Stability of a Particle Levitated in an Acoustic Field

Luke Wortsmann

(Dated: 12 May 2016)

The Gor'kov potential for acoustic radiation pressure on a sphere is a lowest order approximation of the forces of a fluid scattering off a spherical boundary. Using the Gor'kov potential and a cylindrical waveguide, I derive some analytic properties of the stable equilibrium of a spherical object levitated in an acoustic field. I demonstrate the validity of these results using numerical simulations and qualitative experimentation with an ultrasonic transducer and particles of polystyrene foam with different geometries.

I. INTRODUCTION

By creating a very high intensity acoustic field, the force of acoustic radiation can overcome the force of gravity and levitate small objects. Using ultrasonic transducers at inaudible frequencies, acoustic levitation is used in many specialized industries for containerless, or non-contact, processing. At a few watts of acoustic power, these acoustic fields would be deafening at audible frequencies so ultrasonic frequencies are typically used.

In the lab, I used an ultrasonic transducer which outputted a signal around 58 kHz $\pm$ 1 kHz. Using this transducer, I was able to levitate pieces of polystyrene foam (Styrofoam) with a variety of geometries and sizes. Even with an unstable driver, the transducer was able to levitate this particles for extended periods of time (effectively indefinitely).

The acoustic radiation force is a second order term and therefore does not arise using the analysis of linearized acoustic theory. Much of the research on acoustic levitation relies on numerical simulations. Using Comsol Multiphysics, I ran a number of numerical simulations to model an ultrasonic transducer, the acoustic field it creates, and the trajectories of particles in this field.

The seminal result in the analytical study of acoustic levitation is the Gor'kov potential. In 1961, L.P Gor'kov derived the lowest order force on a spherical particle in an acoustic field. From this potential, (equation 1), I derived some results analytically for an acoustic field that approximates the one generated by the transducer in lab. The Gor'kov potential is also implemented in Comsol for some simulations.

$$U = 2\pi R^3 \left(\frac{\langle p^2 \rangle}{3\,\rho_0\,c^2} f_1 - \frac{\rho_0\,\langle u^2 \rangle}{2} f_2 \right) \tag{1}$$

$$f_1 = 1 - \frac{\rho_0\,c^2}{\rho_s\,c_s^2} \tag{2}$$

$$f_2 = 2\left(\frac{\rho_s - \rho_0}{2\rho_s + \rho_0} \right) \tag{3}$$

II. THEORY

The Gor'kov potential, equation 1, describes the radiation pressure on a sphere of radius R and density ρ_s in an acoustic field. c_s is the speed of sound in the sphere and is typically much greater than the speed of sound in air, thus I will take $f_1 = 1$ to simplify calculations. The density of air, (or the density of the fluid the sphere is in) is ρ_0, for air at STP, $\rho_0 = 1.225$ kg/m^3 and $c = 343$ m/s.

The acoustic field is described by the pressure, p, and particle velocity, u. The mean square deviation of these terms, $\langle p^2 \rangle$ and $\langle u^2 \rangle$ are independent of time in a harmonic field and both are functions of position. The particle velocity and the pressure of the acoustic field can both be described using a velocity potential wavefunction φ:

$$p = -\rho \frac{\partial \varphi}{\partial t} \tag{4}$$

$$u = \nabla \varphi \tag{5}$$

Where the velocity potential is an eigenstate of the Laplacian operator for a harmonic field:

$$\nabla^2 \varphi = -\left(\frac{2\,\pi\,f}{c} \right)^2 \varphi$$

We are looking for radially isotropic solutions, where the real-valued wavefunction takes the form:

$$\varphi = a_1\, J_0\,(a_2\, r)\, \cos\,(a_3\, z)\, \sin\,(\omega\, t)$$

Where a_1, a_2, and a_3 are undetermined constants and J_0 is the zeroth Bessel function. In numerical simulation and in the lab, it appears that the wavelength of the field is about what it would be in free air. Thus $a_3 = f\,\pi/c$, where f is the frequency of the transducer. Then, to satisfy Laplace's equation, a_2 must equal $\sqrt{3}\,f\,\pi/c$. To have a_1 be a function of the maximum particle velocity, v_0, a_1 must equal $-\frac{c\,v_0}{\pi\,f}$. Thus our approximation for the acoustic field is:

$$\varphi = -\left(\frac{c\,v_0}{\pi\,f} \right)\, J_0 \left(\frac{\sqrt{3}\,\pi\,f\,r}{c} \right) \cos\left(\frac{\pi\,f\,z}{c} \right) \sin\,(\omega\,t) \tag{6}$$

