Philosophical Issues in Newtonian Mechanics

The Relation of Mathematics to Physics

It was Galileo who said, "The book of nature is written in the language of mathematics", but Newton who was the first to put this dictum into effect quantitatively. It isn't clear that even in the early 21st century the symbiosis is yet perfect, but physics and mathematics are indeed so intimately intertwined that most physics text books don't distinguish consciously between those steps in their reasoning which follow by purely mathematical deduction and those which require some physical intuition or assumption (in an ideal world, they would be required to use different types for these two kinds of steps!)

What is the nature of the mathematical "truth"? The Greeks were, reasonably, impressed by the certainty of mathematical demonstration, as exemplified par excellence in Euclidean geometry, and it is not surprising that this made them excessively fond of the idea that one could obtain, if not complete knowledge about the physical world, at least to a very high degree of understanding of it from purely a priori considerations. (This applies particularly to Pythagoras, Plato, and their disciples). In modern times one can perhaps distinguish three major classes of view:

1. David Hume held that all the truths of mathematics are analytic, that is, true by virtue of the meaning of the concepts which enter them. (The traditional paradigm of a (non-mathematical) analytic statement is one such as "All bachelors are unmarried", which clearly conveys no new information to anyone who understands the standard meaning of the term bachelor).

2. John Stuart Mill held, on the contrary, that mathematics tells us true empirical facts about the world that it is justified by that alone. In other words, the truths of mathematics are "synthetic" (the opposite of "analytic"), but by that same token are not necessarily infallible.

3. To Immanuel Kant, mathematics was the paradigm of a kind of knowledge that he called "synthetic a priori"; that is, knowledge which, because it is an intrinsic consequence of the way in which we perceive the world, can be known to be true without detailed inspection of the external world, but which is nevertheless more than simply a reflection of the meaning of the concepts involved. One of Kant's favorite examples of a synthetic a priori statement about the physical world is that it satisfies the axioms of Euclidean geometry-this is as we shall see when we come to general relativity, in retrospect, an unfortunate choice.

A possible, somewhat commonsensical compromise view, is that while the truth of mathematical statements is self-contained and needs no reference to the physical world to verify it, such statements are only useful in the context of physics if there happen to exist classes of objects in the world which obey the fundamental axioms of the mathematical structure in question. For example (Hawkins), in a world which consisted only of clouds it is unlikely that we would find the concept of the natural numbers very useful; conversely, we should not be particularly surprised if, having postulated a set of axioms which we believe to be in fact satisfied by a class of objects in the physical world which is of interest to us, we find that these axioms are also satisfied by other classes of objects which may or may not correspond to something "out there" and/or of practical interest (Hawkins, The Language of Nature, pp. 20-1, gives the example of the Peano axioms for the natural numbers).

Ideas of Space and Time

(*Sklar*, pp. 15-25)

A major subject of controversy in the late seventeenth century was the nature of space and time. As we have seen, (1.6), Newton was a firm believer both in the idea of absolute space ("the sensorium of God") and that of absolute time. In contrast, his German contemporary Leibniz (a coinventor with Newton of the branch of mathematics we now know as calculus) held a "relational" view of both; space is essentially nothing more than a set of relations between material objects, and time similarly a set of relations between events. This clash of viewpoints was made explicit in the famous correspondence between Leibniz and Newton's disciple Clarke.**

