General relativity: equivalence principle and space-time curvature.

The origins of general relativity can be traced to two considerations which were well known to Galileo 300 years earlier, but for 250 or so of them don't seem to have been regarded as worthy of special attention:

(1) (In the absence of air resistance, etc.): All bodies, irrespective of their weight, composition, shape etc., fall at the same rate in a gravitational field (inertial mass = (passive) gravitational mass: compare the Eötvös and "5th-force" experiments).

(2) There exists (in both Galilean mechanics and special relativity) a "special" class of reference frames, namely the inertial frames.

Before special relativity-in the latter half of the nineteenth century, Ernst Mach took the second point seriously and asked: What makes the "special" class of inertial frames special? Weinberg's "pirouette" experiment and Newton's "bucket" experiment both demonstrate that there does appear to be something "special" about the frame in which the so-called "fixed" stars appear to be at rest. However, Mach objected that no-one really knew what would happen, if in the latter, the thickness of the walls of the bucket were increased to several kilometers: "no-one is competent to say …". He postulated that inertial frames are determined by the rest frame of the (local) matter of the universe ("Mach's principle") .

Connected with this is the so-called "principle of gravitational induction". Circulating electric currents produce a magnetic field, which "favors" rotation in the sense that the magnetic force can be removed by going to a rotating frame. Can we do the same for gravity? The gravitational force between two static bodies (the analog of the Coulomb force in electromagnetism) has the form $F_{\mathrm{stat}} = G\,\frac{m_1 m_2}{r_{12}^2}$: suppose there also exists the analog of the Ampere force between currents, i.e. $F_{\mathrm{dyn}} = G\,\frac{m_1 m_2}{r_{12}^2}\,\frac{v_1 v_2}{c^2}$.

Now, we know that in a rotating system of coordinates (e.g. on a merry-go-round) there exists, in addition to the "centrifugal" force, a "'Coriolis" force proportional to velocity.

$$F_{Cor} = 2m\omega v$$

where ω is the angular velocity of rotation; this Coriolis force is in a direction perpendicular both to the axis of rotation and to the velocity. Is it possible that this is just F_{dyn} in disguise? Well, suppose that in the formula for $F_{dyn}$the subscript 1 refers to "us" (i.e. a massive object on Earth) and 2 refers to the fixed stars, and we assume for the moment that we are at their "center"; then, if we go into a frame of reference rotating with angular velocity ω, the fixed stars will appear to move with respect to us with velocity $v_2 = \omega r_{12}$ where r_{12} is some kind of average radius. Hence the "gravito-magnetic" force F_{dyn} would be given (at least in order of magnitude) by

$$F_{dyn} \sim \frac{Gm_1 m_2 v_1 \omega r_{12}}{r_{12}^2 c^2} \quad = \quad \frac{Gm_1 m_2 v_1 \omega}{r_{12} c^2}$$

In the same notation the Coriolis force is (leaving out the 2)

$$F_{Cor} \sim m_1 v_1 \omega$$

So if the two are to be equal we must have

$$\frac{Gm_2}{r_{12}c^2} \quad \sim \quad 1.$$

But to the extent that we can define the "radius" $R \sim r_{12}$ of the Universe, its mass m_2 is $\sim \rho R^3$, where ρ is the average mass density: hence we would predict that

$$G\rho R^2/c^2 \quad \sim \quad 1$$

Remarkably, the cosmological evidence does seem to suggest that to the extent that R can be defined (cf. lecture 26) this is about right (to an order of magnitude!). The direction of F_{dyn} is also that of F_{Cor} (though this may not be immediately obvious).

Einstein postulated: No *mechanical* experiment in a freely falling elevator can reveal the fact that it is falling rather than floating freely in space in the absence of gravity. Alternatively, one can consider the "box-with-rope" experiment (Einstein p. 66): an inhabitant of the box will experience the acceleration as *equivalent to a gravitational field.*

Generalization: *All* physical phenomena in a freely falling frame are identical to what they would be in free space ("equivalence principle"). Conversely, all (local) effects of acceleration can be attributed to a gravitational field. Thus, the role of *inertial* frames in special relativity is replaced by *freely falling* frames in general relativity. (Note: it is not trivial, but nevertheless true, that any "smooth" local frame can be transformed into a freely falling one).

