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Physics 398 project, Fall 2000

Abstract:

The sustain of a guitar, that is, the amount of time a note will resonate before fading away due to natural dampening effects, has been the subject of much conjecture and endless speculation for as long as acoustic and electric guitars have been played. Due to the importance of sustain in determining the overall sound quality and playability of a guitar, however, such speculation and debate surrounding the issue of guitar sustain is justifiable. Unfortunately, much of this speculation derives from the inexact qualitative methods characteristically used to describe sustain, usually based on the human ear and a perceived sense of how strong a note sounds for a given span of time. Indeed, many claims have been made concerning the effect that adding weight to the headstock of a guitar, playing with new strings, playing in certain kinds of weather, etc., has on increasing the sustain of a guitar. Although such reports give a basis for some interesting speculation, many of them suffer for lack of quantifiable evidence.

This experiment seeks to establish at least the beginnings of using objective and quantifiable measurements taken in a laboratory setting to begin the scientific investigation of confirming or refuting many of these claims. Although the setup of the equipment used to make these measurements, and, more importantly, the writing of the code used in the computer program developed to take the essential data, was the most time consuming step in the process of doing the experiment, the underlying principle used to measure the sustain of a guitar was quite simple. Put concisely, when attached to the leads of the guitar pickup, the equipment and program allowed us to take direct measurements of the oscillating voltages produced by an electric guitar after exciting its strings. As with all electric guitars, the amplitudes of these oscillating voltages are directly proportional to the magnitude of the vibrations of the strings of the guitar, and the subsequent measuring of this voltage amplitude as it decreased with time then yielded the raw information needed to do calculations related to the sustain of the guitar, both overall (picking multiple strings at once) and on a single string-by-string basis. However, due to the desirability of keeping this experiment as simple and controlled as possible, only the sustain of single strings was measured in this experiment. In particular, voltage data were collected for each of the vibrating open guitar strings, that is, for notes low E, A, D, G, B, and high E. Also, the effect on dampening caused by raising or lowering the height of the strings above the guitar pickup, i.e. adjusting the action, was also measured for an open D string.

A Simple Illustration of what Fourier (Harmonic) Analysis is:

In order to construct an objective means for measuring the sustain of a guitar string, it is not enough to simply measure one overall amplitude of the voltages produced by the string vibrations and see how this single voltage amplitude decreases with time. If this were the case (see Figure **(1)**), we would effectively be saying that a vibrating guitar string has only one local maxima and that it looks the same as, for example, a single jump rope being swung back and forth between its two fixed ends. Instead, an actual vibrating guitar string looks more like the adding together (superposition) of multiple jump rope segments (see Fig.**(2)**, each with varying amplitudes and lengths that are contained within a largest jump rope. This superposition principle, that is, the adding together of multiple jump rope segments to form a more complicated pattern, can be seen in Fig.**(3)** and **(4)**. In viewing Fig.**(3)** and **(4)** it is important to note that the length of the smaller internal "jump ropes" must be fractional multiples (1/2, 1/3, 1/4, . . . ,1/n) of the length of the largest "jump rope". Correspondingly, the wavelengths and therefore frequencies of each of the smaller jump rope segments must be multiples of the fundamental frequency.

Fictitious Single Jump Rope Model of Vibrating String:

More Realistic Model of Vibrating String:

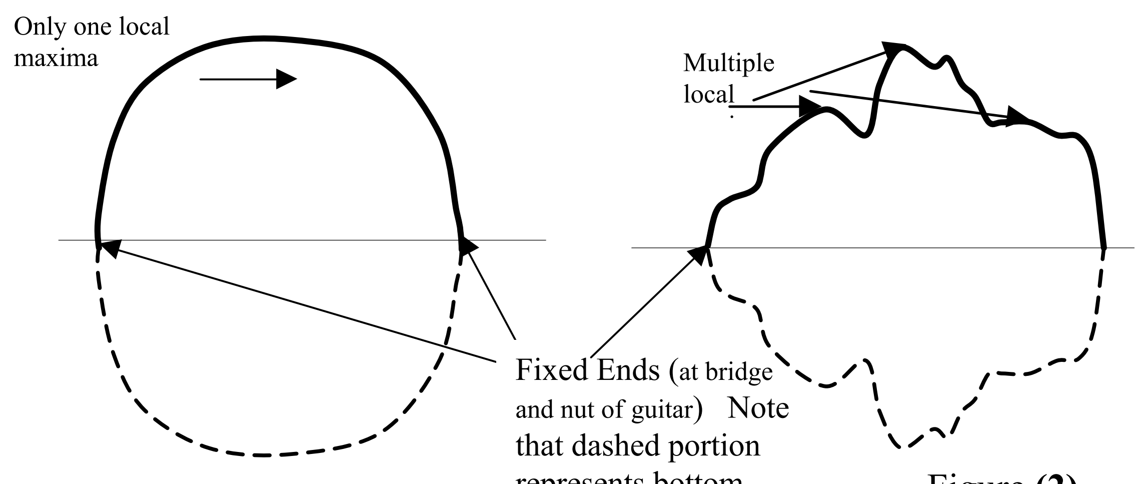
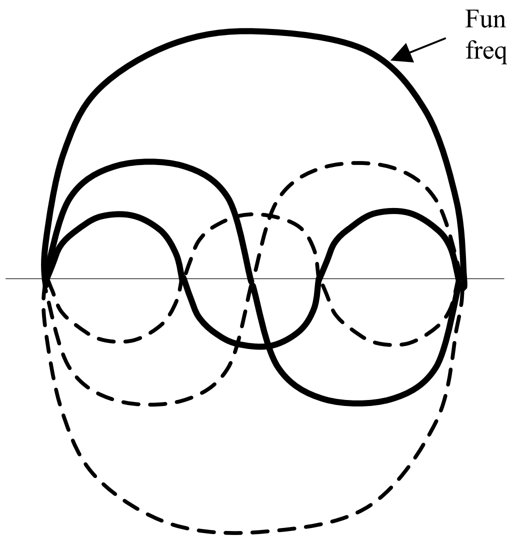
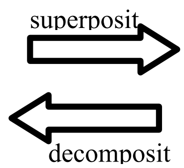