Evidently each point of view faces some obvious problems. While Newton him-self evidently believed that the Universe is infinite in spatial extent, this view was not universal among his contemporaries, and if it is finite the obvious difficulty is: what determines the position in "absolute" space at which the Universe sits? Leibniz presses this issue§ using two principles, which he calls "identity of indiscernibles" and "sufficient reason". The first states that if two objects are absolutely identical in all their properties (hence, "indiscernible", i.e., impossible to tell apart) then they must be the same thing; there are no two different physical objects which are exactly identical (and hence, according to Leibniz, no "atoms" in the sense of Leucippus and Democritus). Thus, the different parts of Newton's "absolute space", which are presumably identical in all respects, must in fact be the same space. The second principle, that of "sufficient reason" states that "in things absolutely indifferent, there is no [foundation for] choice... since this choice must be founded on some reason, or principle". Thus, Leibniz seems to be arguing, even if the different regions of Newton's absolute space could somehow (contrary to the first principle) be truly different, God would have had no reason to place the Universe in one region of this space rather than another. This kind of argument may not at first sight look that convincing, but we shall see later that it has some remarkably useful applications in modern quantum and statistical physics statistical physics). Even for a Newtonian, the problem still arises in a somewhat milder form: what determines where in infinite space the solar system sits? On the other hand, the Leibnizian view has its own problems: (a) what is the status of "times" at which no event occurs, or points in space at which there is no material object – do we have to introduce the idea of "potential" events or bodies? (b) if (e.g.) space is a set of relations between material bodies, what is it exactly that constrains this set of relations (e.g. so that Pythagoras' theorem is obeyed)? It is ironical that with hindsight Newtonian mechanics seems to be more naturally formulated in relationist rather than an absolutist framework.

Space and Time Coordinates

It's convenient at this point to introduce the idea of an "event" and its time and space "coordinates". (This idea, although certainly implicit in the work of Newton and his successors, was only made completely explicit by Einstein two centuries later; but it makes the subsequent discussion a bit easier). An "event" is simply some "happening" which occurs at a definite time and place: e.g. the firing of a gun, a proposal of marriage, the triggering of a Geiger counter by a cosmic ray . .. and so forth. (Obviously a certain amount of idealization is involved here, as so often in physics: each of these "events" in real life lasts a finite time and cannot be exactly localized in space, but for purposes of illustration they will do). Having unambiguously identified the "event" of interest, we

**For a readable summary of the correspondence as it pertains to this issue, see H. Erlichson, *Am. J. Phys.* 35, 89 (1967).

§ Fourth letter, paras. 1-6.

now want to specify its time and space "coordinates" – that is, give numbers which tell us exactly when and where it occurred.

To this end we introduce the idea of a "standard clock" and a "standard ruler". The standard clocks used in the history of physics have been of various types: the King's heartbeat, a sandglass, a pendulum clock, an atomic maser .... In just about every case there are some problems: for example, if a pendulum clock is mover to an upstairs room, it will go slightly slow as compared to a clock of identical construction downstairs.* Let us imagine that we have solved this kind of problem by some arbitrary rule, e.g. that the standard clock must operate at sea level. Then we can define the time differences between two events 1 and 2 (to a good approximation) simply as the number of ticks of the clock between 1 and 2. An immediate (and apparently trivial) consequence of this definition is that if the time difference between event 1 and the (later) event 2 is say 5 minutes, and the time difference between 2 and the (later) event 3 is 6 minutes, then automatically the time difference between 1 and 3 is 11 minutes; or more generally and formally, the time difference Δt_{ij} between two event i and j satisfies the simple additivity relation

$$\Delta t_{31} = \Delta t_{32} + \Delta t_{21} \qquad (*)$$

Notice that in the above discussion we have nowhere assumed that the two events which are compared occur at the same point in space; they may or may not. If they do not, then there may be some practical difficulties in evaluating the time difference, but these can apparently be overcome. In particular there is nothing (so far!) to prevent us establishing that two events occur at different places at the same time.

If we wish to define not only the difference in time between two events, but also the absolute time of a single event, then in addition to our standard clock we need to choose some standard origin of time. This choice is purely conventional, and while most people nowadays use a reckoning of hours, minutes, and seconds with the "zero" at midnight, the "origin" of the year still differs throughout the world (the U.S. and Europe uses the birth of Christ, the Muslim world the Hegira, and the Japanese the accession of the current Emperor).