The maximum particle velocity as a function of the effective sound power level ($P_{\text{effective}}$) is:

$$v_0 = \sqrt{\frac{2\,P_{\text{effective}}}{A\,c\,\rho_0}} \tag{7}$$

Where A is the surface area of the transducer. The nodes where the particle is stable and levitated are along $r = 0$, and along this axis the mean square deviations are:

$$\langle u^2 \rangle = \frac{P_{\text{effective}}}{A\, c\, \rho_0} \sin^2\left(\frac{\pi\, f\, z}{c}\right) \tag{8}$$

$$\langle p^2 \rangle = \frac{4\, c\, \rho_0\, P_{\text{effective}}}{A} \cos^2\left(\frac{\pi\, f\, z}{c}\right) \tag{9}$$

Since force is related to potential $F = -\nabla U$, the Z component of the force per volume along the $r = 0$ axis is:

$$F_z = \left[\frac{\pi\, f\, P_{\text{effective}}\, (\rho_0 + 11\, \rho_s)}{2\, A\, c^2\, (\rho_0 + 2\rho_s)}\right] \sin\left(\frac{2\pi f z}{c}\right) \tag{10}$$

The maximum value of the force is the term inside the bracket, thus the maximum particle density - ρ_s - that can be levitated is implicitly:

$$g\, \rho_s = \frac{\pi\, f\, P_{\text{effective}}\, (\rho_0 + 11\, \rho_s)}{2\, A\, c^2\, (\rho_0 + 2\rho_s)}$$

Solving for ρ_s and assuming f is large, we obtain an upper limit for the density of a particle that can be levitated:

$$\rho_s \leq \frac{11\, \pi\, f\, P_{\text{effective}}}{4\, A\, c^2\, g} - \frac{9\, \rho_0}{22} \tag{11}$$

Or the power needed to levitate an object of some density:

$$P_{\text{effective}} \geq \frac{2\, A\, c^2\, g\, (9\, \rho_0 + 22\, \rho_s)}{121\, \pi\, f} \tag{12}$$

Figure 1 plots the Gor'kov for this acoustic field along with the force vectors for numerical values similar to the experimental setup in the lab. The bright red regions in the plot are equilibrium points, The elongated ones in the blue areas of the plot are stable equilibriums, where the particle can be stably levitated.

I also analyzed the frequency of oscillation around these stable equilibrium points in the absence of gravity. These stable points are approximately at $z_0 = (4\, c\, n + c)/(4\, f)$ where n is an integer. Around these points I looked at oscillations in the r direction and the z direction, these frequencies are:

$$\omega_z = \left[\frac{\pi\, f}{c}\sqrt{\frac{P_{\text{effective}}}{A\, c\, \rho_s}}\right]\sqrt{\frac{\rho_0 + 11\rho_s}{\rho_0 + 2\rho_s}} \tag{13}$$

$$\omega_r = \left[\frac{\pi\, f}{c}\sqrt{\frac{P_{\text{effective}}}{A\, c\, \rho_s}}\right]\sqrt{\frac{3\, (25\rho_s - \rho_0)}{4\, (2\rho_s + \rho_0)}} \tag{14}$$

In the limit were ρ_s is the maximum that can be levitated by $P_{\text{effective}}$, the bracketed term in equations 13 and 14 becomes:

$$\sqrt{\frac{2\, \pi\, f\, g\, (9\, \rho_0 + 22\, \rho_s)}{c\, \rho_s}}$$

For numerical values corresponding to the experimental setup: $\omega_z = 102$ and $\omega_r = 133$. This fairly high frequency implies that the potential well is very stable, with the particle oscillating around the equilibrium point at a very high rate. In the lab, I found that the particle remained almost stationary, and when perturbed by by a slight breeze, frequently returned to a motionless state fairly quickly.

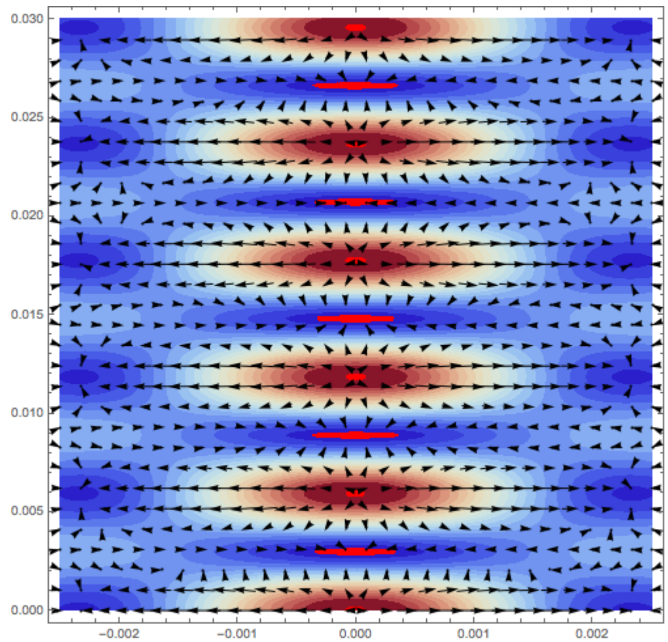

FIG. 1. The Gor'kov potential and force vectors for the acoustic field described and using numerical values corresponding to the experimental setup in lab. Scale is in meters, the r-axis is about 2mm and the y-axis is about 3cm.