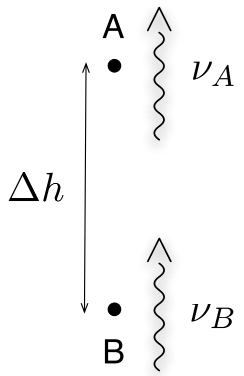
So the prescription is: to work out how things happen *locally*, go over to a freely falling frame, do the calculations as in Minkowski space, then transform back to the original frame!

Immediate applications:

- Gravitational redshift.

  Imagine a source B, emitting some kind of periodic signal (one usually thinks of a light wave, but it need not be), which is picked up by A who is sitting in a gravitational potential which is higher by an amount $g\Delta h$. Suppose A is using a standard "clock" (e.g. a cesium atom) and B compares the frequency of the signal as reported to him by A with that of his own exactly similar standard clock. Obviously, if no gravitational field were involved they should see the same.

  To find the correct result we must visualize the situation as viewed by a freely falling observer. He sees A and B accelerating *upwards* with acceleration g. Suppose we



arrange (as we certainly can) that he is stationary with respect to B at the moment B emits the signal. Then if the signal is received (according to him) by A at a time Δt later, by that time A will be moving upwards with velocity $v = g\Delta t$, and hence will see the signal Doppler-shifted downwards in frequency by an amount which (for $v \ll c$) is $-v/c = -g\Delta t/c$. Now if v is indeed $\ll c$, then Δt is approximately $\Delta h/c$, and thus:

$$\frac{\Delta\nu}{\nu_0} \simeq -g\Delta h/c^2 = -\Delta\phi/c^2$$

which is just the difference in gravitational potential divided by c^2. So, A will see the signal emitted by B "*redshifted*"*. This prediction has been verified, e.g. in the Harvard Tower experiment (and also in astrophysics).

The general formula for the redshift, under conditions when the difference in gravitational potential $\phi_A - \phi_B$ is not necessarily small, turns out to be

$$\nu_A/\nu_B = \sqrt{\frac{1 + 2\phi_B/c^2}{1 + 2\phi_A/c^2}}$$

I quote this result† without proof because it will be useful later.

- Bending of light (qualitative).

  Imagine a source of light, a screen with a slit, and a fixed screen on which the light is received, all inside an elevator. According to the principle of equivalence, if (and only if) the elevator is in free fall, the light will appear to travel in a straight line, i.e. it will traverse the spatial points S, B, C. But for an observer fixed on the earth, these points at the times the light traverses them will *not* appear to lie in a straight line: in fact, the light ray will appear to accelerate downwards, with acceleration g! Thus, light should be bent in the field of massive bodies like

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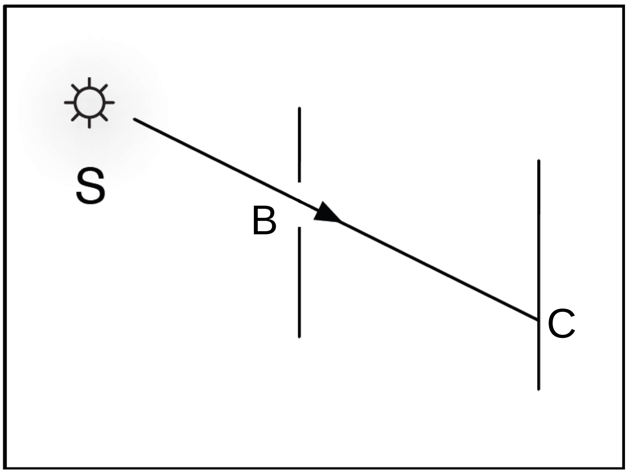
*Caution: This has nothing to do directly with the cosmological redshift.

†The alert reader will notice that if the difference $\phi_A - \phi_B$ is small but ϕ_B is not itself small, this result does not reduce to the one stated above. I hope to have time to discuss this; it is associated with the question of the choice of an "absolute" zero for the gravitational potential.



the Sun. (This effect was already predicted, on the basis of a "particle" picture of light, by Sollner in 1802: but as we shall see there is a catch!)

Digression:

Why are these effects not seen in everyday life? Answer: Generally they are of order ϕ/c^2 which is small for any reasonable "everyday-life" situation (For the Harvard Tower experiment ($g \sim 10\,\text{m/s}^2$, $\Delta h \sim 70\,\text{ft.}$, $c^2 \sim 10^{17}\,\text{m/s}$) it is of order of 10^{-15}! The predicted order of magnitude of the bending of light by the Sun is ~ 1 second of arc. One might reasonably ask what happens when ϕ/c^2 becomes of order 1: See below.