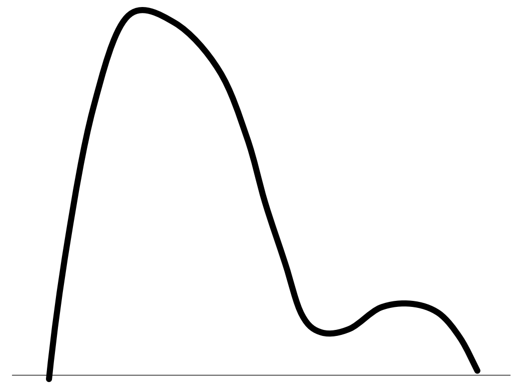
Figure **(1)**

Figure **(2)**

Example of Superposition/Decomposition:
(shows how a wave such as in Fig.**(2)** might be built up and/or taken apart)







bottom (dashed) portion excluded for simplicity

Figure **(4)**

Figure **(3)**

In principle, just as the "jump rope" pattern in Fig.**(4)** above was put together by adding the multiple jump rope segments of Fig.**(3)** together, the same jump rope pattern (henceforth called the "waveform") can conversely undergo a process whereby it is decomposed in order to retrieve its individual jump rope segments as seen in Fig.**(3)**. The point of this jump rope analogy is that each of these individual jump rope segments can be considered as an individual harmonic (a.k.a. vibrational mode) of the overall waveform. Moreover, the point of Fourier analysis is to effectively decompose a complicated waveform in such a way as to extract its constituent "jump rope" segments. In a mathematical expression, these jump rope elements can be represented as a mixture of sine and cosine terms of varying frequencies, which, when added together (superimposed) will yield a function f(x) that, using an infinite number of harmonics, exactly replicates the appearance of the original waveform. The formula used for this superposition of sine and cosine terms was developed by Joseph Fourier in 1822 and, astonishingly enough, it can be used to replicate any well-behaved periodic function, of which a repeating waveform is only one example. The most basic form of this formula is as follows:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos nx + b_n \cdot \sin nx \right)$$ where n is any positive non-zero integer

Equation **(1)**

As a matter of keeping the right terminology, it is important to remember in the above discussion that the "jump rope" corresponding to n = 1 in Equ.**(1)** above represents the 1st or fundamental harmonic, also called the fundamental harmonic, the "jump rope" corresponding to n = 2 represents the 2nd harmonic, the "jump rope" corresponding to n = 3 represents the third harmonic, and so on.

Furthermore, we can deduce from Equ.**(1)** that each cosine and sine term in the infinite series must have a different periodicity, e.g. sin(2x) for n = 2 has half the period (twice the frequency) of sin(x) for n = 1. Also, it is evident that each cosine and sine term is multiplied by its own coefficient, a_n and b_n .

Since we are modeling a vibrating guitar string of a given length, Equ.**(1)** can be put into a more usable form. If the function is periodic over spans of length 2*L*, (rather than having a period of 2π radians as in Equ.**(1)** above), and letting x measure length along the guitar string, Equ.**(1)** can be scaled over the entire length *L* of the guitar string as follows:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cdot \cos\left(\frac{pnx}{L}\right) + b_n \cdot \sin\left(\frac{pnx}{L}\right)\right] \qquad \text{Equation } \mathbf{(2)}$$

In Equ.**(1)** and **(2)**, although it is easily seen that the periodicity of the sine and cosine terms each decreases by predictable fractional amounts as n increases, what isn't obvious in the above summation is how to find the magnitudes of the coefficients a_n and b_n that multiply each sine and cosine term. The phrase "Fourier analysis" refers to the method used to solve for the magnitudes of the coefficients a_0, a_1, a_2, a_3 . . . a_n ; b_1, b_2, b_3, . . . b_n. Especially as concerns this experiment, these coefficients are important because they are the quantities that decrease with time and which can consequently be used to mathematically model the overall dampening of the guitar string vibrations for each harmonic. Rearranging Equ.**(2)** above and using the mathematical properties of inner products, the magnitudes of a_n and b_n can be solved for according to the following formulas:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\cdot dx \qquad \text{Equation } \mathbf{(3a,b,c)}$$

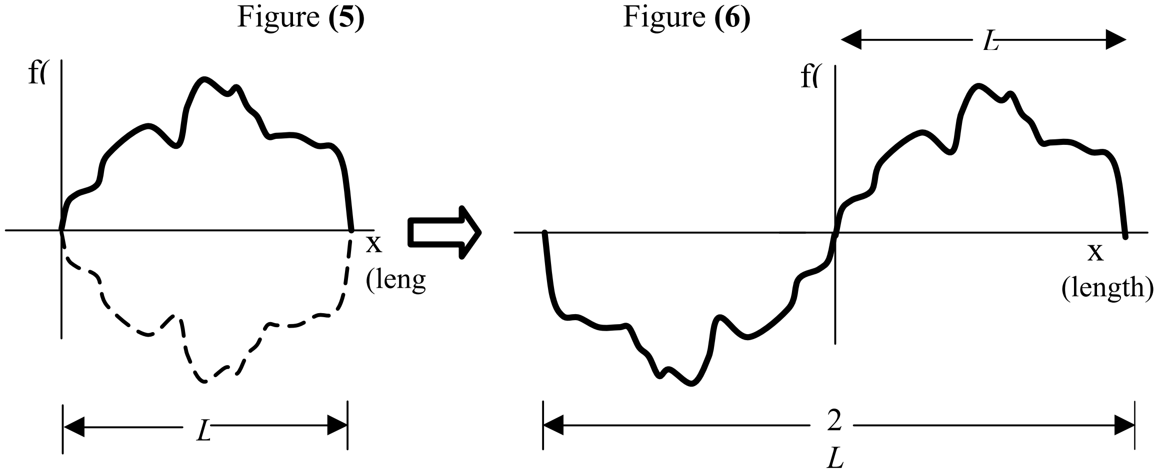
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot \cos\left(\frac{pnx}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot \sin\left(\frac{pnx}{L}\right)dx$$

Thus if we have some idea of what f(x) may be (such as, for example, the symmetric triangular wave produced the instant after the guitar string is plucked as shown in Fig.**(7)** below), we can then iteratively use Equ.**(3)** to find as many a_n and b_n's as we like (more and more iterations allows us to find the coefficients a_n and b_n of higher and higher harmonics, which in turn contributes to a greater degree of precision when decomposing the waveform). Although f(x) changes with time, the shape of the waveform can usually be elicited at any time t and then we can perform a Fourier analysis to obtain the magnitudes of a_0 , a_n , and b_n to see how they decrease with time.