Next consider the choice of a "standard ruler" to define spatial coordinates. For the moment let us restrict ourselves to events occurring at the same time. Then we could for example choose our "ruler" the King's foot, the standard meter rod in Paris or the wavelength of the light emitted in a particular atomic transition. Again, the conditions need to be carefully specified: e.g. the meter rod must be kept at a certain temperature, humidity, etc. However, given these conditions the length of the ruler is taken by definition to be independent of e.g. the time of day or its orientation (the direction in which it lies). We can now define, approximately, the space interval (distance) between two events occurring simultaneously by laying (a set of replicas of) our standard ruler between them and seeing how many we need. Do we have additivity similar to that for time intervals? If the three events 1,2,3 which we consider happen to lie in a straight

*(For the cognoscenti only): This effect (or at least the major part of it) has nothing to do with general relativity; it is simply a consequence of the fact that the gravitational acceleration decreases with distance from the Earth's center.

line, the answer is yes: if event 2 is to the right of 1 by 2 meters and 3 is to the right of 2 by 3 meters, then 3 is to the right of 1 by 5 meters, or more formally

$$\Delta x_{31} = \Delta x_{32} + \Delta x_{21} \qquad \text{(one dimension)}$$

However, even in this case there seems intuitively to be a difference between time and space: somehow, "later" and "earlier" seem more fundamental categories than "to the right of" or "to the left of"! This may be because by changing our point of view (i.e., moving in real 3-D space) we can reverse the "left-right" distinction, whereas no such reversal seems possible for the "earlier-later" one. But this already indicates that it is essential to generalize the notion of space coordinates.

The world we live in is three-dimensional (or at least appears to be so!). This is equivalent to saying that we need three independent numbers to specify the position of an event. At this point it is convenient to proceed in a different order from that used above for the time, and already define an origin of space coordinates. Just as in the case of time, this is purely conventional; geographers generally use the North Pole, but for local purposes in Urbana the "conventional" choice is the intersection of Race and University. Having chosen our origin, we now set up a "grid", that is a set of imaginary rods parallel to three mutually perpendicular directions. What do we mean when we say two direction are "mutually perpendicular"? We can define this by drawing a circle and dividing the circumference into four equal parts with marks; the lines from the center to two neighboring marks then define mutually perpendicular axes. An apparently equivalent alternative definition is via Pythagoras' theorem: two axes are mutually perpendicular if and only if, when we construct a triangle by taking unit intervals along the two axes, the length of the hypotenuse is $\sqrt{2}$. (Actually, as we shall see when we come to general relativity, the two definitions are equivalent in general only under the assumption that the geometry of space is "Euclidean"; fortunately, even in general relativity this is true in the limit is infinitesimal triangles). The "absolute" orientation of the three axes, as distinct from their mutual perpendicularity, is a matter convention: in ordinary life, one usually chooses one axis to be vertical (i.e. parallel to the local gravitational field) and the two horizontal ones to be north-south (N-S) and east-west (E-W).

Having chosen our standard ruler and origin and set up our "grid", we now define the space coordinates x, y, z, of a given event simply as the number of standard rulers we need to place along each (arbitrarily labeled) axis to reach from the origin to the event in question. Thus, for example, if we choose our origin at ground level at the intersection of Race and University and arbitrarily choose x as the "N-S" coordinate, then the "event" which is taking place in this lecture room now would have space coordinates approximately given by

$$x = -500\text{m}, \quad y = -1500\text{m}, \quad z = -5\text{m}$$

Note we still have additivity for each of the coordinates separately (e.g. if 1 is 2m W of 2 and 2 is 3m W of 3, then irrespective of their N-S and vertical coordinates 1 is 5m W of 3!) i.e. we still have equation (∗) and also

$$\Delta y_{31} = \Delta y_{32} + \Delta y_{21}, \quad \Delta z_{31} = \Delta z_{32} + \Delta z_{21}.$$