Geodesics

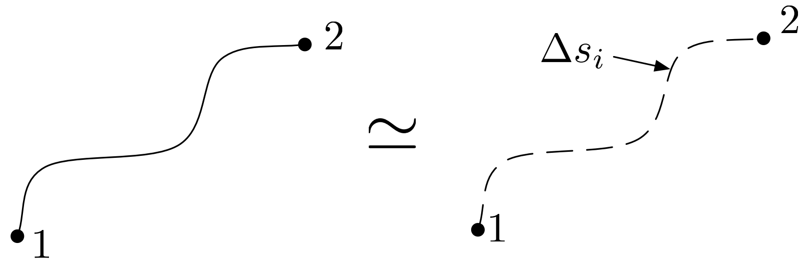
A statement which is true for both massive bodies and light, in both Newtonian physics and special relativity, is the following: (Newton's first law, plus)

As viewed from any *inertial* frame, a body on which no forces act travels in a straight line with constant velocity: so does light. Special relativity adds the statement that the velocity of light is always isotropic and equal to c.

A way of reformulating this statement, which at first sight (only!) looks rather artificial, is the following: Define the (Minkowskian) *space-time interval* between any two events in the standard way (for 3 space dimensions)

$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2}$$

We know that this (the numerical value of Δs^2) is agreed upon by all inertial observers. Consider now a body which starts at point (x_1, y_1, z_1) at time t_1 and ends at (x_2, y_2, z_2) at time t_2, but can follow any path between these two spacetime points. Suppose it does follow some convoluted path (I draw only the "space" part!) and break it up into small



parts of length of order ϵ: define a "total spacetime path length" as

$$\Delta s = \lim_{\epsilon \to 0} \sum_i \Delta s_i, \quad \Delta s \equiv \sqrt{\Delta x_i^2 + \Delta y_i^2 + \Delta z_i^2 - c^2 \Delta t_i^2}$$

Then the above statement of Newton's first law is simply: The actual path followed between spacetime points 1 and 2 is such that the total path length Δs is a minimum (such a path is called a "geodesic"). In addition, for light Δs is zero.

Let's verify that this does give back Newton's first law. First, assume that we "forget about" the time variable: then we simply have to minimize the actual path length in space between the two points. But the shortest path between two points in (Euclidean) space is precisely a straight line! Hence part (1) of Newton's first law follows, and we have (e.g. in two dimensions) simply

$$\Delta y_i = \text{const} \, \cdot \, \Delta x_i$$

When we know x_1, y_1 and x_2, y_2 the constant is of course uniquely determined as $(y_2 - y_1)/(x_2 - x_1)$.

To investigate the second part, let's assume we choose our coordinate system so that motion is only in the x direction. Then, we have

$$\Delta s_i = \sqrt{\Delta x_i^2 - c^2 \Delta t_i^2}$$

This expression differs from the "Pythagorean" one only by the factor of c^2 and the $-$ sign: both can be formally removed by defining a "pseudo – Euclidean" coordinate $\tau \equiv ict$, so $\Delta s_i = \sqrt{\Delta x_i^2 + \Delta \tau_i^2}$. It is then clear from the analogy to ordinary Euclidean space that the optimum path is the "straight line" $\Delta x = \text{const}\ \Delta\tau$ or $\Delta x = \text{another const.}\ \Delta t$, i.e.

$$(x_2 - x_1) = \text{const}\ (t_2 - t_1) \equiv v(t_2 - t_1)$$

which is just the statement that the body moves at constant velocity. In the special case of light (in special relativity) we know that this velocity has to be c, and we therefore recover the result $\Delta s = 0$.

