Simply Degenerate Frequencies in Square Waveguide

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Trivially, all eigenmodes in a square waveguide are degenerate. These eigenvalues typically take on a form: $k \propto m^2 + n^2$ where m and n are positive integers, for example, the harmonic frequencies of acoustical waves with square boundary conditions are given by: $f_{m,n} = \frac{c}{2L}\sqrt{m^2+n^2}$. The trivial degeneracy is $f_{m,n} = f_{n,m}$, however multiple degeneracies can also arise. I analyze the ratio of simply degenerate states to all states and show that this ratio appears to asymptotically approach 0.288 following a power law.

I. INTRODUCTION

In a general rectangular waveguide the frequencies of the harmonics, or eigenstates, are given by:

$$f_{m,n} = \frac{c}{2}\sqrt{\left(\frac{n}{L}\right)^2 + \left(\frac{m}{W}\right)^2}$$

In the case of a square waveguide, $W = L$ and all of the frequencies are degenerate as:

$$f_{m,n} = f_{n,m} = \frac{c}{2L}\sqrt{m^2+n^2}$$

Define $k \equiv m^2 + n^2$ so $f_{m,n} = \frac{c}{2L}\sqrt{k}$. Thus the problem of analyzing eigenfrequencies reduces to analyzing values of k. k is a legitimate eigenstate if it is an integer composed of the sum of two integers squared. If k is can be expressed as the sum of two squares, then it will always be degenerate but only sometimes be simply degenerate. For k to be simply degenerate, m and n are the unique integers that, when squared, sum to k. For example, $k = 50$ is not simply degenerate because $1^2 + 7^2 = 5^2 + 5^2 = 50$.

Interestingly, the angular momentum of the acoustic field of two simply degenerate states at the center of the square can be significant. A small, free spinning rotor can be made to spin when placed in the center of the square and a pair of simply degenerate harmonics are driven.

II. DETERMINING DEGENERATE EIGENSTATES

One can determine if any integer k can by formed from the sum of two integers squared through an extension of Fermat's theorem on sums. Consider the prime factorization of k:

$$k = 2^{\alpha}\prod_i p_i^{\beta_i}\prod_j q_j^{\gamma_j}$$

Where p_i is a prime congruent to 1 mod 4 and q_j is a prime congruent to 3 mod 4.

If all of the γ_j exponents are even, then k is expressible as the sum of two integers. It then follows that k can be a harmonic frequency of the square waveguide. To show that those two integers are unique, and thus the eigenstate is simply degenerate, one must examine a result from Minkowski's geometry of numbers.

The expression $k = n^2 + m^2$ suggests a circular geometric representation. For a solution to exist over the integers, the circle of radius $\sqrt{k}$ must intersect an integer grid point (m, n). The number of intersections, where both m and n are positive, is given by:

$$\prod_i (1 + \beta_i)$$

For a simply degenerate eigenstate, there are only two intersections and thus only one β_i is equal to one, the rest equaling zero. However, if k is a square number, then additionally there are two more intersections: $(\sqrt{k}, 0)$ and $(0, \sqrt{k})$. So if k is square, then all β_i must equal zero.

III. COUNTING DEGENERATE STATES

The above algorithm enables the rapid computation of harmonic frequencies. I implemented this algorithm in a Mathematica script, which rapidly counts degenerate k up to $k \approx 10^7$. For each integer, the script first determines if it is degenerate, then determines if it is simply degenerate. These parameters are stored and a running count of degenerate, and simply degenerate states is kept. Let the number of degenerate numbers less than k, as a function of k, be: $n_d(k)$, and the number of simply degenerate numbers less than k, also as a function of k, be $n_s(k)$. The asymptotic behavior of the ratio: $r(k) = n_s(k)/n_d(k)$ as $k \to \infty$ appears to converge to a constant value. Using Mathematica, I ran a regression analysis on calculated values of $r(k)$ using a generalized power law model:

$$r(k) \approx \frac{\alpha}{(k+\sigma)^{\gamma}} + \epsilon$$

When k approaches infinity, this model approaches ϵ, thus the ratio of all simply degenerate states to all degenerate states is near ϵ. Running the regression in Mathematic on calculated values of $r(k)$ with $k = 1000$ to 10^6 returns the following fitted values:

	Estimate	Standard Error	Confidence Interval
α	1.01283	9.434×10^{-6}	[1.01281, 1.01286]
σ	677.314	0.123	[676.998, 677.631]
γ	0.121192	1.568×10^{-6}	[0.121188, 0.121196]
ϵ	0.287921	2.345×10^{-6}	[0.287915, 0.287927]

The confidence interval is at 99%. So as $k \to \infty$, we expect the ratio of simply degenerate states to all degenerate states to follow:

$$r(k) \approx \frac{1.01283}{(k+677.314)^{0.121192}} + 0.287921$$

This implies that the total number of simply degenerate states accounts for only 28.8% of all states, Though at more physical and observable values, this ratios appears to be closer to (or greater than) 50%. Below are graphs from the regression calculation of the model and the residuals of the fit.

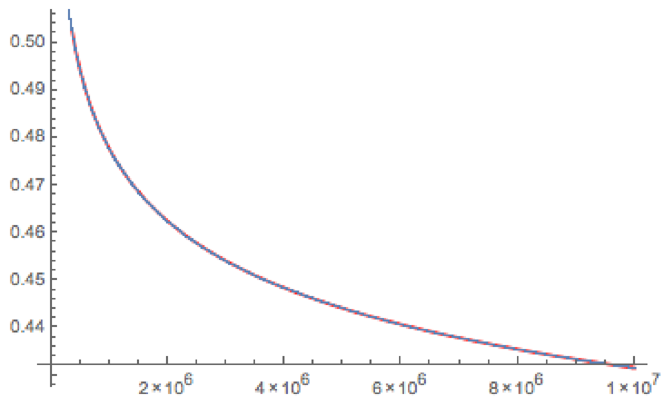

FIG. 1. A plot of the calculated values of $r(k)$ as well as the fitted power model (indistinguishable).

