Consequences of special relativity.

The "length" of moving objects.

Recall that in special relativity, "simultaneity" depends on the frame of reference of the observer. Thus, we are allowed to synchronize only those clocks which are at rest with respect to one another (and, for the moment at least, at rest in an inertial frame). Now, what do we mean by the "length" of an object? If the object is at rest in our reference frame, there is no problem: we simply mark off the points of our frame with which its ends coincide, and determine the length $\Delta x \equiv L$ between them, which by hypothesis is independent of time. Call this the "proper" length.

What if the object is moving with respect to us? We then need to extend the *definition* of (apparent) "length". The most natural definition is this: Consider two lights, say, attached to the front and back of the moving object. Suppose they each emit a sequence of (arbitrarily closely spaced) flashes, so that two of these flashes ("events") occur simultaneously *as judged by us.* Then we *define* L_{app} as the spatial separation judged by us to occur between these two "simultaneous" events. I.e. "length is the distance between events occurring at the front and back *at the same time*". Now we can apply the Lorentz transformation directly: let Δx, Δt be the separation of the two events as judged by an observer sitting on the bar, (note Δt is unknown so far!) and $\Delta x'$, $\Delta t'$ be the separation judged by us, then we can apply the Lorentz transformation in reverse: if v is the velocity of the object relative to us, then $-v$ is our velocity relative to it, so (applying a Lorentz transformation with velocity $-v$)

$$\Delta x = \frac{\Delta x' + v\Delta t'}{\sqrt{1 - v^2/c^2}}$$

and another equation for Δt which we do not need for present purposes. However, $\Delta t'$ is by construction zero, and Δx is the distance between the front and back events as judged by an observer with respect to whom the rod is at rest, i.e, exactly what we mean by the "proper" (true) length L. Thus,

$$\Delta x \equiv L = \Delta x'/\sqrt{1 - v^2/c^2}, \text{ or since } L_{\text{app}} \equiv \Delta x'$$

$$\boxed{L_{\text{app}} = L\sqrt{1 - v^2/c^2}}$$

Thus, "moving rods appear shorter"-the celebrated *Lorentz contraction.* Note this was also obtained in the Lorentz theory in which the contraction is a real physical effect of motion through the ether (see Lecture 11): in special relativity it is simply a consequence of the revised definition of simultaneity.

The *"pole-in-barn" paradox*: A man carrying a $20'$ pole rushes into a $15'$ barn at 0.8 of the speed of light, so that $\gamma(v) \equiv 1/\sqrt{1 - v^2/c^2}$ is 5/3. Does the pole fit into the barn? According to an observer at rest with respect to the barn,

$$L_{\text{app}} = L/\gamma(v) = \frac{3}{5}L = 12'$$

i.e. the pole looks only $12'$ long, so it does fit into the $15'$ barn. But according to an observer traveling with the pole (who is an equally good inertial observer!), the pole is its original length, $20'$, and it is the barn which is contracted (to $3/5 \times 15' = 9'$)! Thus the pole certainly does not fit into the barn. Who is right?

Answer: both, or neither! The point is that the concept "fit into" implicitly requires a definition of simultaneity: if A and B are events occurring "simultaneously" at the back and front ends of the pole, and C and D events occurring simultaneously with A and B at the near and forward ends of the barn, then the statement that "the pole fits into the barn" is equivalent to the statement that there exists a time t such that for events occurring at that time,

$$x_C \leq x_A, \quad x_B \leq x_D$$

and this statement is *not Lorentz invariant**, since "simultaneity" depends on the reference frame. (Note the smallness of the effect: for a spaceship, the escape velocity from Earth $v_{\text{esc}} \approx 11\,\text{km/sec}$, so at this velocity the difference between $(1 - v^2/c^2)^{-1/2}$ and 1 is only a factor of about 10^{-9}).

Time dilation.