So far, so good: we have really learned nothing we didn't know. But we note a great advantage of the "geodesic" formulation: it is totally independent of what coordinate system we choose to employ! (we can not only use different inertial frames, but having chosen a frame can use e.g. spherical polar coordinates instead of Cartesian ones). Thus, we can for example choose to use the frame of reference appropriate to an observer who is stationary on the earth's surface. This is *not* a freely falling frame, and hence we expect that in general the trajectories of "free" (i.e, not acted on by any nongravitational forces) bodies would not be straight lines and/or would not be traversed at constant velocity: this is, of course, what we observe. However, the Minkowskian expression for the space-time interval, $\Delta s^2 = \Delta x^2 - c^2\Delta t^2$, automatically leads to linear motion with constant velocity: thus, *the expression for the interval in this noninertial frame must be non-Minkowskian.*

Let's assume for the sake of simplicity that since the horizontal motion is not affected, the bit of Δs^2 which is $\Delta x^2 + \Delta y^2$ is not changed. Then, since the situation is independent of time (and xy-coordinates), the most general expression we are allowed to have is

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + f(z)\Delta z^2 - g(z)c^2\Delta t^2$$

Now, the redshift experiment immediately allows us to infer $g(z)$. For consider the time interval between two flashes emitted by B, as observed by A and B. Both agree that the x, y and z coordinates of the two events are the same, so $\Delta x^2 = \Delta y^2 = \Delta z^2 = 0$. But they disagree about Δt: if B (who is actually carrying the clock emitting the flashes) sees an interval τ, then A reckons the interval to be $\tau\sqrt{\frac{1+2\phi_A/c^2}{1+2\phi_B/c^2}}$. If, then, they are to measure the same value of the space-time interval, we must have

$$g(z_B)\tau^2 = g(z_A)\,\frac{1+2\phi_A/c^2}{1+2\phi_B/c^2}\,\tau^2$$

i.e.

$$g(z) = \frac{\text{const}}{1+2\phi(z)/c^2}$$

Since the constant only defines the unit of time, we can set it equal to 1. Thus,

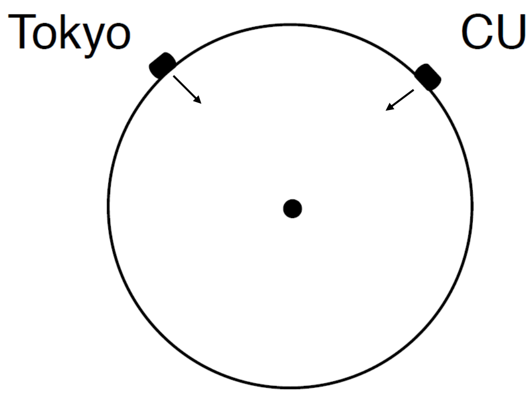
$$\Delta s^2 = \Delta x^2 + \Delta y^2 + f(z)\Delta z^2 - \frac{c^2\Delta t^2}{1+2\phi(z)/c^2}$$

We do not yet know what $f(z)$ is: let us for the moment tentatively set it equal to 1. A standard variational calculation* then shows that provided $|\phi(z)| \ll 1$, an observer using this expression and requiring that Δs be a minimum will indeed see the trajectories predicted by "Newtonian gravity".

Now, it might be objected that all of the last two pages is a sledgehammer to crack a nut. After all, in this case it is perfectly straightforward to transform to a freely falling frame (which is the same everywhere in space), simply posit that in this frame free bodies move in straight lines, and then transform back to the frame of the stationary observer. To see the advantage of the "metric" (or "geodesic") formulation we have to consider a somewhat more complicated problem, e.g. the gravitational field of the spherical earth.

*which however requires more math than assumed in the course.

Since the direction of the gravitational acceleration is *different* in Tokyo from that in Champaign-Urbana, it is clear that it is impossible to transform *globally* to a single freely falling (inertial) frame, and thus we would expect that there exists no frame in which motion is globally in a straight line. We *must* then use the more general expression, with $\phi(z) \to \phi(r) = -GM/r$. This begins to suggest that space-time might be "intrinsically curved".



Digression on curved space (–time)

We know that the three-dimensional space around us is (at least to a very good approximation) Euclidean. But imagine for a moment that we were ants crawling on the two-dimensional surface of a spherical football, and had no means of knowing about the direction perpendicular to the surface. We would then measure all distances as "distances traversed along the surface", and we could find a number of "non-Euclidean" features:

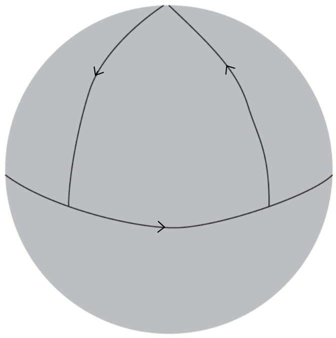
1. The angles of a triangle need not add up to 180° (consider e.g. a triangle based on the North pole and the equator).

