# Consequences of Special Relativity

[Recap of Michelson-Morley experiment, Lorentz transformation, invariance of spacetime interval: $\gamma(v) \equiv 1/\sqrt{1 - v^2/c^2}$.]

## The "length" of moving objects

Recall that in special relativity, "simultaneity" depends on the frame of reference of the observer. Thus, we are allowed to synchronize only those clocks that are at rest with respect to one another (and, for the moment at least, at rest in an inertial frame). Now what do we mean by the "length" of an object? If the object is at rest in our reference frame, there is no problem: we simply mark off the points of our frame with which its ends coincide, and determine the length $\Delta x \equiv L$ between them, which by hypothesis is independent of time. Call this the "proper" length.

What if the object is moving with respect to us? We then need to extend the *definition* of (apparent) "length". The most natural definition is this: Consider two lights, say, attached to the front and back of the moving object. Suppose they each emit a sequence of (arbitrarily closely spaced) flashes, so that two of these flashes ("events") occur simultaneously *as judged by us*. Then we *define* $L_{\text{app}}$ as the spatial separation judged by us to occur between these two "simultaneous" events. I.e. "length is the distance between events occurring at the front and back *at the same time*". Now we can apply the Lorentz transformation directly: let $\Delta x$, $\Delta t$ be the separation of the two events as judged by an observer sitting on the bar, (note $\Delta t$ is unknown so far!) and $\Delta x'$, $\Delta t'$ be the separation judged by us, then we can apply the Lorentz transformation in reverse: if $v$ is the velocity of the object relative to us, then $-v$ is our velocity relative to it, so (applying a Lorentz transformation with velocity $-v$)

$$\Delta x = \frac{\Delta x' + v\Delta t'}{\sqrt{1 - v^2/c^2}}$$

and another equation for $\Delta t$ which we do not need for present purposes. However, $\Delta t'$ is by construction zero, and $\Delta x$ is the distance between the front and back events as judged by an observer with respect to whom the rod is at rest, i.e, exactly what we mean by the "proper" (true) length $L$. Thus,

$$\Delta x \equiv L = \Delta x'/\sqrt{1 - v^2/c^2}, \text{ or since } L_{\text{app}} \equiv \Delta x'$$

$$\boxed{L_{\text{app}} = L\sqrt{1 - v^2/c^2}}$$

Thus, "moving rods appear shorter" – the celebrated *Lorentz contraction*. Note this was also obtained in the Lorentz theory in which the contraction is a real physical effect of motion through the ether (see Lecture 11): in special relativity it is simply a consequence of the revised definition of simultaneity.

The *"pole-in-barn" paradox*: A man carrying a $20'$ pole rushes into a $15'$ barn at 0.8 of the speed of light, so that $\gamma(v) \equiv 1/\sqrt{1 - v^2/c^2}$ is 5/3. Does the pole fit into the barn? According to an observer at rest with respect to the barn,

$$L_{\text{app}} = L/\gamma(v) = \frac{3}{5}\,L = 12'$$

i.e., the pole looks only $12'$ long, so it does fit into the $15'$ barn. But according to an observer traveling with the pole (who is an equally good inertial observer!), the pole is its original length, $20'$, and it is the barn which is contracted (to $3/5 \times 15' = 9'$)! Thus the pole certainly does not fit into the barn. Who is right?

Answer: both, or neither! The point is that the concept “fit into” implicitly requires a definition of simultaneity: if $A$ and $B$ are events occurring “simultaneously” at the back and front ends of the pole, and $C$ and $D$ events occurring simultaneously with $A$ and $B$ at the near and forward ends of the barn, then the statement that “the pole fits into the barn” is equivalent to the statement that there exists a time t such that for events occurring at that time,

$$x_D \leq x_C, \quad x_D \geq x_B$$

and this statement is *not Lorentz invariant,*[*] since “simultaneity” depends on the reference frame. (Who shuts the barn door?) Do note the smallness of the effect: for a spaceship, the escape velocity from Earth $v_{\text{esc}} \approx 11\,\text{km/sec}$, so at this velocity, the difference between $(1 - v^2/c^2)^{-1/2}$ and 1 is only a factor of about $10^{-9}$.