Consider two events associated with the same physical object and occurring at the same point with respect to it, e.g. two successive ticks of a clock. The "proper time" elapsed between these two events is defined to be the time difference as measured in the frame with respect to which the clock is at rest ("traveling with the clock"). How will these events be separated in time as judged by an observer with respect to whom the clock is moving at speed v? We now apply the Lorentz transformation directly

$$\Delta t' = \frac{\Delta t - v\Delta x/c^2}{\sqrt{1 - v^2/c^2}}$$

but $\Delta x = 0$ by construction, hence

$$T_{\text{app}} = T_{\text{proper}}/\sqrt{1 - v^2/c^2} > T_{\text{proper}}$$

i.e. moving clocks appear to run slow! ("Fitzgerald time dilatation": again, asserted in pre-relativistic theory as a "physical" effect of motion relative to the aether.) This prediction is experimentally verified by observing the apparent rate of decay of muons incident on Earth with velocities comparable to c; the rate is appreciably slower than that of the same muons when at rest in the laboratory.

The "twin" paradox: Imagine two identical twins, say Alice and Barbara. Alice stays at home (assumed in this context to be an inertial frame!): Barbara embarks in a spaceship, accelerates to a high velocity v, travels for a long time at that velocity, then switches on the rocket engines to reverse her velocity, returns to earth and finally

*A statement is said to be "Lorentz invariant" if its truth (or falsity) is independent of the inertial frame in which it is asserted.

decelerates to rest to join Alice. When they compare notes, have they aged the same amount? (Assume that Barbara's biological processes are not affected *physically* by the process of acceleration and are determined by the *proper* time elapsed, that is, the time measured by a clock traveling with her).

This is a slightly tricky problem, because it is tempting to argue that there must be exact symmetry between A and B: according to B, it is A who has traveled and returned, so if A can legitimately say to B "you don't look a day older!" B should equally well be able to say the same to A, hence they must have aged at the same rate. But this argument is fallacious: Alice has not at any point accelerated relative to an inertial frame, whereas Barbara has. Thus there is no a priori reason why we should not get an asymmetry. Alice is certainly an inertial observer, so we can trust her conclusions. Once Barbara is well under way at a steady velocity v, she (Alice) can certainly argue that Barbara's clock runs slow compared to her own, by a factor $\gamma^{-1} = \sqrt{1 - v^2/c^2}$. But what of the periods when Barbara is accelerated? Alice can argue that even if there is an effect associated with these periods (actually there isn't) it should be independent of the total time elapsed (i.e, of the length of the constant-velocity phase), and hence should be negligible in the limit $T \to \infty$. Thus B *really has* aged, on her return, less than A by a factor $\sqrt{1 - v^2/c^2}$.

$\star$ Problem: why can't Barbara make the same argument about the effects of (her own) acceleration?

Experimental confirmation of (something related to) the twin paradox: clocks carried around the world in an airliner. ($v/c \sim 10^{-6}$!)

Relativistic Doppler effect

Suppose a source S and a receiver R are in uniform relative motion with velocity v (of S away from R). The source emits flashes (or crests ofa light wave, etc.) at frequency ν_0 as measured in its own frame. What is the frequency "seen" by R? I.e. at what frequency does R *receive* them?

According to R, if $T_0 \equiv 1/\nu_0$ is the period between flashes as seen by S, then R "sees" them *emitted* at intervals[†] ($\Delta x \equiv 0$, $\Delta t \equiv T_0$)

$$T' = T_0/\sqrt{1 - v^2/c^2}$$

and moreover he sees that their spatial separation $\Delta x'$ (if S is moving away) as $+vT_0/\sqrt{1 - v^2/c^2}$. Consequently, the time interval between *receipts* is

$$T_{\text{rec}} = T' + \frac{\Delta x'}{c} = T_0 \, \frac{1 + v/c}{\sqrt{1 - v^2/c^2}} = T_0 \sqrt{\frac{1 + v/c}{1 - v/c}}$$

[†]Primed (unprimed) variables refer to quantities measured in the frame of $R(S)$.

and so (for a source moving away)

$$\boxed{\nu = \nu_0 \sqrt{\frac{1 - v/c}{1 + v/c}}} \tag{1}$$

(and for motion "towards", $v \to -v$). Note that the nonrelativistic formula $\nu = \nu_0 \frac{1+v_\mathrm{r}}{1+v_\mathrm{s}}$ (Lecture 9) agrees with this for $v_\mathrm{r}, v_\mathrm{s} \ll c$ (in this limit the $\frac{1+v_\mathrm{r}}{1+v_\mathrm{s}}$ is approximately $1 - \frac{v_\mathrm{s}-v_\mathrm{r}}{c} = 1 - v/c$, which is also the approximate value of $\sqrt{\frac{1+v/c}{1-v/c}}$ for $v \ll c$). The best-known application of the result (1) is to the famous "Hubble red shift" in astronomy.