At this point, it is helpful to recall some of the physical parameters of a vibrating guitar string. First, one quickly notices that all cosine terms must vanish because the ends of the strings are fixed and so the amplitudes of the string at position x = 0 and x = *L* must equal 0 for all time t, therefore $a_n = 0$ for all n. As a result, the Fourier series modeling the vibration of the string given in Equ.**(2)** can therefore be reduced to a series of sine terms, which makes it an odd function. However, even though we have thus reduced Equ.**(2)** to a more simplified version, this does not at all mean that a mathematical expression can be used to model the physical shape, i.e. waveform, of a vibrating guitar string. In fact, a mathematical trick must be used so that this series of sine terms can be equated to the function f(x) representing the actual physical shape of the waveform, which in present form is not periodic (the shape of the waveform does not repeat itself over the length of the guitar string), nor can it necessarily be called an odd function. To make the waveform and the Fourier sine series compatible, the waveform can be theoretically reflected about the origin, thereby making it an odd function and giving it a periodicity of 2*L* (see Figures **(5)** and **(6)** below):

Figure **(5)** Figure **(6)**



Note that, in hindsight, since Equ.**(2)** can only be applied to periodic functions, this reflection about the origin is the "behind the scenes" technique that is necessary to make Fourier analysis at all applicable to the problem of decomposing the vibrating string into its constituent harmonics.

A second deduction easily made by considering the physical parameters of a vibrating guitar string is that $a_0 = 0$ in Equ.**(2)**. The coefficient $a_0/2$ represents the offset distance on the ordinate axis of the Fourier series in Equ.**(2)**, i.e., if $a_0/2$ is positive then the entire function moves upwards a distance $a_0/2$. Since there is no offset distance in the modeling of a guitar string vibration (no DC offset in the oscillating voltages induced by the vibrating string), then we know $a_0 = 0$.

Taken together, these considerations yield a simplified version of Equ.**(2)**:

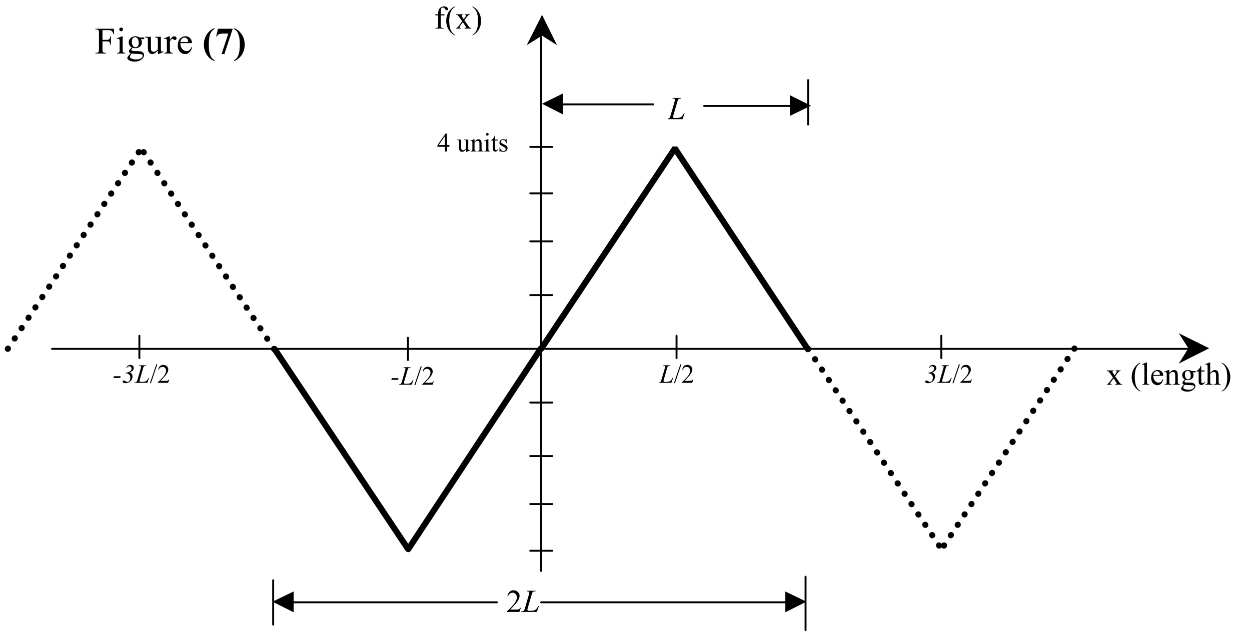
$$f(x) = \sum_{n=1}^{\infty}\left[b_n \cdot \sin\left(\frac{pnx}{L}\right)\right] \qquad \text{Equation } \mathbf{(4)}$$

from which we need only employ Equ.**(3c)** to find the b_n's for as many harmonics as we see fit.

Example of obtaining Fourier series for a waveform

As an example of calculating an actual Fourier series for a given waveform at t=0 (the instant after the string is plucked), suppose that an open guitar string is plucked right at its midpoint (above the 12th fret) so that, after reflection about the origin, the initial waveform has the following initial profile (known as a triangle wave):

Figure **(7)**



From the above diagram, the waveform can be constructed by the following well-behaved piecewise function:

f(x) = 8x/*L* for $0 \leq x \leq L/2$
f(x) = -8x/*L* + 8 for $L/2 \leq x \leq L$

To get the Fourier sine series, we must solve for b_n in Equ.**(4)** using Equ.**(3c)**. Using Formula **(3c),** after noting that both f(x) and sin(πnx/*L*) are odd functions and that the product of two odd functions is an even function which permits us to halve the limits of integration and multiply by a factor of 2, we have the following:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot\sin\left(\frac{pnx}{L}\right)dx \;=\; \frac{2}{L}\int_{0}^{L} f(x)\cdot\sin\left(\frac{pnx}{L}\right)dx$$

$$= \left[\frac{2}{L}\int_{0}^{L/2}\frac{8x}{L}\cdot\sin\left(\frac{pnx}{L}\right)dx\right] + \left[\frac{2}{L}\int_{L/2}^{L}\left(\frac{-8x}{L}+8\right)\cdot\sin\left(\frac{pnx}{L}\right)dx\right]$$

After performing the necessary but messy integration we compute b_n as follows: (for a more detailed version of how to calculate b_n see "Fourier Analysis III: Examples of the Use of Fourier Analysis" by Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, IL, copyright 2000)

$b_n = 0$ for n = 2, 4, 6, 8 , . . .etc.
$b_n = +8\cdot(2/n\pi)^2$ for n = 1, 5, 9, 13, . . . etc.
$b_n = -8\cdot(2/n\pi)^2$ for n = 3, 7, 11, 15 . . . etc.