## Time dilation

Consider two events associated with the same physical object and occurring at the same point with respect to it, e.g., two successive ticks of a clock. The “proper time” elapsed between these two events is defined to be the time difference as measured in the frame with respect to which the clock is at rest (“traveling with the clock”). How will these events be separated in time as judged by an observer with respect to whom the clock is moving at speed $v$? We now apply the Lorentz transformation directly (*unprimed frame* = that of clock; *primed one* = ours):

$$\Delta t' = \frac{\Delta t - v\Delta x/c^2}{\sqrt{1 - v^2/c^2}}$$

but $\Delta x = 0$ by construction, $\Delta t = T_{\text{proper}}$, and $\Delta t' = T_{\text{app}}$; hence:

$$T_{\text{app}} = T_{\text{proper}}/\sqrt{1 - v^2/c^2} > T_{\text{proper}}$$

I.e., *moving clocks appear to run slowly!* (“FitzGerald time dilation”: again, asserted in pre-relativistic theory as a “physical” effect of motion relative to the aether.) This prediction is experimentally verified by observing the apparent rate of decay of muons

[*] A statement is said to be “Lorentz invariant” if its truth (or falsity) is independent of the inertial frame in which it is asserted.

incident on Earth with velocities comparable to $c$; the rate is appreciably slower than that of the same muons when at rest in the laboratory.

*The "twin" paradox*: Imagine two identical twins, say Alice and Barbara. Alice stays at home (assumed in this context to be an inertial frame!): Barbara embarks in a spaceship, accelerates to a high velocity $v$, travels for a long time at that velocity, then switches on the rocket engines to reverse her velocity, returns to Earth and finally decelerates to rest to join Alice. When they compare notes, have they aged the same amount? (Assume that Barbara's biological processes are not affected *physically* by the process of acceleration and are determined by the *proper* time elapsed, that is, the time measured by a clock traveling with her.)

This is a slightly tricky problem, because it is tempting to argue that there must be exact symmetry between $A$ and $B$: according to $B$, it is $A$ who has traveled and returned, so if $A$ can legitimately say to $B$ "you don't look a day older!" $B$ should equally well be able to say the same to $A$, hence they must have aged at the same rate. But this argument is fallacious: Alice has not at any point accelerated relative to an inertial frame, whereas Barbara has; thus, there is no a priori reason why we should not get an asymmetry. Alice is certainly an inertial observer, so we can trust her conclusions. Once Barbara is well under way at a steady velocity $v$, she (Alice) can certainly argue that Barbara's clock runs slow compared to her own, by a factor $\gamma^{-1} = \sqrt{1 - v^2/c^2}$. But what of the periods when Barbara is accelerated? Alice can argue that even if there is an effect associated with these periods (actually there isn't) it should be independent of the total time elapsed (i.e, of the length of the constant-velocity phase), and hence should be negligible in the limit $T \to \infty$. Thus $B$ *really has* aged, on her return, less than $A$ by a factor $\sqrt{1 - v^2/c^2}$.

$\star$ Problem: why can't Barbara make the same argument about the effects of (her own) acceleration?

Experimental confirmation of (something related to) the twin paradox: clocks carried around the world in an airliner. ($v/c \sim 10^{-6}$!)

## Relativistic Doppler effect

Suppose a source $S$ and a receiver $R$ are in uniform relative motion with velocity $v$ (of $S$ away from $R$). The source emits flashes (or crests of a light wave, etc.) at frequency $\nu_0$ as measured in its own frame. What is the frequency "seen" by $R$? I.e., at what frequency does $R$ *receive* them?

According to $R$, if $T_0 \equiv 1/\nu_0$ is the period between flashes as seen by $S$, then $R$ "sees" them *emitted* at intervals† ($\Delta x \equiv 0$, $\Delta t \equiv T_0$)

$$T' = T_0/\sqrt{1 - v^2/c^2}$$

and moreover he sees that their spatial separation $\Delta x'$ (if $S$ is moving away) as $+vT_0/\sqrt{1 - v^2/c^2}$.

†Primed (unprimed) variables refer to quantities measured in the frame of $R(S)$.

Consequently, the time interval between *receipts* is

$$T_{\rm rec} = T' + \frac{\Delta x'}{c} = T_0 \frac{1+v/c}{\sqrt{1-v^2/c^2}} = T_0 \sqrt{\frac{1+v/c}{1-v/c}}$$

and so (for a source moving away)