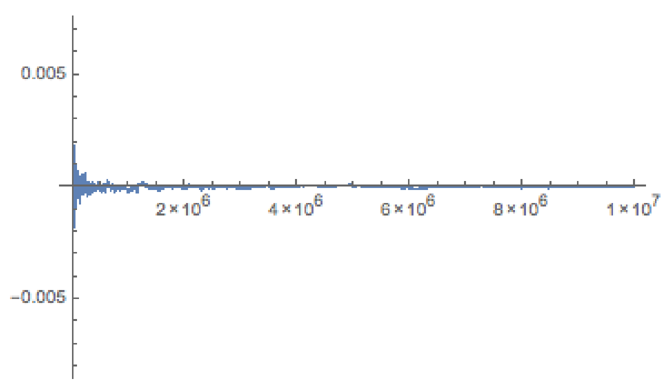

FIG. 2. A plot of the residuals of the model and the calculated values of $r(k)$.

IV. EXPERIMENT

In lab, I attempted to replicate the results of Schroeder[1] and use a simply degenerate frequency pair to measure the angular momentum of the field, however, I was unsuccessful in my attempt. To generate enough acoustic radiation pressure, a very loud sound source is needed and at audible frequencies this can be difficult to work with in the lab. A very light, free moving rotor is also need to spin in the acoustic field. I could not find such a rotor that would work and Schroeder[1] seemed to use a makeshift yogurt cup and needle, something I could not get to work.

The goal of the experiment was to drive a pair of simply degenerate eigenstates of a square box by driving two speakers, placed on adjacent edges of the box, 90 degrees out of phase at the frequency of the lowest simply degenerate harmonic. I used a box with a side length of 47cm, the lowest simply degenerate k with nonzero angular momentum at the center of the square is 5, so the driving frequency was: $f = \frac{c}{2L}\sqrt{5} = 815.93$ Hz. With two function generators attached to a oscilloscope, we could monitor and adjusted the phase difference between the function generators. The signals where amplified by two amps then sent to two computer speakers in the square box.

While Schroeder[1] performed the above experiment after discussing the relationship between simply degenerate and degenerate harmonics, I do not particularly see the connection and the results obtained from counting degenerate eigenstates cannot be easily measured to my knowledge (especially at high frequencies).

V. CONCLUSION

Following Schroeder[1], I implemented an algorithm derived from number theory to count degenerate frequencies in a square waveguide. Using this algorithm, a Mathematica script analyzed the asymptotic ratio of simple degenerate to degenerate frequencies for values of k up to 10^7. I find that, while at low frequencies - which are inherently more physical and measurable - there are more simply degenerate states than non-simply degenerate, asymptoticly it appears as if only about 28% of harmonic frequencies are simply degenerate.

While this result is interesting, I am not sure it is significant. Acoustic frequencies with magnitudes measured in megahertz do not exist, the linear models of acoustics break down before megahertz frequencies. This result might be applicable to electromagnetic radiation or observable of a quantum particle in a square well, however I am not sure what observables would arise in a simply degenerate eigenstate versus a multiply degenerate eigenstate.

It is possible that in a quantum system, the density of these states - as measure by the ratio of simply degenerate states - could have some significance, especially as a limit as the value of the observable approaches infinity. In this case it is interesting that at quantum levels, the density of simply degenerate states is much greater than at much larger and more classical levels.

VI. CODE APPENDIX

Below is the Mathematica code used to count degenerate states and run the regression. The function call simpleDegenerate[n] returns:

- $\{n, 1, 1\}$ if n is uniquely expressed as the sum of a unique pair of squares.
- $\{n, 1, 0\}$ if n is not uniquely expressible as the sum of a unique pair of squares.
- $\{n, 0, 0\}$ if n is not expressible as the sum of two squares

```
simpleDegenerate = Compile[{{n, _Integer}},
   Module[{factors, f, f4, c1, c2, c3},
    factors = FactorInteger[n];
    c1 = 0;
    c2 = Mod[Sqrt[n], 1] == 0;
    c3 = True;
    Do[
     f4 = Mod[f[[1]], 4];
     If[f4 == 1, c1 += f[[2]]];
     If[f4 == 3, c3 = c3 && EvenQ[f[[2]]]];
     , {f, factors}];
    {n, If[c3, 1, 0],
     If[c3 && ((c1 == 1 && c2 == False) || (c1 == 0 && c2)),
      1, 0]}
    ]];

sdn = ParallelTable[simpleDegenerate[n], {n, 1, 10^7}];
sdnR = Transpose[{sdn[[;; , 1]],
    N[Accumulate[sdn[[;; , 3]]]]/N[Accumulate[sdn[[;; , 2]]]]}];

(* Drops the first cutoffN values of n *)
cutoffN = 1000;
model = NonlinearModelFit[sdnR[[cutoffN ;;]],
   a (n + s)^(-g) + e, {{a, 1}, {s, 0}, {g, 0}, {e, 0}}, n,
   MaxIterations -> Infinity, ConfidenceLevel -> 0.99];
model["ParameterConfidenceIntervalTable"]
```

The NonlinearFit function is probably not ideal to use with a dataset of 10^7 points, and it should be noted that the confidence intervals and standard error parameters are distorted with this magnitude of points.

VII. REFERENCES

[1] M. Schroeder, “Circularly polarized acoustic fields: the number theory connection,” Acta Acustica united with Acustica **75**, 94–98 (1991).