III. SIMULATION

To establish the validity of these analytic models, I used Comsol Multiphysics to numerically model the ultrasonic transducer and glass reflector setup. I modeled harmonics of the acoustic field for several different geometries, the Gor'kov potential for these geometries and harmonics, and particle trajectories for several different size and density particles. Below are some plots of the Gor'kov potential for three different geometries, note how figure 3 resembles the analytic field plotted in figure 1.

In addition to numerically solving for the harmonics, I also used Comsol to simulate particle trajectories for large particles. Analytic results from the Gor'kov potential should break down when the particle is larger than the wavelength of the field. Comsol solves for the particle trajectory in this regime by numerically solving the Navier-Stokes equations and must include higher order effects including reflections off the particle. Figure 4 plots the position of some large particle at different times.

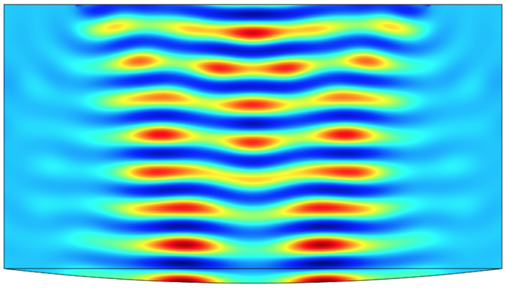

FIG. 2. The Gor'kov potential for an ultrasonic transducer (top) reflecting off a glass reflector (bottom).

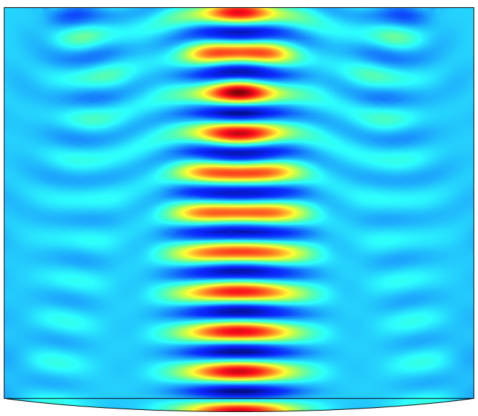

FIG. 3. Similar to Figure 2.

IV. EXPERIMENTATION

To verify the validity of the analytical analysis and numerical simulation, I used a ultrasonic transducer to levitate different sized fragments of styrofoam. The transducer has a driving frequency of 58 kHz $\pm$ 1 kHz, and a radius of 3cm, thus A or the surface area of the transducer is 28.27 cm^2. The particles of polystyrene foam used varied in geometry but were all about equal to or slightly less than 5mm in diameter. The density of polystyrene foam, ρ_s, is approximately 50 kg/m^3. While I could not directly measure the power of the acoustic field generated because the frequency is too high to be detected by the available microphones, I estimate it to be around 1 watt. This implies the maximum density that can be levitated is 153 kg/m^3, or about 3 times that of polystyrene foam or 15% the density of water. I attempted to levitate drops of water and some other dense material to no avail.

I found that the stability of the particle is not particularly sensitive to the height between the reflector and transducer, especially when the distance between the reflector and transducer is small. This result is confirmed in numerical simulations, a standing wave is generally created when the reflector and transducer are fairly close (see figures 2 and 3). I also has some success levitating the particle in nodes not on the $r = 0$ axis, for example the low potential regions in the simulation plotted in figure 2.

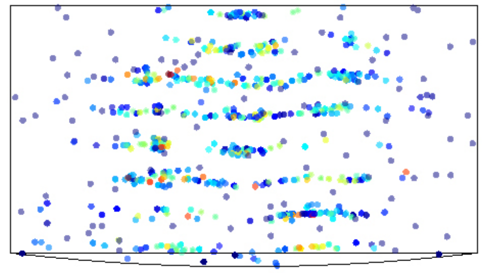

FIG. 4. Some large, light particles in an acoustic field. Note the grouping directly in the center.

