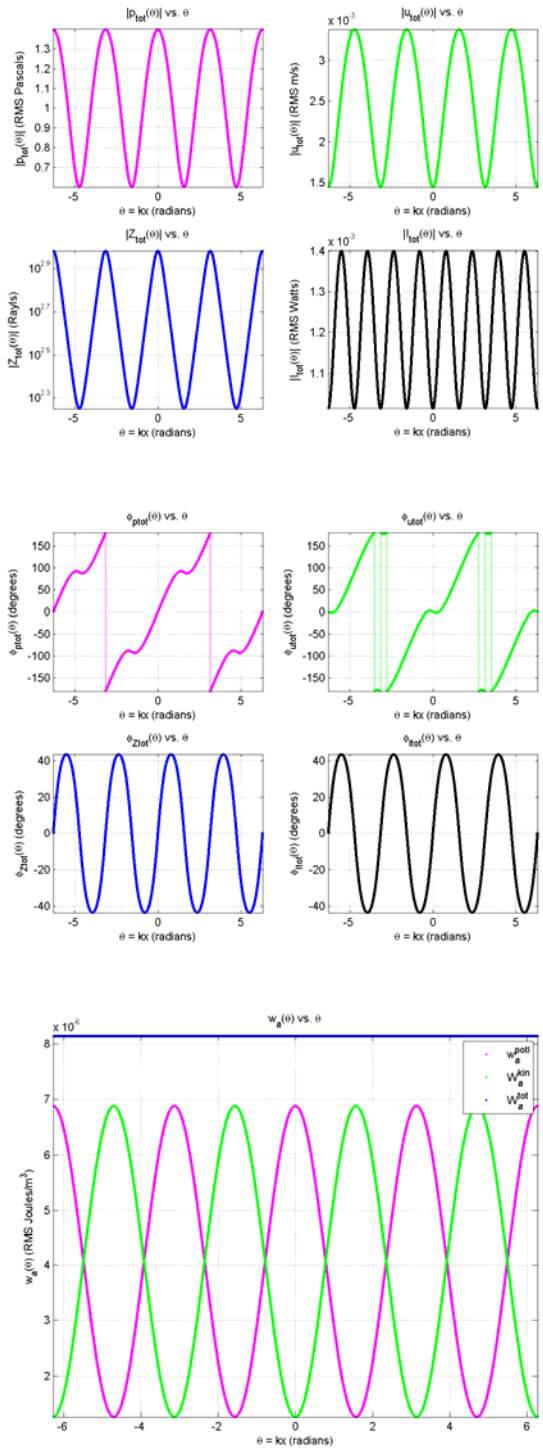


Figure 6. $\left|\tilde{R}\right| = 0.5$

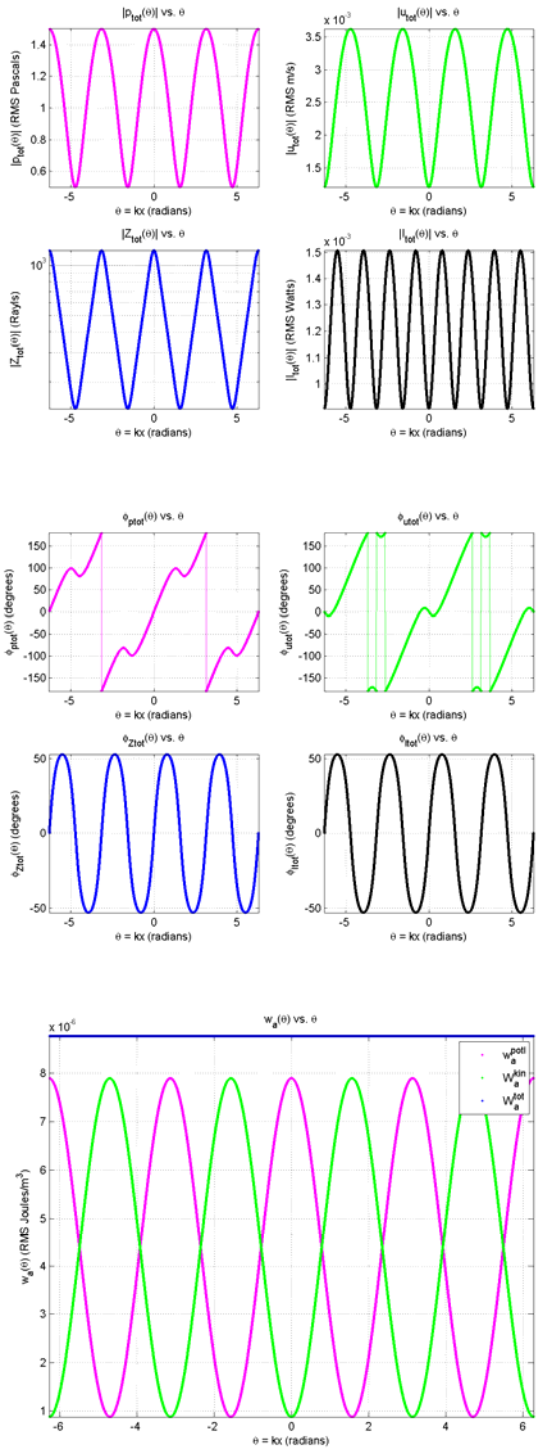