However, let's consider the total distance ΔS_{ij} between two events (or for that matter two points), assuming for simplicity that they are both on the earth's surface so that $\Delta z_{ij} = 0$. We have by Pythagoras' theorem

$$\Delta S_{ij} = \sqrt{(\Delta x_{ij})^2 + (\Delta y_{ij})^2}$$

and it is easy to convince oneself that ΔS_{ij} does not satisfy the additivity relation (e.g. if Loomis Lab is 500m from the Union and the Union is 500m from the Library, it does not follow that Loomis is 1000m from the Library-this would be true only if the three buildings were in a straight line).

Choice of Reference Frame

So far, so good; the definition of the space coordinates of simultaneous events appears to involve no particular difficulties. But what about events which occur at different times? So long as we are content always to refer events to the same reference frame (e.g. that whose origin is fixed at ground level, at Race and University, with axes vertical, N-S and E-W) we can proceed just as above and there is no extra difficulty. However, there are cases in which we may not wish to do this.

Imagine a train raveling at a constant velocity† (speed) v, and a passenger who drops two stones successively at an interval of ten seconds from the same window. How will these two events be described (a) by an observer at the side of the track, and (b) by the train passenger himself?

(a) The observer at the side of the track will say "The first stone was dropped at 12 o'clock at this crossing; the second stone was dropped at 10 seconds past 12 o'clock, some distance up the line". How far up? Well, if for example the train is traveling at 30m/sec, then it will be 300m up. More generally, the distance between the two events as measured by a "ground" observer (call it Δx_{gr}) will be

$$\Delta x_{\text{gr}} = (\text{distance moved by the train relative to the ground}) = v\Delta t$$

where Δt is the time interval between the two events. (b) One the other hand, the natural frame of reference for the passenger himself is that fixed on the train, i.e. he will say (unthinkingly, perhaps) "I dropped stone 1 at 12 o'clock, and stone 2 ten seconds later, from the same point". I.e. the distance between the two events as measured by someone on the train, Δx_{tr}, is simply zero.

More generally, suppose the passenger had walked a distance Δx (say 10m forward up the train between dropping the stones. Then obviously we will have

$$\Delta x_{\text{tr}} = \Delta x \quad (= 10\text{m})$$

†What exactly, does this mean? We can, if necessary, define velocity by the ensuing argument, see below. For the moment we will take the concept as intuitively obvious.

whereas the observer on the ground will now say that the distance $\Delta x_{\rm gr}$ between the two events is 300m+10m=310m. More generally we have

$$\Delta x_{\rm gr} = \Delta x_{\rm tr} + (\text{distance moved by the train})$$

and it is clear that this formula is valid whether or not the train is moving with constant velocity. Suppose it is, then we have

$$\Delta x_{\rm gr} = \Delta x_{\rm tr} + v_{\rm tr}\Delta t \quad (v_{\rm tr} = \text{constant})$$

Note that we can now, if we wish, as it were reverse the argument and define the velocity of the train $v_{\rm tr}$ by the statement that if $\Delta x_{\rm tr} = 0$ (stones dropped from the same window), then $v_{\rm gr} \equiv \Delta x_{\rm gr}/\Delta t$. (cf. last lecture.)

It is clear that this argument is a very general one and it applies to any two observers who are moving with constant relative velocity: If A is moving with constant velocity v relative to B, and we denote quantities as observed by A by unprimed symbols and those observed by B by primed ones, then we have quite generally

$$\Delta x' = \Delta x + v\Delta t.$$

Note carefully that it is implicit in the argument that both observers measure the same time interval between the events, i.e., formally (and apparently trivially!)

$$\Delta t' = \Delta t.$$

The transformation between frames of A and B given by the two boxed equations is called a Galilean transformation.