Minkowski space: space-time diagrams

Digression: rotation of coordinate systems in ordinary Euclidean space. To set up a coordinate system in ordinary 3D space, we must do two things:

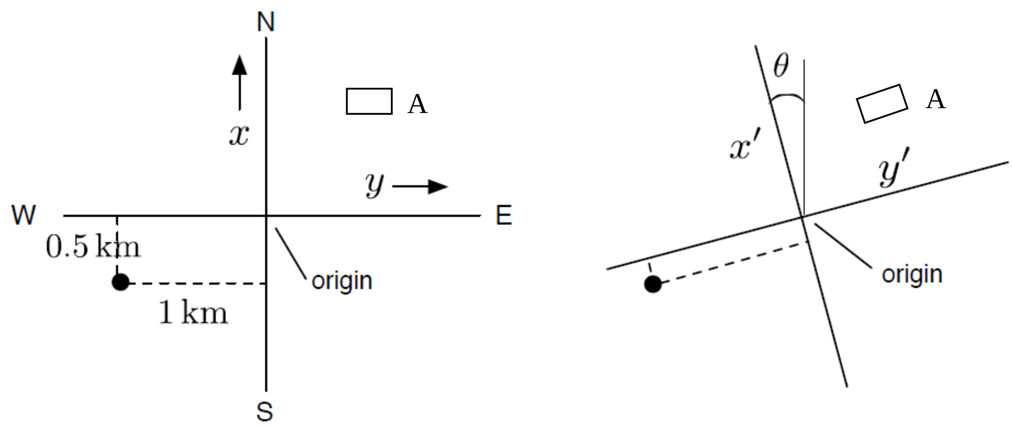
(1) choose an "origin" of coordinates – e.g. the intersection of University and Race at ground level,

(2) choose a system of three mutually perpendicular axes, e.g. $x =$ NS, $y =$ EW, $z =$ vertical.

Then (e.g.) my position now is approximately (-0.5 km, -1 km, -2 m). From now on I neglect the z-coordinate. My *distance* from the origin is $\approx \sqrt{(-1)^2 + 0.5^2} \approx 1.1\,\mathrm{km}$.

Suppose now that we decide to keep the origin fixed but make a new choice of axes, which must however remain mutually perpendicular. E.g. leaving z fixed, we choose a new x-axis at an angle θ to the W of the original x-axis (N): to preserve perpendicularity we must then have the new y-axis θ N of E .

It is clear that my x- and y-coordinates are now changed[‡]:



[‡]Technically: $x' = x\cos\theta + y\sin\theta$, $y' = y\cos\theta - x\sin\theta$.

my new x-coordinate (x') is still negative but less than x, and my y-coordinate (y') is somewhat increased. But obviously my distance from the origin (which is unshifted!) has not changed. This is guaranteed by Pythagoras's theorem, which tells me that

$$s = \sqrt{x^2 + y^2} = \sqrt{x'^2 + y'^2}$$

and one can verify explicitly that indeed $x^2 + y^2 = x'^2 + y'^2$. Note also that the above rotation is area-preserving: $A = \Delta x \Delta y$, but equally $A = \Delta x' \Delta y'$[§]. Neither coordinate system is "privileged", each is as good as the other.