2. The circumference of a circle is in general less than $2\pi \times$ its radius.

3. More than one "straight line" can be drawn through the same two points.

4. The space is in an obvious intuitive sense "closed" (in fact its area is $4\pi R^2$, though the ants would not know that R is a "radius").

5. "Parallel transport is nontrivial": if e.g. I start from the North Pole, carrying a stick which always points in the some direction as viewed in the local coordinate system (e.g. always points south), walk as far as the equator, turn east and go along the equator for a few thousand miles, then turn back N. and return to the North pole — is the direction of the stick the same as it started?



Now of course *we* know, though the ants don't, that the two-dimensional surface is "embedded" in a three-dimensional Euclidean continuum. But in fact it is possible to describe the "intrinsic" properties of the surface *independently* of this knowledge; the ants could infer everything about it by making sufficiently careful measurements, and indeed there are some two-dimensional "surfaces" which *cannot* be embedded in Euclidean space. So the question is: Is our "four-dimensional" space-time in some sense "curved", and if so, what is it that determines the degree and nature of the curvature?

Einstein's answers were as follows: (1) yes; (2) the presence of mass. In fact, in a frame of reference where all mass is stationary, one can say crudely that

curvature is proportional to mass density

It should be emphasized, here, that the "curvature" which occurs in this statement is the appropriately defined curvature of *four-dimensional spacetime*. Thus this is non-zero inside (e.g.) the earth (mass density $\neq 0$) but zero in the free space outside. It then turns out (lots of math!) that this determines the function $f(r)$ to be the inverse of $g(r)$, that is, the correct expression for the metric outside a spherical massive body as a function of r (independent of t!) is[§]

$$\Delta s^2 = \Delta s_\perp^2 + \left(1 - \frac{2GM}{rc^2}\right)\Delta r^2 - \frac{c^2 \Delta t^2}{1 - \frac{2GM}{rc^2}}$$

– the famous *Schwarzschild metric.* Note that the (appropriately defined) four dimensional curvature of this metric is zero. However, let us consider the three dimensional space obtained by "slicing" at a definite time (for the observer in question) (i.e, considering events with $\Delta t = 0$). It turns out that this" three dimensional space *does* have a finite curvature! This is no different from the fact that in "everyday life", while ordinary three dimensional space is (very nearly) Euclidean (zero curvature), the two dimensional subspace corresponding to a "slice" at $r =$ const (e.g. the surface of a football) has finite curvature.

[§]Here $\Delta s_\perp^2$ the "distance on the surface of the sphere" (technically, in spherical polar coordinates we have $\Delta s_\perp^2 = r^2(\Delta\theta^2 + \sin^2\theta\Delta\phi^2)$).

Some consequences of the Schwarzschild metric:

1. bending of light in the field of the Sun (the result turns out to be a factor of 2 times the result of the "naïve" (Sollner-type) argument.): This was verified in the famous Royal Society expedition to West Africa to observe the 1919 eclipse.

2. precession of perihelion of Mercury (space curvature contributes 1/2 of the effect of the redshift).

3. effects in radar echoes between planets.

One obvious question: What happens if we have so large a mass, in so small a radius, that $2GM/rc^2 = 1$? (so that the redshift becomes "imaginary")? Answer: a black hole!

Application to cosmology: Suppose the Universe is filled with some kind of matter whose average density, in its rest frame, is ρ. Then at "radius" R we have $2GM/R = (8\pi/3)\, G\rho R^2 \equiv \Omega$. It looks therefore as if it ought to "matter" whether the actual "radius" of the Universe[‖] is such that this parameter is greater or less than 1. In fact, this is true: in the simplest model of the Universe (the so-called Friedmann-Robertson-Walker model, see lecture 26), we have the following situation

$\Omega < 1$ the Universe is a space of constant negative curvature, and is infinite. ("open" Universe)

$\Omega = 1$ the Universe is a space of flat space-time, infinite. ("flat" Universe)

$\Omega > 1$ the Universe is a space of constant positive curvature and is hence the three dimensional analog of a spherical surface. ("closed" Universe)

What is the experimental value of Ω? (See lecture 26)

Finally, a note: If the mass is stationary, the freely falling bodies have no "rotation". However, if the mass is *rotating*, in general they do: there is a tendency to "drag" projectiles into rotation though it is generally not complete. (More in next lecture.)

[‖]Or more generally the "scale factor" (see lecture 26).