One immediate and apparently trivial consequence of the above arguments, is the additivity of velocities (or more precisely components of velocity).[‡] It is more or less intuitively obvious that if the train is traveling at 30m/sec, and I walk forward along the train at 1m/sec, then the observer on the ground sees me traveling at 31m/sec. More generally and formally: Suppose that a given body (e.g. me) is present at two events (e.g. the two stone droppings). We can then define my velocity as seen by a particular observer as the ration of the space interval Δx measured by that observer to the time interval Δt measured by him. But, from the Galilean transformation, the relation between the rations seen by the "primed" and "unprimed" observer is

$$\frac{\Delta x'}{\Delta t'} \equiv \frac{\Delta x'}{\Delta t} = \frac{\Delta x + v\Delta t}{\Delta t} = \frac{\Delta x}{\Delta t} + v$$

i.e. my velocity as seen by B is that seen by A plus the velocity of A relative to B, or

$$v_{\rm B} = v_{\rm A} + v_{\rm A\,rel\,to\,B}.$$

Note that this relation applies to the different components of velocity. It does not in general apply to the total velocity; e.g. if a boat is traveling at 4 mph relative to shore, and I walk at 3 mph across the boat, my velocity relative to the shore is not 7 mph (actually in this particular case it is 5 mph by Pythagoras' theorem).

[‡]That it did not seem "trivial" to all of Newton's contemporaries is indicated by the fact that he felt it necessary to spend some time on this point (*Principia*, p. 7).

Galilean Invariance

We now return to Newton's laws, taking for granted (a) the definition of a common "time" which can be agreed upon by all observers, and for the moment also (b) that we have an independent criterion (independent, that is, of the laws themselves) of when the external forces are acting or not. (E.g. with a good approximation we can assume that for a puck moving horizontally on smooth ice, there is no appreciable horizontal force).

For a body whose mass is constant in time and on which no external force acts, Newton's first law simply states

$$\text{velocity} = \text{constant}.$$

But we didn't yet specify our reference frame! In fact, the velocity will be different as viewed from different frames. However, it is easy to see that as long as those frames are moving uniformly with respect to one another, this doesn't matter. Recall the result

$$v_{\mathrm{B}} = v_{\mathrm{A}} + v_{\mathrm{A\,rel\,to\,B}}$$

Suppose then that the velocity seen by A is constant, and moreover A is moving uniformly relative to B, i.e. $v_{\mathrm{A\,rel\,to\,B}} = \text{const}$. Then it immediately follows that B also sees a velocity which, though different from that seen by A, is also constant. Thus, if Newton's first law is valid in some frame of reference S, then it is also valid in any other frame moving with constant velocity relative to S, i.e. related to it by a Galilean transformation. Thus, the first law defines not a single unique reference frame, but a class of frames, which we call inertial. We postpone for a moment the question of what exactly it is which picks out the class of inertial frames.

It is now immediately clear that not only Newton's first law but also his second is invariant under Galilean transformation. If we assume that the mass of the body in question is constant (true in the overwhelming majority of cases of practical interest), then N2 can be written

$$\text{acceleration} \equiv \frac{\Delta v}{\Delta t} = \frac{\text{applied force}}{\text{mass}} \qquad (\star)$$

Now from the above relation $(\star)$ the changes over the time interval Δt of the velocity as seen by different observers are related by

$$\Delta v_{\mathrm{B}} = \Delta v_{\mathrm{A}} + \Delta(v_{\mathrm{A\,rel\,to\,B}})$$

However, if A is moving uniformly relative to B the last term is by definition zero, so $\Delta v_{\mathrm{B}} = \Delta v_{\mathrm{A}}$ and the acceleration seen by B is the same as that seen by A. Thus, if Newton's second law is valid in some frame S, it is equally valid in any frame related to S by a Galilean transformation; thus N2, like N1, defines a class of frames. Since N1 is a special case of N2, the class so defined must be the inertial frames. Notice however that there is a crucial assumption implicit here, namely that the applied force seen by A is the same as that seen by B. For forces of the type given as examples by Newton (percussion, air pressure . . .) this assumption gives rise to no particular difficulty; electromagnetism, as we shall see in lecture 10, is a different matter.