Figure 7. $\left|\tilde{R}\right| = 0.6$

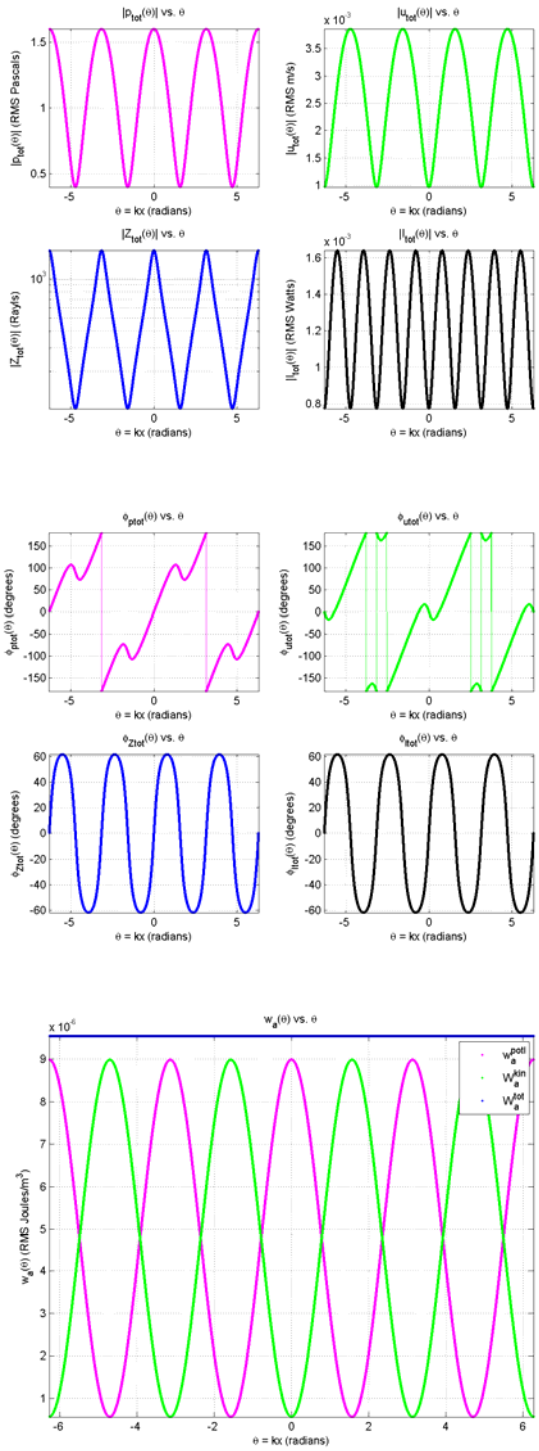


Figure 8. $\left|\tilde{R}\right| = 0.7$