After plugging these values for b_n into Equ.**(4)**, the entire Fourier series can then be represented compactly as follows:

$$f(x) = 8\sum_{odd-n}^{n=\infty}(-1)^{(n+1)/2}\left(\frac{2}{np}\right)^2 \sin(npx/L)$$ Equation **(5a)**

Expanding the above over the first few terms we have:

$$f(x) = \frac{32}{p^2}\left\{\sin(px/L) - \frac{1}{9}\sin(3px/L) + \frac{1}{25}\sin(5px/L) - \frac{1}{49}\sin(7px/L) + \frac{1}{81}\sin(9px/L) - \ldots\ldots\right\}$$

Equation **(5b)**

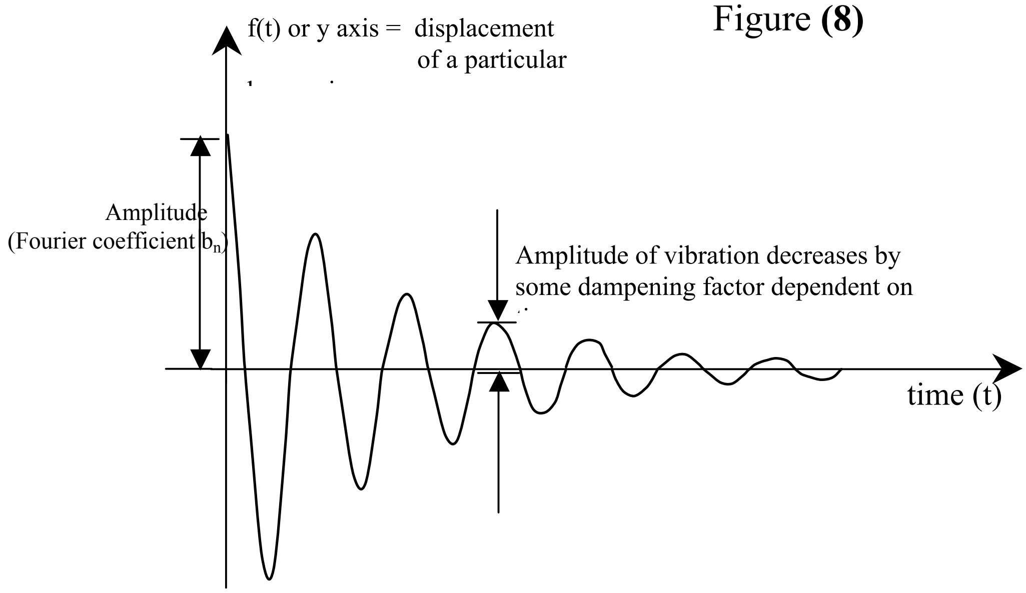
Note that this example shows how the magnitude of b_n decreases by a factor of $1/n^2$ for increasing (odd) values of n. Thus, given that the Fourier coefficients $b_1, b_2, b_3, \ldots, b_n$ represent the amplitudes of the 1st, 2nd, 3rd,. . . ,nth harmonic, respectively, it can be seen from this example (and similarly for other examples of the same type such as an asymmetrical triangular curve) that the amplitudes decrease very quickly with increasing harmonic number n for a waveform at a given time t.

So going back to the point of doing Fourier analysis, Fourier analysis must be performed on a given waveform, i.e., the waveform must be represented as a Fourier series, in order to make any sense out of the decay in amplitude of the individual harmonics of the waveform as a function of time. Since the overall waveform is theoretically nothing more than the superposition (summation) of its harmonics, the decreasing amplitude of each harmonic can be tracked by applying Fourier analysis to the waveform. Measuring the decrease in the amplitudes of the harmonics over time is the only precise way of objectively measuring the sustain of vibrating guitar strings.

Measuring Decay Rates

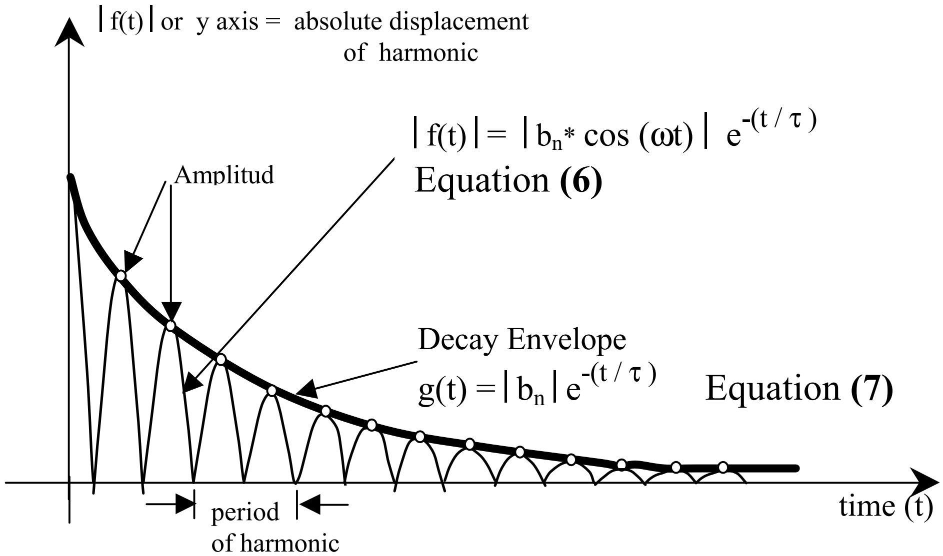
Next, in the interests of making quantitative measurements to monitor and compare different rates of guitar string dampening (e.g. to compare decay rates among different harmonics of the same string, different open strings on the same guitar, and (perhaps in later experiments) the same strings on different guitars) we need to set a baseline standard and/or develop a mathematical tool that allows us to make such comparisons. The formulation of a time constant τ (Greek letter Tau) gives us just such a tool. Say we want to graph the vibrations of a particular harmonic as a function of time. Then we would expect the magnitude of the vibrations, which initially starts as b_n (the Fourier coefficient for the nth harmonic), to decrease with time. The graph of the lateral displacement of a general nth harmonic should look something like this:

Figure **(8)**



Although the overall displacement of the harmonics oscillates in a sinusoidal pattern, we are primarily interested in monitoring how the maximum heights (amplitudes) of the oscillations decrease with time. With this in mind, we can form a “decay envelope” by making a best-fit curve going through the amplitudes of each oscillation. Furthermore, since the negative parts of the oscillation also contain negative amplitudes to be considered as valuable data, we can take the absolute value of each of the oscillations, thereby producing the effect of reflecting the negative parts of the oscillations about the x axis and generating more amplitude points which will ultimately yield a more precise decay curve. (seen in Fig.**(9)** below)

Figure **(9)**



In the above figure, note that the dampening factor $e^{-(t/\tau)}$ in Equ.**(6)** and **(7)** arises out of the assumption that the amplitudes of an (ideal) vibrating string decay exponentially, which in fact is consistent with theories already developed to model dampening in string vibrations (for a more exhaustive explanation of more realistic (non-ideal) string dampening effects, see “*Waves II: Vibrations of* Real Strings” by Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, IL, copyright 2000). From Equ.**(7)**, the time constant τ therefore represents the amount of time that must elapse for the amplitude to decrease by a factor of 1/e = 0.368, i.e., after one time constant has elapsed ($t = \tau$) only 36.8% of the original amplitude (b_n coefficient) remains. The usefulness of τ can further be exploited if we manipulate Equ.**(7)** by taking the natural logarithm of g(t) to form a new function h(t):

$$\text{let}\ \ h(t) = Ln[g(t)]\ ;\ \ \text{then}\ \ h(t) = Ln\big[|b_n|\exp(-t/t)\big] = Ln|b_n| + Ln[\exp(-t/t)]$$

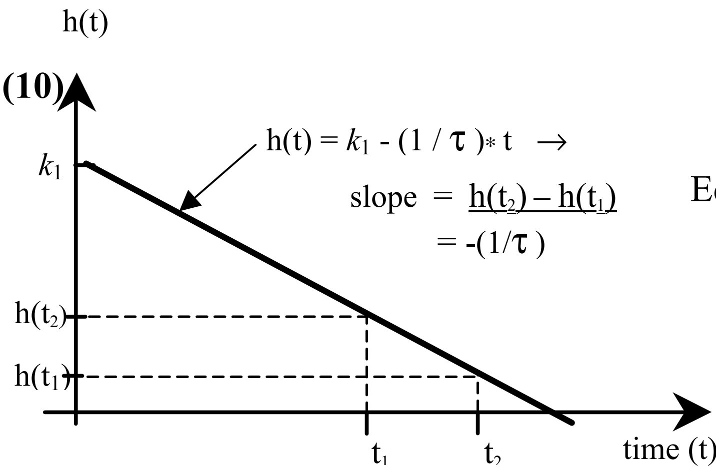
$$= k_1 - (t/t)$$

$$= k_1 - (1/t) * t$$

Equation **(8)**

Where we are replacing the natural log of the Fourier coefficient, which is a constant, with another constant k_1. Note that Equ.**(8)** has effectively reduced the function g(t) for the exponential decay curve to an equation for a straight line with slope $-(1 / t)$ and y-intercept k_1. The graph of the function h(x) looks like this:

Figure **(10)**



Equation **(9)**

As can be seen from the decay line above (which is a semi-logarithmic plot of the original decay curve), the negative reciprocal of τ appears in the expression for the slope of the function h(x) and is independent of the y-intercept k_1. Since k_1 is directly proportional to the magnitude of the Fourier coefficient b_n (which is the displacement at time = 0 before any dampening occurs), we see that the magnitude of the Fourier coefficient has no bearing on the rate of decay, i.e. slope, of the decay line. Therefore the value of Tau remains independent of the initial displacement of the guitar string (it is controlled by the same natural dampening processes for a given vibration no matter what the initial value of the displacement). This feature of Tau was of utmost importance in conducting this experiment because it meant that a standard way of plucking the guitar string in order to achieve the same initial displacement did not have to be developed, i.e., the string could be plucked with little force or with great force without affecting the slope of the decay line and the resulting calculation of the time constant (see caveat below). Monitoring differences in the time constant indeed gives us a baseline standard for drawing comparisons among decay rates (sustains) for different guitar strings and even different harmonics of the same string. ***Caveat***: Although the above description of Tau is useful for understanding the utility of Tau, the above description assumes that Tau is constant over the entire time interval during which the guitar string vibrates. With this assumption in place, we could easily calculate a Tau for any two readings taken of b_n separated by any arbitrary time interval. In reality, however, different dampening processes (such as dampening caused by the stiffness of the strings, dampening caused by the air viscosity, etc.) are dominant only at certain time durations during the string's vibration. This leads to a more jagged decay line in Fig.**(10)** having varying slopes for different time intervals, each with its own characteristic Tau (see Discussion section below). Thus in this experiment, in order to find specific Tau's for each time interval in hopes of relating them to a specific dampening process dominant during that time interval, it is necessary to sample the amplitudes of the harmonics at many different times after the string is initially plucked.

Experimental Methods and Procedure:

Monitoring Each Harmonic by Frequency

At this point, it is appropriate to briefly explain the correlation between harmonics and the frequencies at which those harmonics occur within the waveform. As commented above in the illustration using jump ropes, the length (or, more accurately, wavelength) through which, for example, the 3rd harmonic of a guitar string vibrates is 1/3 the wavelength (1/3**L*) at which the 1st harmonic vibrates. Consequently, because of the fundamental relationship that holds between a wave's velocity, frequency, and wavelength according to the equation

$f = \frac{v}{l}$ where *f* is the frequency , λ is the wavelength of the wave, and *v* is the velocity at which the wave travels.