Finally, let's consider Newton's third law: if the masses of the bodies involved are constant, then this can be written (Δv now indicates the change in velocity either over a given time interval, or in a complete collision)

$$m_1 \Delta v_1 = -m_2 \Delta v_2$$

Because Δv_1 and Δv_2 (as distinct from v_1 and v_2 themselves) are unaffected by Galilean transformations, this means that if N3 is valid in a frame S then it is valid in any frame moving uniformly with respect to S. Again, for consistency it is necessary to choose the class of frames as defined as the inertial frames.

Thus, the laws of nature (N1-3) look the same in all inertial frames. This doesn't of course mean e.g. that we can't tell whether we are moving with respect to the earth (we can tell this not only by direct inspection but by effects such as air resistance etc.), but it does mean that when this kind of possibility is shut off, as in Galileo's "ship's cabin" though-experiment, there is no way of telling. On the other hand, if the frame is not inertial (e.g. is rotating) we can most certainly tell: try walking across a merry-go-round!

What exactly is it that makes a particular class of frames the "inertial" frames? According to Newton, it is because there is a particular one of the inertial frames which is that of "absolute space", while later Mach suggested that the inertial frames are defined by the mean behavior of the matter (i.e. stars, etc.) in our neighborhood in the universe. Since Newton believed that the fixed stars are at rest in absolute space, the outcome is the same. More on that later... (Newton's bucket: Weinberg's "pirouette").

Are Newton's Laws Definitions of Force?
(*Hesse*, pp. 134–143)

There are two obvious possibilities.

(a) Suppose that we regard the existence of an externally applied force (e.g. muscular or elastic) as something recognizable independently of its accelerating effect. Then,

1. if the forces in question are between different bodies, then N3 is independent of N1 (which does not by itself state that the total momentum of a system of (interacting) bodies is conserved); thus, N3 has to be regarded as an empirical statement.
2. as regard different parts of the same body (on which no external forces act), N1 as applied to that body implies that N3 applies to interactions (if any) between the parts (otherwise the body as a whole would accelerate even though no externally applied forces act on it, contrary to N1).

(b) Alternatively, we can regard N1 (and N2) as definitions, i.e. any acceleration is to be automatically regarded as evidence for the existence of force. In this case it is an empirical fact that not only systems of bodies but also single bodies whose parts are interacting satisfy conservation of total momentum, i.e. that N3 applies.

Hesse (*loc. cit.*) argues that Newton took viewpoint (b): for our purposes (a) is more convenient. If so, then clearly we need (inter alia) to ask: "What is the (identifiable) force that is responsible for the vertical acceleration of bodies on earth, and for the acceleration of the planets?" i.e. we must postulate a gravitational force.

The Gravitational Force and Action at a Distance

The feature of Newton's mechanics which both he and his contemporaries liked least was that the gravitational force is supposed to act at a distance, and moreover instantaneously. The whole idea of action at a distance, instantaneous or not, was strongly resisted by 17th century thinkers; it was associated in their minds with "occult properties", and where such effects seemed to be indubitably observed, as in magnetism, people went to extreme lengths to try to provide a "local" physical mechanism (i.e. one which worked by contact-cf. Aristotle!), e.g. Descartes with his "vortex" theory.

Newton himself certainly didn't believe in action at a distance:

> "It is inconceivable, that inanimate brute matter should, without the mediation of something else, which is not material, operate upon, and affect other matter without mutual contact .... And this is one reason, why I desired you would not ascribe innate gravity to me. That gravity should be innate, inherent, and essential to matter, so that one body may act upon another, at a distance through a vacuum, without the mediation of anything else, by and through which their action and force may be conveyed from one to another, is to me so great an absurdity, that I believe no man who has in philosophical matters a competent faculty of thinking, can ever fall into it. Gravity must be caused by an agent acting constantly according to certain laws; but whether this agent be material or immaterial, I have left to the consideration of my readers."

and the idea was severely criticized by others, e.g. by Leibniz.