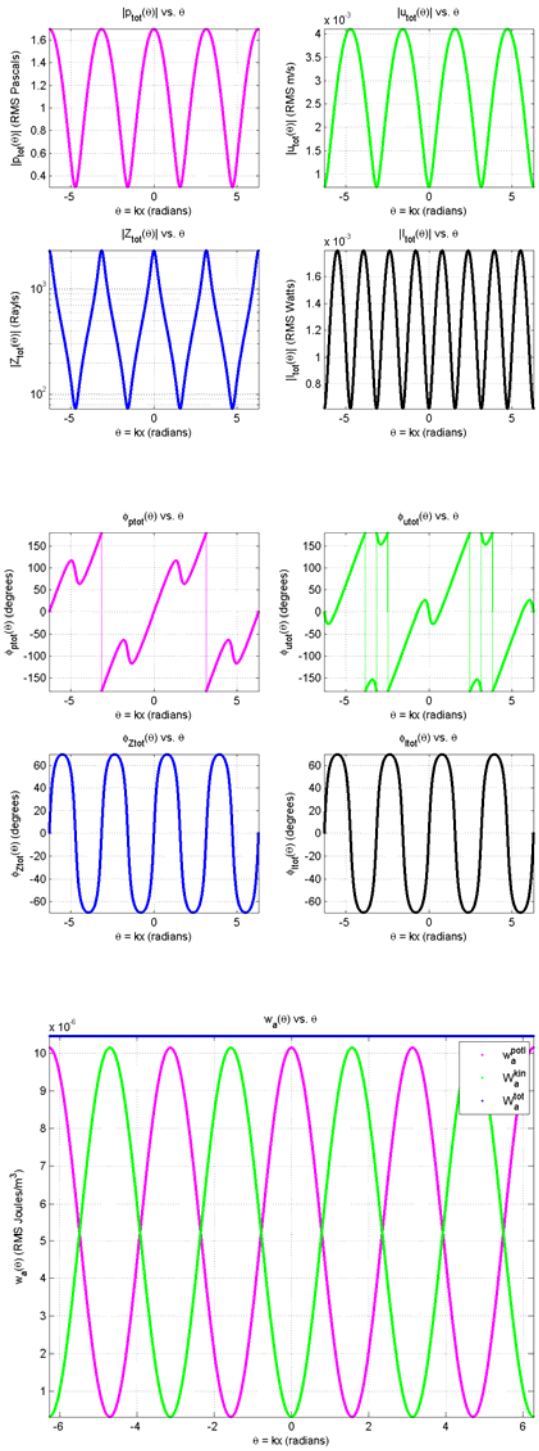


Figure 9. $\left|\tilde{R}\right| = 0.8$

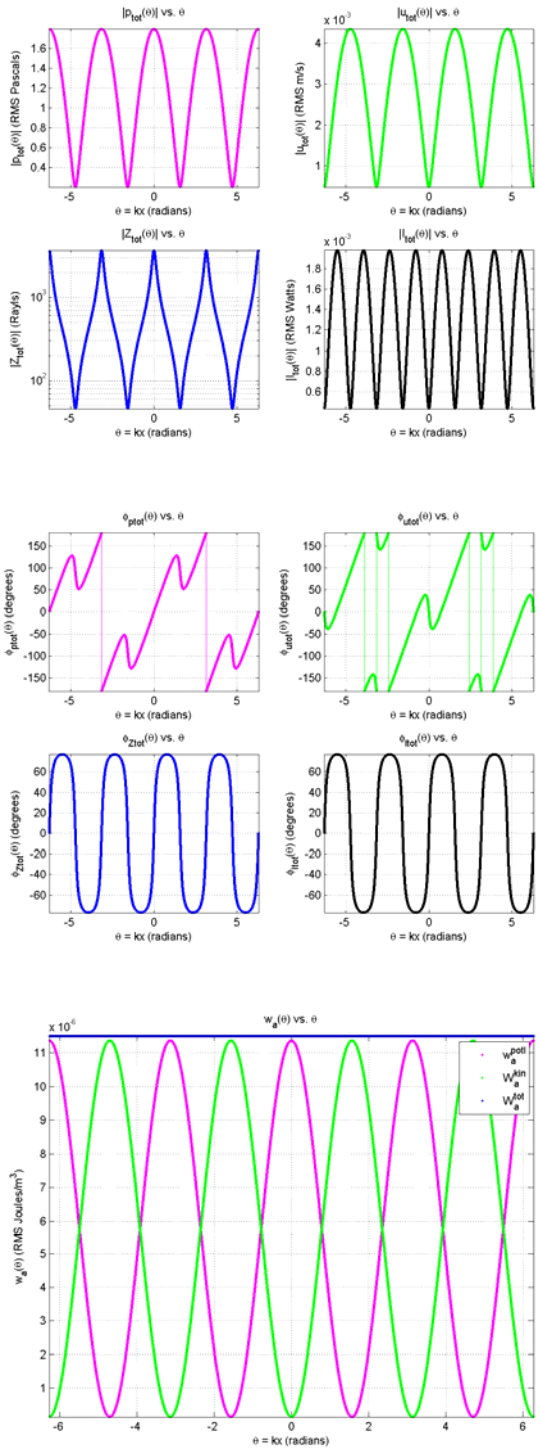