It is obvious that frequency is inversely related to wavelength, and so a correlation can be made between these two physical parameters of a vibrating string. (Note that since *v* is only a function of the string's tension and mass per unit length, and neglecting the slight increase in tension caused by plucking the guitar string and the ensuing vibration that follows, *v* is constant for any given string.) Since then each harmonic has its own amplitude b_n, and, furthermore, each harmonic vibrates at its own specific frequency, it logically follows that each frequency can be assigned its own specific amplitude b_n (which, incidentally, can be positive or negative). The computer program used in this experiment used frequency to keep track of the amplitudes of each corresponding harmonic.

As a quick example, it is a well-known fact that a properly tuned low E string vibrates at a frequency of 82.4 hertz (tuning the low E string to this frequency actually means the 1st (fundamental) harmonic of the open string is tuned to this frequency). Given this information, it is easy to compute that the 3rd harmonic, since it has a wavelength 1/3 that of the 1st harmonic, will vibrate at a frequency of 3*82.4 hertz = 247.2 hertz. Similarly, all higher harmonics will vibrate at increasing multiples of the frequency at which the 1st harmonic vibrates. After performing Fourier analysis, we know, for example, what the value of b_3 is that corresponds to the amplitude at which the 3rd harmonic vibrates.

The Connection Between String Displacements and Induced Voltages

As mentioned earlier, the electromagnetics of an electric guitar effectively converts the mechanical energy present in a vibrating guitar string into an oscillating voltage (as the guitar strings vibrate over the magnetic pickup of the guitar, the time-varying magnetic flux through the metal strings produces an oscillating voltage inside the coil of the magnetic pickup). It turns out that, at least within the ideal dampening effects assumed in this experiment, electromagnetic theory predicts that the displacements of the guitar string are directly proportional, neglecting some phase change, to the oscillating voltages induced by these vibrations. Thus, after including some constant of proportionality β, Equ.**(6)** and **(7)**, which show the decay rate of harmonic displacement, can easily be recast into equations showing the decay rate of induced voltage oscillations for each harmonic:

$$|V(t)| = \beta * |f(t)| = \beta * |b_n \cos(\omega t)| \; e^{-(t/\tau)} = A_0 \; |\cos(\omega t)| \; e^{-(t/\tau)}$$

Equ.**(6a)**

where β is some positive constant and $A_0 = \beta * b_n$

similarly,

$$|V_{amp}(t)| = \beta * g(t) = \beta * |b_n| \; e^{-(t/\tau)} = A_0 \, e^{-(t/\tau)}$$

Equ.**(7a)**

where $V_{amp}(t)$ = the amplitude of the voltage induced by a particular harmonic's contribution to the overall string vibration

Note that if Equ.**(7a)** were to be graphed as in Fig.**(10)**, although the y-intercept would change from k_1 to some other constant k_2, (where $k_2 = Ln\,(A_0)$) , the graph would remain essentially unchanged and, most importantly, the slope $-(1/\tau)$ would remain unchanged.

As per the caveat mentioned at the end of the "Measuring Decay Rates" section, finding two voltages for a given harmonic and then using Equ.**(9)** was not sufficient for finding a localized Tau. Instead, our experiment entailed using a computer program to repetitively calculate, harmonic by harmonic, the Fourier coefficients over incremental time intervals so that an entire list (array) of decreasing A_0's could be obtained for each harmonic. Specifically, the computer program was set up to calculate Fourier coefficients over a range of 1,200 frequencies, with each of these 1,200 coefficients calculated every 0.4 seconds over a run time of 10.4 seconds. The computer program therefore constructed a 1,200×26 array using this methodology. After a row of 26 (decreasing) A_0's had been found for a particular harmonic, a rate of decay for that harmonic, i.e. a localized τ, was easily calculated using two of the 26 values for A_0. This was done using Equ.**(8)** with some slight modification made to reflect the fact that we are working with amplitudes of voltages instead of amplitudes of harmonic displacement. Exchanging A_{0final} for g(t) and $A_{0initial}$ for b_n in Equ.**(8)** (absolute value signs excluded here for clarity; the computer program automatically took the absolute value of the oscillating voltage amplitudes) led to the following calculation enabling us to find τ:

$$\begin{aligned} Ln\left|A_{0\,final}\right| = Ln\left[\left|A_{0ini}\right|\exp(-t/t)\right] &= Ln\left|A_{0ini}\right| + Ln\left[\exp(-t/t)\right] \\ &= k_2 - (t/t) \\ &= k_2 - (1/t) * t \end{aligned}$$

As an example, using actual data collected from an open low E string vibrating at 220.0 Hertz (corresponds to the third harmonic on a very untuned guitar) was as follows:

Time Elapsed :	0.4 seconds	0.8 seconds	1.2 seconds	1.6 seconds	2.0 seconds
A_0 from F-Analysis:	0.0682 mV	0.0409 mV	0.0277 mV	0.0202 mV	0.0144 mV

Plugging in consecutive values for A_0 at t = 0.4 sec and A_0 at t = 0.8 sec into Equ.**(8)** gives:

$$Ln(.0409) = Ln[(.0682)\exp(-t/t)] = Ln(.0682) + Ln[\exp(-t/t)]$$
$$= -2.69 - (t/t)$$
$$-3.20 = -2.69 - (1/t) * t \quad where: t = 0.4\sec$$
$$\tau = 0.782 \text{ sec}$$

Plugging in values for A_0 at t = 0.4 sec and A_0 at t = 2.0 sec into Equ.**(8)** gives:

$$Ln(.0144) = Ln[(.0682)\exp(-t/t)] = Ln(.0682) + Ln[\exp(-t/t)]$$
$$= -2.69 - (t/t)$$
$$-4.24 = -2.69 - (1/t) * t \quad where: t = 1.6\sec$$
$$\tau = 1.03 \text{ sec}$$

Methods and Materials:

To find the Fourier coefficients, the software program LabView 5.0 was used. The actual hardware was hooked up and the software configured as follows:

The positive and ground terminal leads of the electric guitar used in this experiment were connected to an ADC board going to the computer that read voltages produced by the vibrations of the guitar strings. These voltage readings were taken by LabView at intervals specified by the sampling rate, which the user was prompted to enter at the LabView user interface panel (e.g. if the user typed in a sampling rate 10,000, LabView would take voltage readings from the ADC every 0.1 millisecond). LabView also prompted the user to enter the total number of samples to be taken, up to a maximum of 64,000 per trial. So the desired total length of the sampling period was determined by the user and was just the number of samples divided by the sampling rate. The sampling rate specified for each of the trials chosen was 64,000, so the duration of each trial was 10.67 seconds. This was thought to be an adequate period over which the decay of the vibrations in the strings could be measured.