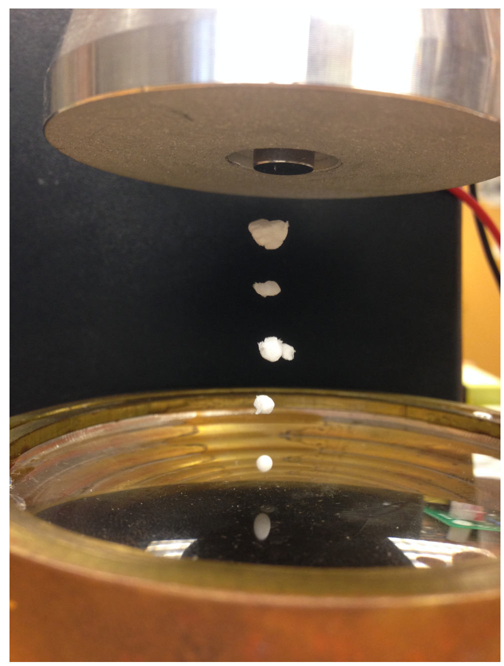

FIG. 5. Photograph of the transducer (top) and reflector (bottom) levitating 5 particles at the stable nodes of the acoustic field.

The levitated particles remained stable and motionless for long periods of time, with the large tolerance of frequency of the transducer, this is surprising. With the analytical model, I plotted the percent difference between the Gor'kov potential with the upper bound of the frequency and the lower bound of the frequency and interestingly I found that this difference is nearest to zero at the equilibrium points of the field at the stated fre-

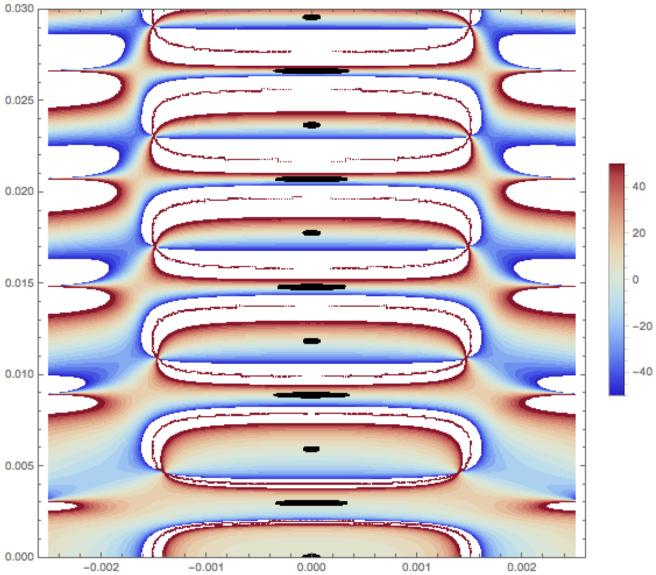

FIG. 6. Percent difference between $f = 57$ kHz and $f = 59$ kHz and the equilibrium points for $f = 58$ kHz. Note how the equilibrium points (black) are all where the percent difference is near 0%

quency. See figure 6 which plots this percent difference and shows the equilibrium points.

While the analytic wavefunction does not fully model the reflector in the system, it does correctly predict the stability of the particle even with a large tolerance of frequency.

V. CONCLUSION

Much of the recent work on acoustic levitation primarily uses numerical simulation to analyze specific specific systems. While this approach might be necessary to get accurate results with complicated systems and a nonlinear effect, I wanted to develop some analytic results to use as a reference point in analyzing the stability of acoustic levitation. By constructing a reasonable velocity potential wavefunction and using the Gor'kov potential, I was able to get some reasonable analytic expression that accurately reflected the results of both numerical simulation and experiment.

I found that an acoustic levitation system is fairly robust to changes in driving frequency as well as transducer height. Slight perturbations to levitated particles are corrected by a significant restoring force, and a low density particle can be levitated stably indefinitely.

VI. REFERENCES

[1] L. Gor'Kov, "On the forces acting on a small particle in an acoustical field in an ideal fluid," in *Soviet Physics Doklady*, Vol. 6 (1962) p. 773.

[2] J. Li, P. Liu, H. Ding, and W. Cao, "Nonlinear restoring forces and geometry influence on stability in near-field acoustic levitation," Journal of Applied Physics **109**, 084518 (2011).

[3] G. T. Silva, "Acoustic radiation force and torque on an absorbing compressible particle in an inviscid fluid," The Journal of the Acoustical Society of America **136**, 2405–2413 (2014).

[4] J.-H. Xie and J. Vanneste, "Dynamics of a spherical particle in an acoustic field: A multiscale approach," Physics of Fluids (1994-present) **26**, 102001 (2014).

[5] N. Pérez, M. A. Andrade, R. Canetti, and J. C. Adamowski, "Experimental determination of the dynamics of an acoustically levitated sphere," Journal of Applied Physics **116**, 184903 (2014).

[6] W. Xie, C. Cao, Y. Lü, Z. Hong, and B. Wei, "Acoustic method for levitation of small living animals," Applied Physics Letters **89**, 214102 (2006).

[7] M. Barmatz and P. Collas, "Acoustic radiation potential on a sphere in plane, cylindrical, and spherical standing wave fields," The Journal of the Acoustical Society of America **77**, 928–945 (1985).