$$\boxed{\nu = \nu_0 \sqrt{\frac{1-v/c}{1+v/c}}} \tag{1}$$

(and for motion "towards", $v \to -v$). Note that in special relativity, the frequency shift cannot depend on the velocity $v_{\rm s}$ of the source and that $v_{\rm r}$ of the receiver separately (velocity relative to what?), but only on their difference $v = v_{\rm s} - v_{\rm r}$, as in equation (1). Note also that the nonrelativistic formula $\nu = \nu_0 \frac{1+\frac{v_{\rm r}}{c}}{1+\frac{v_{\rm s}}{c}}$ (Lecture 9) agrees with this for $v_{\rm r}, v_{\rm s} \ll c$ (in this limit, the $\frac{1+\frac{v_{\rm r}}{c}}{1+\frac{v_{\rm s}}{c}}$ is approximately $1 - \frac{v_{\rm s}-v_{\rm r}}{c} = 1 - v/c$, which is also the approximate value of $\sqrt{\frac{1+v/c}{1-v/c}}$ for $v \ll c$). The best-known application of the result (1) is to the famous "Hubble red shift" in astronomy.

## Addition of velocities in special relativity

Suppose that system $S'$ moves in the positive $x$-direction relative to $S$ with velocity $u$, and $S''$ moves in the positive $x$-direction relative to $S'$ with velocity $v$. With what velocity does $S''$ move relative to $S$?

At first sight the answer is obvious: with velocity $(u+v)$! But then, e.g. if $u = v = 0.8c$, $(u+v) > c$, and so the denominator in the Lorentz transformation formulae would be imaginary. So something is wrong.[‡]

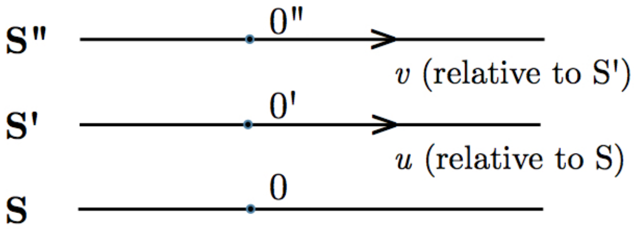



Figure 1: Comparative velocities

What is the velocity of S$''$ relative to S? (Call it $w$.)

Consider (any) two events as viewed from S, S$'$ and S$''$:

[‡]We know that we "cannot catch up with light", since $c$ is the same in all frames of reference.

L.T. (S $\to$ S$'$):

$$\Delta t' = \frac{\Delta t - u\Delta x/c^2}{\sqrt{1-u^2/c^2}}$$

$$\Delta x' = \frac{\Delta x - u\Delta t}{\sqrt{1-u^2/c^2}}$$

L.T. (S$'$ $\to$ S$''$):

$$\Delta t'' = [\text{not needed}]$$

$$\Delta x'' = \frac{\Delta x' - v\Delta t'}{\sqrt{1-v^2/c^2}}$$

$$\equiv \frac{(\Delta x - u\Delta t) - v(\Delta t - u\Delta x/c^2)}{\sqrt{(1-u^2/c^2)(1-v^2/c^2)}}$$

$$\equiv \frac{\Delta x(1+uv/c^2) - (u+v)\Delta t}{\sqrt{(1-u^2/c^2)(1-v^2/c^2)}}$$

Suppose two events are, e.g., light flashes emitted by $0''$, so $\Delta x'' = 0$; then

$$\frac{\Delta x}{\Delta t} = \frac{u+v}{1+uv/c^2}$$

But $\Delta x/\Delta t$ is by definition just $w$, so

$$w = \frac{u+v}{1+uv/c^2}$$

Which is $< c$ for any $u, v < c$.

It is consistent, then, to assume that the speed of light is a limiting velocity for the relative motion of any two inertial frames. Indeed, it is easy to see that if $u$ and $v$ both approach $c$, then $w$ also approaches $c$ – but never quite gets there. We can indeed never “catch up” with light!

## Minkowski space: Space-time diagrams

Digression: rotation of coordinate systems in ordinary Euclidean space. To set up a coordinate system in ordinary 3D space, we must do two things:

(1) choose an “origin” of coordinates – e.g. the intersection of University and Race at ground level,

(2) choose a system of three mutually perpendicular axes, e.g. $x =$ NS, $y =$ EW, $z =$ vertical.

Then (e.g.) my position now is approximately (-0.5 km, -1 km, -2 m). From now on, I neglect the $z$-coordinate. My *distance* from the origin is $\approx \sqrt{(-1)^2 + 0.5^2} \approx 1.1\,\text{km}$.

Suppose now that we decide to keep the origin fixed but make a new choice of axes, which must however remain mutually perpendicular. E.g., leaving $z$ fixed, we choose a new $x$-axis at an angle $\theta$ to the W of the original $x$-axis (N): to preserve perpendicularity we must then have the new $y$-axis $\theta$ N of E .