Results:

The results of the experiment were plotted and are seen in Appendix A and Appendix B. For each of the trials done, it is important to note that the guitar string was plucked close to the bridge of the guitar (therefore producing a "brighter sound" characteristic of higher harmonics with relatively large amplitudes). Appendix A shows the decreasing amplitude of the 1^{st} and 2^{nd} harmonics for each open string over an interval of about 10 seconds, during which the dampening of the vibrations is clearly seen. The plots of the decay curves for both the 1^{st} and 2^{nd} harmonics are shown on both a regular and on a semi-log scale. Appendix B shows for each open string the relative magnitudes of the vibrations for each string at a specific time measured over a frequency spectrum ranging from 2.5 to 3,000 hertz divided into 2.5 hertz intervals. In these plots the relative magnitudes of the first five harmonics for each open string can be easily seen. Finally, time constants for each of the vibrating strings were calculated after the initial plucking of the string over three specific time intervals: 0.8 to 3.2 seconds, 2 to 6 seconds, and 6 to 8.8 seconds. The results for the time constants are summarized in the table below:

Discussion:

As mentioned in the caveat above, the presence of different dampening processes will cause the values measured for Tau to change depending on which dampening process is most dominant for a specific time interval of the string's vibration. These dampening processes include, but are not limited to, energy losses due to the viscosity of air, the energy transferred from the string vibrations to produce sound waves in air, the energy transferred to overcome the stiffness of the strings, energy transfer to the non-rigid nut and bridge of guitar, and a wide number of other energy losses due to electromagnetic dissipation caused by the electric and magnetic interactions of the metal guitar strings vibrating over a magnetic pickup. Each one of these energy loss mechanisms has its own characteristic time constant, and the total time constant can be formulated as the sum of the reciprocals of each of the individual time constants for each energy loss mechanism. This can be represented as follows:

$$\frac{1}{t} = \sum_{i=1}^{n} \frac{1}{t_i} = \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + ... + \frac{1}{t_n}$$ where each τ_i is the time constant for an individual energy loss mechanism

Equ.**(6a)** therefore becomes:

$$|V(t)| = \beta * |f(t)| = \beta * |b_n \cos(\omega t)| \; e^{-(t/\tau)}$$

$$= A_0 \; |\cos(\omega t)| \; e^{-t[(1/\tau_1) + (1/\tau_2) + (1/\tau_3) + ... + (1/\tau_n)]}$$

As seen from the plots of the 1st and 2nd harmonics in Appendix A, it is noticeable that, for the open Low E, D, and B strings (strings for which the data seems reliable), the 2nd harmonic seems to display varying time constants while Tau for the 1st harmonic seems to be relatively constant. For the Low E, D, and B strings, the decay line of the 2nd harmonic seems to have three distinct regions: a fairly steep segment (small Tau) shortly after the string is plucked, an intermediate segment of lesser slope (greater Tau), and then near the end of its timed vibration another segment of different slope. In the case of the open B string shown in Appendix A4, the decay line for the 1st harmonic shows a rather interesting behavior. Immediate after the string is plucked the vibration shows very quick decay, but after this immediate decay (after about 2.5 seconds has elapsed) the magnitude of the 1st harmonic oscillations actually increases for a period of time. Although this does not make physical sense in the case of an ideal vibrating string (the string cannot gain energy with time to increase the magnitude of its oscillations), in the non-ideal case there exists the possibility that the oscillation may be amplified by the coupling of the string vibration with the vibrations of the non-rigid end supports.

As a general trend, the plots of the decay curves for each of the strings show that the 2nd harmonic generally dies out faster (has a shorter time constant) than the 1st harmonic. Although this may seem intuitive from Equ.**(5b)**, where the initial amplitude of the 2nd harmonic is predicted to be 1/9 of the amplitude of the first harmonic, the fact is that Equ.**(5)** does not take into account how the amplitudes of the harmonics also depend on time. Equ.**(5)**, cannot therefore predict the *rates* of decay and the time constants must therefore be elicited here by experiment (see “Waves II” by Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, IL, copyright 2000 for a more in-depth theoretical approach to quantitatively predicting how the magnitudes of the oscillations can be calculated as a function of time, i.e., how to predict the time constants for different dampening processes).

Another interesting feature of this experiment can be seen in the relative magnitudes of the 1st and 2nd harmonic initially after the string was plucked. Because the guitar strings were plucked in this experiment close to the bridge of the guitar, it makes sense that the initial amplitudes of the higher harmonics might be initially greater than the 1st harmonic (producing a brighter sound than if the string was plucked at the 12th fret and/or midpoint of the neck, in which case the 1st harmonic is excited the most and produces a characteristic mellow tone). Correspondingly, compared to the plots of the 1st harmonic, greater magnitudes of the 2nd harmonic can be seen in the decay curve plots of the open Low E, A, and B strings.

Along with the plots shown in Appendix A, the fact that the guitar strings were plucked at the bridge is also apparent by looking at the frequency spectrum plots shown in Appendix B. In these plots the first five harmonic frequencies appear as unmistakable spikes. Along with showing that the harmonic frequencies are separated by regular frequency intervals because the resonate harmonic frequencies of the higher harmonics are integral multiples of the 1st harmonic, the heights of the spikes show the relative magnitudes of the oscillations for the different harmonics. Although this is true mostly for the frequency spectrum plots taken soon after the plucking of the guitar, when all five harmonics are visually apparent, the frequency spectrum plots taken after a long period of time, e.g., after 6 or 8 seconds has elapsed, are also useful in showing some trends. Some of these trends varied depending on the string being tested. For example, the frequency spectrum plot for the Low E string in Appendix B1, shows how the magnitude of the 1st harmonic is generally preserved for a longer period of time during while the higher harmonics fade out after the first 4 seconds. This is in contrast to the lengthy preservation of all five harmonics as seen in the plot of the open A string vibration in Appendix B2. This in turn indicates that the preservation of the higher harmonics may be string dependent or subject to other conditions that coincidently vary dramatically by string. Also, another trait apparent in several of the plots in Appendix B2 is the presence of a weak resonate frequency at 60 hertz caused by the ambient electric circuitry common in all AC-powered household and laboratory appliances, usually apparent during the last few seconds of measurements when the voltages caused by the vibrating string are no higher than the ambient 60 hertz voltages. To some degree, the presence of this "household" AC frequency at 60 hertz can help us gage the accuracy of our experimental apparatus. Lastly, another trait seen by viewing the frequency spectrum plots in Appendix B is that the magnitudes of the different harmonics cannot be measured purely by the height of the spikes produced on the plot. Rather, as indicated in Appendix B9 for the open D string, a more accurate method for measuring the relative strengths of the different harmonics would require integrating the voltages of the frequencies immediately surrounding the main resonate frequency seen for each harmonic.