> "A body is never moved naturally, except by another body which touches it and pushes it; after that it continues until it is prevented by another body which touches it. Any other kind of operation on bodies is either miraculous or imaginary."

As late as 1730, Bernoulli won a prize from the (French) Academy of Sciences by explaining Kepler's third law in terms of a "vortex" hypothesis. It was only in the late 18th century, as more and more successes of Newton's law of universal gravitation were chalked up, that people finally came to terms with the idea-only to have it shattered, as we shall see, by later developments.

Determinism

Let's first consider the motion of a single body in a situation when, given its position, we know exactly the forces acting on it. Good approximations include the motion of a cannon ball fired from a gun (the force is simply the gravitational force in the vertical direction due to the earth*), or a planet moving around the Sun (the force is the gravitational force due to the Sun, which as we have seen is in the direction of the latter and proportional to the inverse square of the distance from it). Suppose that at some initial time (call it t_i) we know the exact value of the position of the body and also the exact value of its velocity. For simplicity of notation only I shall consider a one-dimensional motion, and therefore denote the position by x_i and the velocity by v_i. Consider now the situation at a slightly later time, $t_i + \Delta t$ (where eventually we are going to let Δt tend to zero). By hypothesis we know the force F_i acting on the body at point x_i and if Δt is small enough it will still be very close to x_i at $t_i + \Delta t$, so we can write that the acceleration over the time interval Δt is approximately

$$a_i = F_i/m$$

But acceleration is just rate of change of velocity, so the change in velocity over the time interval Δt is just $a_i\Delta t$, and so at $t_i + \Delta t$ we have

$$v_i(t_i + \Delta t) \;=\; v_i + a_i\Delta t \;=\; v_i + (F/m)\Delta t$$

and is exactly known (in the limit $\Delta t \to 0$ when the approximation becomes exact). What about the position? Since the velocity averaged over the interval Δt is close to the initial one v_i, we have approximately for the position at time $t_i + \Delta t$

$$x(t_i + \Delta t) = x_i + v_i\Delta t$$

and again is exactly known. In other words, in the situation considered

If we know the position and velocity of the body exactly at time t_i, then (in the limit $\Delta t \to 0$) we also know it exactly at time $t_i + \Delta t$

It is clear that we can now iterate this argument, and thereby reach the conclusion:

If we know the position and the velocity at some initial time t_i, then we know it at all subsequent times!

If this theorem applied only to single bodies like the cannon ball or the planet, it would perhaps not be very interesting. But it is easy to see that it also applies to an arbitrary collection of bodies (provided that not only the external forces but the forces acting between them are exactly known), and moreover in 3 dimensions just as well as in one: So under the same conditions:

Complete knowledge of the positions and velocity of all the particles of the system at an initial time t_i determines the complete behavior at any later time.

Does this mean that the past uniquely determines the future (but not vice versa)? Well, yes and no! We can just as well run the argument backwards: if we know the values x_f and v_f of position x and velocity v at some "final"time t_f, then we equally know them at a slightly earlier time $t_f - \Delta t$, and so on to an arbitrary previous time. Again, if we know them at some intermediate time, we can determine both the future

*For the moment we will neglect the frictional resistance due to the air, although we shall have to return to this question in a later lecture.

and the past behavior. In fact it even turns out (though this is less obvious) that if we know just the position of the body at any two different times, we can determine the complete behavior. [For cognoscenti only: what we are saying here is that since Newton's second law is second order in time, any two independent pieces of information suffice to determine the complete solution.] Again, these statements generalize to an arbitrary collection of bodies and to 3 dimensions.

So, does the past determine the future? Or the future the past? Or ...