Now consider the possibility of treating time on an equal footing with the space coordinate (consider one space dimension for simplicity). We start with a given reference system and consider an event with space coordinate x, at time t. Under a Lorentz transformation to a moving coordinate system (keeping the origin fixed) we have

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \quad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

The first thing we must do is to measure time in units of (distance/c) (or vice versa, distance in terms of $c \times$ time): in these units v is measured in units of c. It is then tempting to think of a Lorentz transformation as a "rotation" of our space-time coordinate system. However, there is an important difference, because of the fact that the Lorentz transformation formulae for both x and t contain minus signs. As a result, we do not have

$$x^2 + t^2 = x'^2 + t'^2$$

but rather

$$x^2 - t^2 = x'^2 - t'^2$$

Then are two obvious ways of handling this difference so as to make an analogy with spatial rotation.

(a) We can introduce in place of t the "imaginary" coordinate $\tau \equiv it$ when i as usual stands for "$\sqrt{-1}$"; then everything is in exact analogy with spatial rotations. This is convenient for formal calculations but doesn't help much with intuition.

(b) We can introduce the following transformation: introduce an angle θ by[¶]

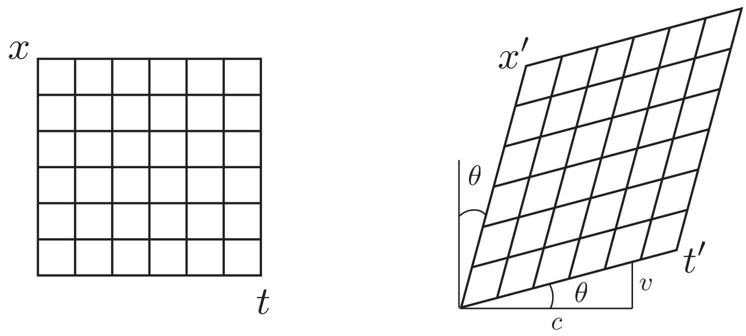
$$\tan\theta = v/c$$

then rotate the x and t axes *towards* one another, each by the angle θ. Note when $\theta = 45°$, i.e. $v = c$, the two new x' and t' axes coincide).

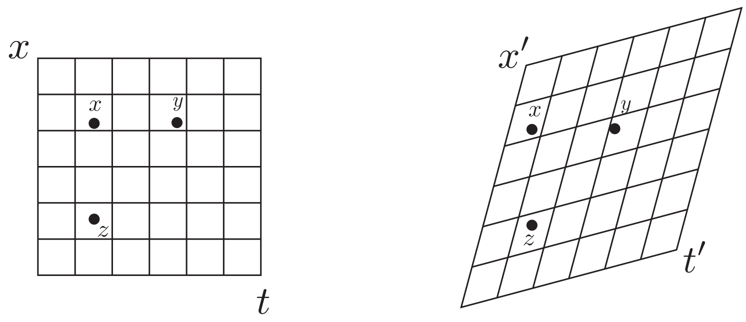
(Note: This does not correspond to $x' = x\cos\theta - y\sin\theta$, etc., because of the scale factor, see below.)

[§] The area of the block marked A cannot be affected by tilting it so that the sides are parallel to the x' and y' axes.

[¶] The function $\tan\theta$ is essentially the slope of a line oriented at angle θ to the horizontal (cf. diagram). If you are unfamiliar with trigonometric functions, just look at the diagram!



Clearly, two events (x, y) which are at the same point in space ($x_1 = x_2$) in the old diagram are not so in the new one, and vice versa.



But equally, events which are simultaneous in S will not be simultaneous in S'.

There is one important catch about the use of this diagram: If we simply mark off, as "unit distance" and "unit time", *the same* interval on the x', t' axes as on the x, t ones, it is easy to see that

(a) area is not preserved, i.e. $A \neq \Delta x' \Delta y'$ and

(b) $\Delta s'^2 \neq \Delta s^2$ (this is easy to see because as $\theta \to 45°$, there is no distinction between $\Delta x'$ and $\Delta t'$ so $\Delta s'^2 \to 0$).

It can be verified that remedying (a) also remedies (b), i.e. we must mark off the "unit intervals" in S' so that unit interval corresponds to unit area ($A = \Delta x' \Delta y'$). It is clear that since the angle of the parallelograms is 2θ, this requires us to expand the "unit interval" along the $\Delta x'$ and $\Delta t'$ axes by a factor $1/\sqrt{\cos 2\theta} \equiv \sqrt{(1 + v^2/c^2)/(1 - v^2/c^2)}$. Generally speaking, this kind of Minkowski space-time diagram is useful for qualitative visualization but rather less so for quantitative calculations.