Figure 10. $\left|\tilde{R}\right| = 0.9$

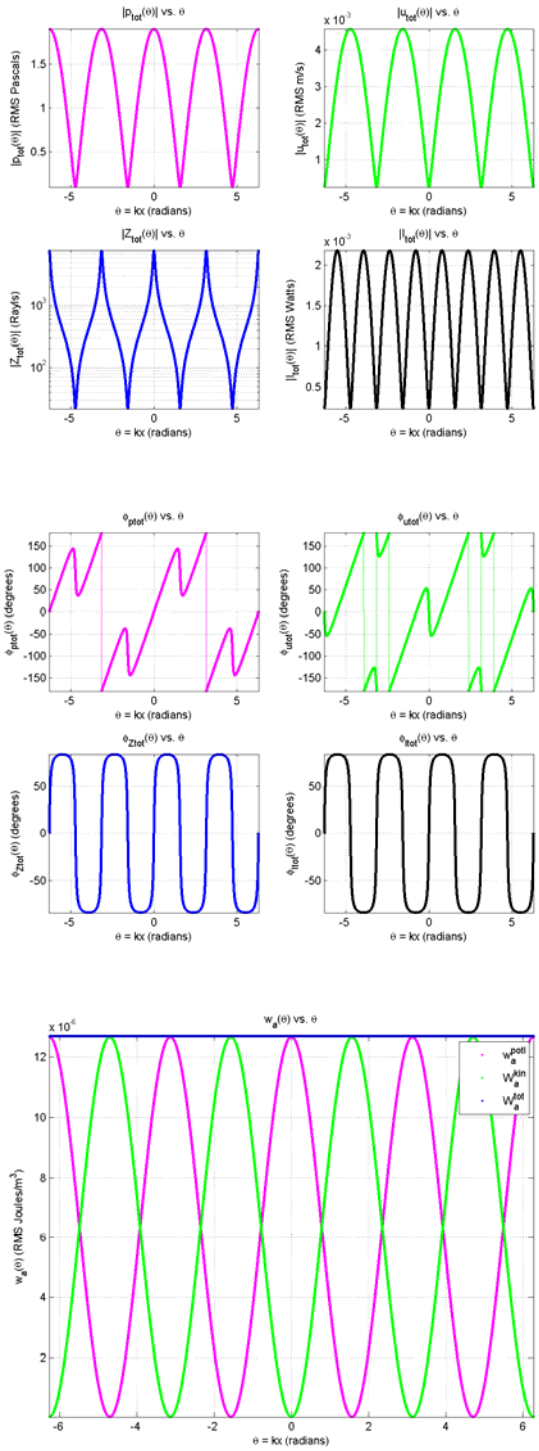
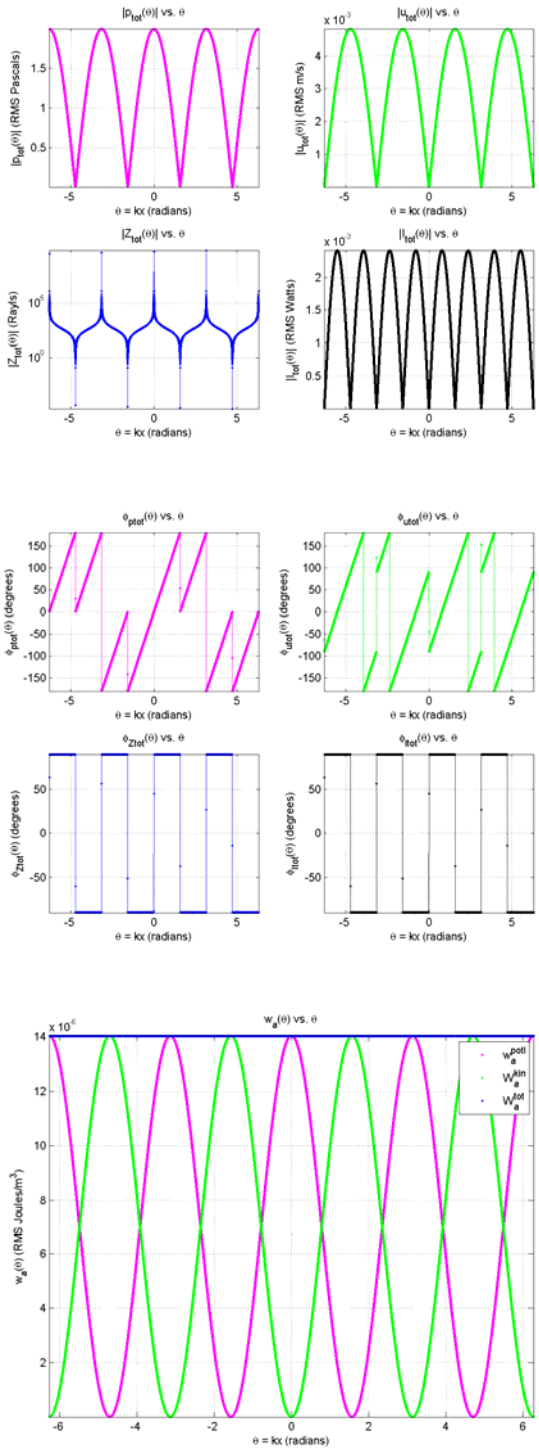