It is clear that my $x$- and $y$-coordinates are now changed:§

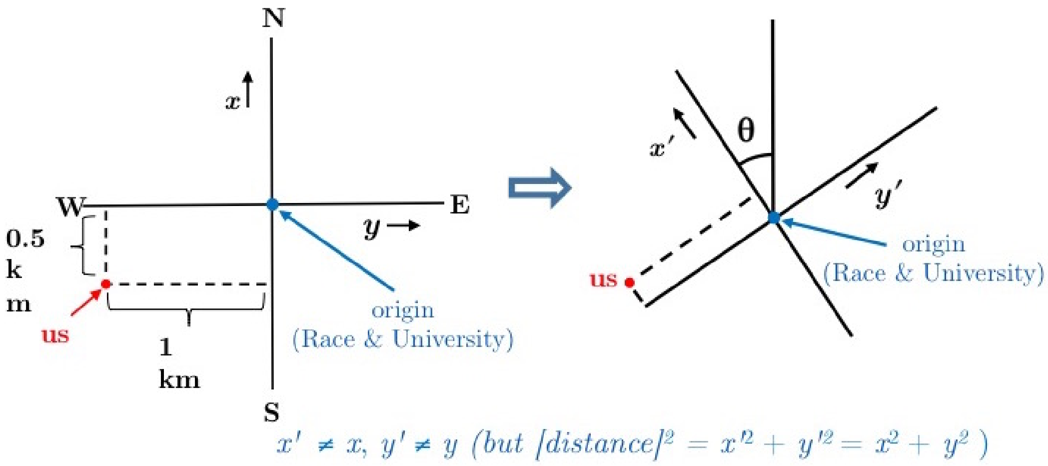



Figure 2: Minkowski diagram
*Analogy: rotation in ordinary [2D] space.*

my new $x$-coordinate ($x'$) is still negative but less than $x$, and my $y$-coordinate ($y'$) is somewhat increased. Moreover, it is clear that in general two points that had the same $x$-coordinate will not have the same $x'$-coordinate, and similarly for $y$ and $y'$. But obviously my distance from the origin (which is unshifted!) has not changed, nor has the distance between any other two points. This is guaranteed by Pythagoras's theorem, which tells me that

$$s = \sqrt{x^2 + y^2} = \sqrt{x'^2 + y'^2}$$

and one can verify explicitly that indeed $x^2 + y^2 = x'^2 + y'^2$. Note also that the above rotation is area-preserving: $A = \Delta x \Delta y$, but equally $A = \Delta x' \Delta y'$.¶ Neither coordinate system is "privileged", each is as good as the other.

Now consider the possibility of treating time on an equal footing with the space coordinate (consider one space dimension for simplicity). We start with a given reference system and consider an event with space coordinate x, at time t. Under a Lorentz

§Technically: $x' = x\cos\theta + y\sin\theta$, $y' = y\cos\theta - x\sin\theta$.

¶The area of the block marked $A$ cannot be affected by tilting it as in the figure so that the sides are parallel to the $x'$ and $y'$ axes.

transformation to a moving coordinate system (keeping the origin fixed) we have

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \quad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

The first thing we must do is to measure time in units of (distance/$c$) (or vice versa, distance in terms of $c \times$ time): in these units $v$ is measured in units of $c$. It is then tempting to think of a Lorentz transformation as a "rotation" of our space-time coordinate system. However, there is an important difference, because of the fact that the Lorentz transformation formulae for both $x$ and $t$ contain minus signs. As a result, we do not have

$$x^2 + t^2 = x'^2 + t'^2$$

but rather

$$x^2 - t^2 = x'^2 - t'^2$$

Then are two obvious ways of handling this difference so as to make an analogy with spatial rotation.

(a) We can introduce in place of t the "imaginary" coordinate $\tau \equiv it$ when $i$ as usual stands for "$\sqrt{-1}$"; then everything is in exact analogy with spatial rotations. This is convenient for formal calculations but doesn't help much with intuition.

(b) We can introduce the following transformation: introduce an angle $\theta$ by‖

$$\tan\theta = v/c$$

then rotate the $x$ and $t$ axes *towards* one another, each by the angle $\theta$. Note when $\theta = 45°$, i.e. $v = c$, the two new $x'$ and $t'$ axes coincide).

(Note: This does not correspond to $x' = x\cos\theta - y\sin\theta$, etc., because of the scale factor; see below.)