As far as the difference in time constants among the various strings of the guitar (Low E, A, D , G, B, and Hi E), a quick look at the table of time constants in Appendix C shows that the wound strings (Low E, A, and D) did generally have a higher Tau for the 1st harmonic than the plain strings. This may be attributed in large part to the greater momentum imparted to the wound strings upon plucking them as compared to the plain strings, which have a significantly smaller mass (lower momentum after initial displacement) than the wound strings. Although the larger, more massive strings probably induced greater dampening forces opposing their motion, these dampening forces were not proportionally large enough to balance the higher momentum given to the wound strings as compared to the plain strings. Correspondingly, the string with the highest mass, Low E, had the highest overall time constant. It is interesting to note, however, that the time constant for the 2nd harmonic was not significantly higher for the wound strings compared to the plain strings.

As mentioned in the beginning of this discussion, one of the main questions upon which this experiment was based was the question of determining which dampening processes were dominant over which specific time intervals of the string vibration. In keeping with this aim, the goal of the last part of this experiment was to monitor any changes in the magnitude of the voltage signal caused by varying the height of the pickups relative to the vibrating strings, i.e., varying the guitar action. Unfortunately, because the guitar pickup came off the first guitar while attempting to adjust its height, a second guitar had to be used that was different from the guitar for which all the beginning work (the voltages for the six open strings) had already been conducted. As such, the voltage signal produced by vibrating the open D string for the second guitar was much weaker than for the first guitar, and the results were probably not as reliable. Despite these difficulties, the time constants calculated and presented in the table of Appendix C for the open D string vibrating over a pickup adjusted very close to the stings showed a slightly lower time constant for the 1st harmonic at time intervals of 0.8 to 3.2 seconds and 2.0 to 6.0 seconds, and a more pronounced difference for the time constant measured during the 6.0 to 8.8 second time interval. Thus, although our prediction was borne out by experiment, namely, that the time constant should be lower for the pickup adjusted close to the guitar strings (more dissipative losses due to increased magnetic dampening), the overall differences in time constant values were on average not very substantial and may or may not be able to substantiate our hypothesis beyond an allowance given for experimental error. Furthermore, the time interval over which this dampening process might have its most dominant influence on overall guitar string dampening could not be determined. Compared to the time constants calculated for the 1st harmonic, the time constants calculated for the second harmonic show a more significant difference, especially during the 2 to 6 second time interval. Again, however, the reliability of the data might be questionable. A quick visual inspection comparing the plots of the decay curves for the 1st harmonic in Appendix A7 to the plot of the decay curve in Appendix A8 can also be made. This shows that the decrease in voltage magnitude per unit time shown by the decay curve for the pickup farther away is noticeably more precipitous than for the pickup adjusted closer to the strings, thus actually contradicting our hypothesis. One thing that might add at least a little more credence to the results, however, is that the initial voltage for the close pickup initially after the string was plucked was greater than for the farther pickup, which would be consistent with what we expect provided the strings were plucked with approximately the same initial displacement. As a final observation, the fluctuations seen in the harmonics during a time of 8 and 10 seconds shown in Appendix B7 might be indicative of the humming phenomena sometimes noticed when using low guitar action.

After conducting this experiment, it was noticed that the code used to take the voltage measurements for each frequency starting at 2.5 hertz and going up to 3000 hertz in 2.5 hertz intervals was a bit unnecessary and wasteful of computer computations. More precise measurements of the voltage, taken more continuously, could have been achieved if the allotted number of 64,000 samples (voltage measurements) taken by the computer program could have been designated for a shorter frequency spectrum. An even better method would have employed the coding of a separate program for each vibrating string so that all the voltage measurements taken could be isolated only around the expected harmonic frequencies of the vibrating string. Lastly, although it would have introduced added complications in diagnosing the pure harmonics of a guitar string, adding an amplifier in series with the ADC board going to the computer could have increased the amplitudes of the harmonics in such a way as to make the amplitudes of the higher harmonics still perceptible even over long decay times.

Conclusion:

This experiment set up a beginning apparatus and method for monitoring the decay rates of each vibrating open string of a guitar. In general, the expectations that the higher harmonics would decay faster than the lower harmonics was confirmed by experiment. Also, while there was only a slight differences in the time constants for the 2^{nd} harmonic among the six strings, the 1^{st} harmonic for the heavier wound strings (Low E, A, and D) showed higher time constants than for the lighter plain strings (G,B, and High E). Lastly, the hypothesis that an open string vibrating closer to a guitar pickup would show a higher decay rate of its harmonics was not quantifiably confirmed in this experiment due to the questionable legitimacy of the data collected during this part of the experiment.

Although the vibration of guitar strings is a non-ideal phenomena with much coupling between many mechanisms of dissipative losses (many varying time constants), the theory used in this experiment was based on ideal strings with only one mechanism of dissipative energy losses (assumed a non-varying time constant) and so it had its limitations, especially when attempting to quantitative evaluate the experimental results. Despite this, different time constants were calculated in this experiment by iteratively measuring the voltage differences at different times during the string vibrations. A more accurate approach for finding a specific time constant for each time interval could have been realized by measuring the "instantaneous" drop in voltage over an infinitesimally small (or as small as experimentally possible) time interval. Also, computer codes tailored to each string being tested could be developed to take as many measurements as possible for a specific harmonic frequency, thus dramatically increasing the precision of finding the time constants and measuring the sustain of the guitar strings.