Figure 11. $\left|\tilde{R}\right| = 1.0$



Listing of the MATLAB code:

```
%==========================================================================
% Simple_Standing_Wave.m
%
% Study of the acoustical physics associated with the linear superposition
% of two counter-propagating 1-D monochromatic plane/traveling waves,
% = a standing wave.
%
%==========================================================================
% Author: Prof. Steven Errede  4/5/2014 07:50 hr
%
% Last Update:                 4/7/2014 10:20 hr {SME}
%==========================================================================
clear all;
close all;

npts = 40000;

thetar = zeros(1,npts);

MgPtot = zeros(1,npts);
MgUtot = zeros(1,npts);
MgZtot = zeros(1,npts);
MgItot = zeros(1,npts);

PhPtot = zeros(1,npts);
PhUtot = zeros(1,npts);
PhZtot = zeros(1,npts);
PhItot = zeros(1,npts);

Wapot  = zeros(1,npts);
Wakin  = zeros(1,npts);
Watot  = zeros(1,npts);

rho0 =  1.204; % Density of air @ NTP (kg/m^3)
   c =  344.0; % Longitudinal speed of sound @ NTP (m/s)
  z0 = rho0*c; % Longitudinal specific acoustic impedance of free air (Rayls)

  p0 =  1.000; % Over-pressure amplitude (RMS Pascals)

% |~R| = |~B|/|~A|
%MgR =  1.000 - 1.0e-7;
 MgR =  0.100;
%MgR =  0.000 + 1.0e-7;

  delphi0_ba  = 0.0; % = phi0b - phi0a   (degrees)
  delphi0_bar = (pi/180.0)*delphi0_ba; % (radians)

  thetar_lo = -2.0*pi + 1.0e-7; % = kx_lo
  thetar_hi =  2.0*pi;          % = kx_hi

  dthetar = (thetar_hi - thetar_lo)/npts;

  for j = 1:npts;
      thetar(j) = thetar_lo + (j-1)*dthetar; % = kx

      MgPtot(j) =  p0          *sqrt(1.0 + (2.0*MgR*cos(2.0*thetar(j) + delphi0_bar)) + (MgR*MgR));
      MgUtot(j) = (p0/(rho0*c))*sqrt(1.0 - (2.0*MgR*cos(2.0*thetar(j) + delphi0_bar)) + (MgR*MgR));
      MgZtot(j) = MgPtot(j)/MgUtot(j);
      MgItot(j) = MgPtot(j)*MgUtot(j)/2.0;

      Pnum      = sin(thetar(j))*(1.0 + (MgR*cos(2.0*thetar(j) + delphi0_bar))) +
                  cos(thetar(j))*       (MgR*sin(2.0*thetar(j) + delphi0_bar));
      Pden      = cos(thetar(j))*(1.0 + (MgR*cos(2.0*thetar(j) + delphi0_bar))) -
                  sin(thetar(j))*       (MgR*sin(2.0*thetar(j) + delphi0_bar));
      PhPtot(j) = (180.0/pi)*atan2(Pnum,Pden);

      Unum      = sin(thetar(j))*(1.0 - (MgR*cos(2.0*thetar(j) + delphi0_bar))) -
                  cos(thetar(j))*       (MgR*sin(2.0*thetar(j) + delphi0_bar));
```

```
      Uden      = cos(thetar(j))*(1.0 - (MgR*cos(2.0*thetar(j) + delphi0_bar))) +
                  sin(thetar(j))*       (MgR*sin(2.0*thetar(j) + delphi0_bar));
      PhUtot(j) = (180.0/pi)*atan2(Unum,Uden);