Addition of velocities in special relativity

Suppose that system S' moves in the positive x-direction relative to S with velocity u, and S'' moves in the positive x-direction relative to S' with velocity v. With what velocity does S'' move relative to S?

At first sight the answer is obvious: with velocity $(u+v)$! But then, e.g. if $u = v = 0.8c$, $(u+v) > c$, and so the denominator in the Lorentz transformation formulae would be imaginary. So something is wrong.[‖]

Let's consider 2 events as observed from S, S', and S'', and apply the Lorentz transformation explicitly: If their space and time separations in S are respectively Δx, Δt then as observed from S' they are

$$\Delta x' = \frac{\Delta x - u\Delta t}{\sqrt{1-u^2/c^2}}, \quad \Delta t' = \frac{\Delta t - u\Delta x/c^2}{\sqrt{1-u^2/c^2}}$$

Then, applying the Lorentz transformation between S' and S'', we have

$$\Delta x'' = \frac{\Delta x' - v\Delta t'}{\sqrt{1-v^2/c^2}} = \frac{\Delta x(1+uv/c^2) - (u+v)\Delta t}{\sqrt{(1-u^2/c^2)(1-v^2/c^2)}}$$

$$\Delta t'' = \frac{\Delta t' - u\Delta x'}{\sqrt{1-v^2/c^2}} = \frac{\Delta t(1+uv/c^2) - (u+v)\Delta x/c^2}{\sqrt{(1-u^2/c^2)(1-v^2/c^2)}}$$

Consider the special case $\Delta x'' = 0$, i.e. we consider the origin of the S'' coordinate system as viewed from S. It clearly moves at a speed w given by

$$\boxed{w = \frac{\Delta x}{\Delta t} = \frac{u+v}{1+uv/c^2}}$$

Thus, it seems plausible that the speed of S'' relative to S is w, and indeed it turns out with a bit of simple algebra that $\Delta x''$, $\Delta t''$ are related to Δx, Δt precisely by a single Lorentz transformation with velocity w.

Since if $u < c$ and $v < c$ then $(u+v)/(1+uv/c^2)$ is always $< c$,[**] it is consistent to assume that the speed of light is a limiting velocity for the relative motion of any two inertial frames. Indeed, it is easy to see that if u and v both approach c then w also approaches c (but never quite gets there). We can indeed never "catch up" with light!

Future, past and elsewhere in special relativity

Consider two events E_1, E_2 with arbitrary spacetime separation. We recall that the quantity

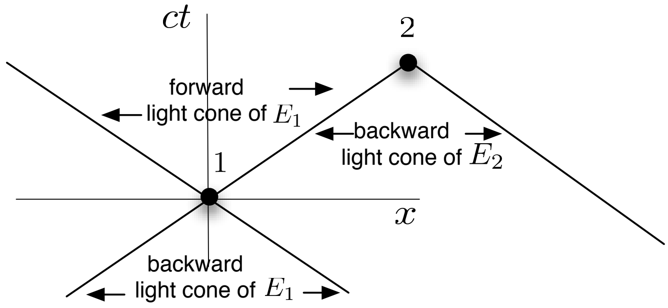
$$\Delta s^2 \equiv \Delta x^2 - c^2\Delta t^2$$

is Lorentz-invariant, that is, it is reckoned to be the same by all observers. But, since Δx and Δt can be anything, Δs^2 can be positive, negative, or 0. What is the significance of this?

[‖] We know we "cannot catch up with light", since c is the same in all frames of reference.