      Znum      = (2.0*MgR*sin(2.0*thetar(j) + delphi0_bar));
      Zden      = (1.0 - (MgR*MgR));
      PhZtot(j) = (180.0/pi)*atan2(Znum,Zden);

      Inum      = (2.0*MgR*sin(2.0*thetar(j) + delphi0_bar));
      Iden      = (1.0 - (MgR*MgR));
      PhItot(j) = (180.0/pi)*atan2(Inum,Iden);

       Wapot(j) = 0.5     *(MgPtot(j)*MgPtot(j))/(z0*c);
       Wakin(j) = 0.5*rho0*(MgUtot(j)*MgUtot(j));
       Watot(j) = Wapot(j) + Wakin(j);
  end

 figure (01);
  subplot(2,2,1);
  plot (thetar,MgPtot,'m',thetar,MgPtot,'m.');
  axis tight;
  grid on;
  xlabel ('{\theta} = kx (radians)');
  ylabel ('|p_{tot}({\theta})| (RMS Pascals)');
   title ('|p_{tot}({\theta})| vs. {\theta}');

  subplot(2,2,2);
  plot (thetar,MgUtot,'g.',thetar,MgUtot,'g.');
  axis tight;
  grid on;
  xlabel ('{\theta} = kx (radians)');
  ylabel ('|u_{tot}({\theta})| (RMS m/s)');
   title ('|u_{tot}({\theta})| vs. {\theta}');

  subplot(2,2,3);
  %plot   (thetar,MgZtot,'b',thetar,MgZtot,'b.');
  semilogy(thetar,MgZtot,'b',thetar,MgZtot,'b.');
  axis tight;
  grid on;
  xlabel ('{\theta} = kx (radians)');
  ylabel ('|Z_{tot}({\theta})| (Rayls)');
   title ('|Z_{tot}({\theta})| vs. {\theta}');

  subplot(2,2,4);
  plot (thetar,MgItot,'k',thetar,MgItot,'k.');
  axis tight;
  grid on;
  xlabel ('{\theta} = kx (radians)');
  ylabel ('|I_{tot}({\theta})| (RMS Watts)');
   title ('|I_{tot}({\theta})| vs. {\theta}');

figure (02);
  subplot(2,2,1);
  plot (thetar,PhPtot,'m',thetar,PhPtot,'m.');
  axis tight;
  grid on;
  xlabel ('{\theta} = kx (radians)');
  ylabel ('{\phi}_{ptot}({\theta}) (degrees)');
   title ('{\phi}_{ptot}({\theta}) vs. {\theta}');

  subplot(2,2,2);
  plot (thetar,PhUtot,'g',thetar,PhUtot,'g.');
  axis tight;
  grid on;
  xlabel ('{\theta} = kx (radians)');
  ylabel ('{\phi}_{utot}({\theta}) (degrees)');
   title ('{\phi}_{utot}({\theta}) vs. {\theta}');
  subplot(2,2,3);
  plot    (thetar,PhZtot,'b',thetar,PhZtot,'b.');
 %semilogy(thetar,PhZtot,'b',thetar,PhZtot,'b.');
  axis tight;
```

```
 grid on;
 xlabel ('{\theta} = kx (radians)');
 ylabel ('{\phi}_{Ztot}({\theta}) (degrees)');
  title ('{\phi}_{Ztot}({\theta}) vs. {\theta}');

 subplot(2,2,4);
 plot (thetar,PhItot,'k',thetar,PhItot,'k.');
 axis tight;
 grid on;
 xlabel ('{\theta} = kx (radians)');
 ylabel ('{\phi}_{Itot}({\theta}) (degrees)');
  title ('{\phi}_{Itot}({\theta}) vs. {\theta}');


figure (03);
 plot (thetar,Wapot,'m.',thetar,Wakin,'g.',thetar,Watot,'b.');
 axis tight;
 grid on;
 xlabel ('{\theta} = kx (radians)');
 ylabel ('w_{a}({\theta}) (RMS Joules/m^{3})');
  title ('w_{a}({\theta}) vs. {\theta}');
 legend ('w_{a}^{potl}','W_{a}^{kin}','W_{a}^{tot}');
```