[**] $1 + \frac{uv}{c^2} - \frac{u+v}{c} = \left(1 - \frac{u}{c}\right)\left(1 - \frac{v}{c}\right) \geq 0$

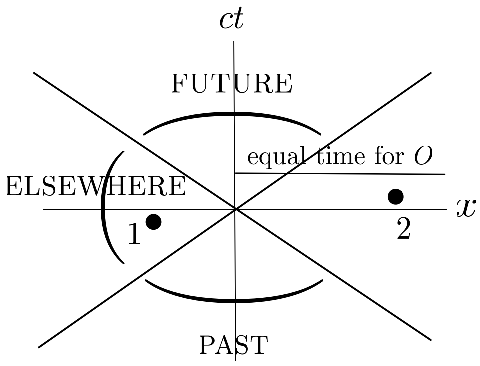
(a) Suppose $\Delta x = \pm c\Delta t$ exactly. Then a light signal sent from event 1 will exactly reach event 2 (i.e. it will reach the space point x at the right time t). In view of the invariance of Δs^2, this is true for all observers. The two events are in this case said to be "light-like separated", or "on one another's light cones". (See graph on the next page). E_2 is on the forward light cone of E_1, and E_1 is on the

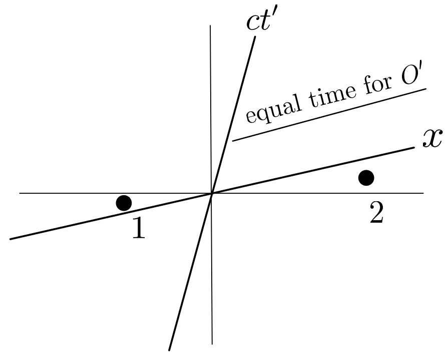


backward light cone of E_2. Note that all observers will agree on the sign of Δt $\left(\Delta t' = \frac{1-v/c}{\sqrt{1-v^2/c^2}} \text{ and } v < c\right)$ (and also, assuming one dimension, on the sign of Δx)

(b) Now suppose $|\Delta x| < c\Delta t$. In this case $\Delta s^2 < 0$ for all observers. It is straightforward to show that we can find a reference system in which $\Delta x = 0$, i.e. the events occur *at the same point* at *different times*. Also Δx can have either sign depending on the observer. The sign of Δt is still unique: if one observer sees E_1 occurring before E_2, so will any other. E_1 and E_2 are said to be "timelike separated".

(c) Finally suppose $|\Delta x| > c\Delta t$. Since now $\Delta s^2 > 0$ for all observers, there is no Lorentz transformation which will put us in a frame in which $\Delta x > 0$. The two events are said to be "spacelike separated". For such events, *different observers may disagree about the time order.*





It is clear that for observer O, event E_2 is later than for E_1, but for O' the reverse is true.

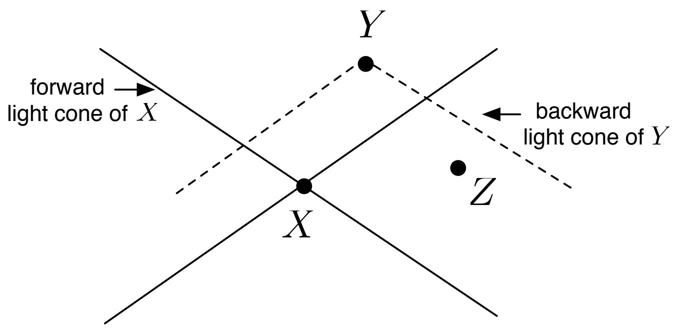
Causality

The word "causality" (or sometimes, "local causality") has a special sense in special relativity. It may embrace a weaker or stronger statement:

(1) Events which lie outside one another's light cones cannot be causally connected (i.e. E_1 cannot exert a causal influence on E_2, nor vice versa).

(2) One event (E_1) can influence another (E_2) only if E_2 lies within or on the future light cone of E_1 .

Note these two claims are actually of a rather different nature: one is independent of the "direction of time", the other depends on it. However, if we accept that causality is "transitive", i.e. if X causes Y and Y causes Z than X causes Z, then by inspection of the diagram a violation of (2) will imply a violation of (1), i.e. (1) implies (2).



(Here Y is in the forward light cone of X, and Z is in the backward light cone of Y, but Z is spacelike separated from X. If not only can X causally influence Y but Y can causally influence Z, and causality is transitive, then X can causally influence Z, i.e. violation of (2) implies violation of (1).)