Clearly, two events $(x, y)$ that are at the same point in space ($x_1 = x_2$) in the old diagram are not so in the new one, and vice versa. But equally, events that are simultaneous in $S$ will not be simultaneous in $S'$.

There is one important catch about the use of this diagram: If we simply mark off, as "unit distance" and "unit time", *the same* interval on the $x'$, $t'$ axes as on the $x$, $t$ ones, it is easy to see that

(a) area is not preserved, i.e. $A \neq \Delta x' \Delta y'$ and

(b) $\Delta s'^2 \neq \Delta s^2$ (this is easy to see because as $\theta \to 45°$, there is no distinction between $\Delta x'$ and $\Delta t'$ so $\Delta s'^2 \to 0$).

We can fix up both (a) and (b) by an appropriate rescaling of the units of length and time, but it is not worth doing so for present purposes; generally speaking, this kind of Minkowski space-time diagram is useful for qualitative visualization but rather less so for quantitative calculations.

‖The function $\tan\theta$ is essentially the slope of a line oriented at angle $\theta$ to the horizontal (cf. diagram). If you are unfamiliar with trigonometric functions, just look at the diagram!

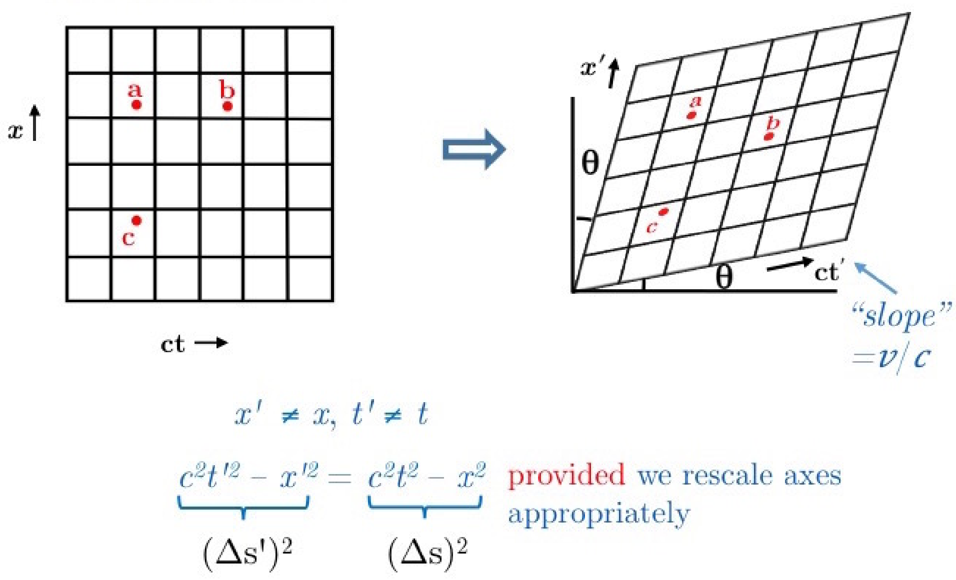



Figure 3: "Minkowski space"

### Future, past and elsewhere in special relativity

Consider two events $E_1$, $E_2$ with arbitrary spacetime separation. We recall that the quantity

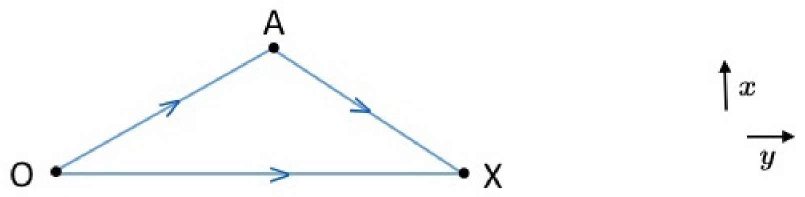
$$\Delta s^2 \equiv c^2 \Delta t^2 - \Delta x^2$$

is Lorentz-invariant, that is, it is reckoned to be the same by all observers. But, since $\Delta x$ and $\Delta t$ can be anything, $\Delta s^2$ can be positive, negative, or 0. What is the significance of this?

(a) Suppose $\Delta x = \pm c \Delta t$ exactly (i.e., $\Delta s = 0$; see figure 5(a)). Then a light signal sent from event 1 will exactly reach event 2 (i.e. it will reach the space point $x$ at the right time $t$). In view of the invariance of $\Delta s^2$, this is true for all observers. The two events are in this case said to be "lightlike separated", or "on one another's light cones". (See graph on the next page). $E_2$ is on the forward light cone of $E_1$, and $E_1$ is on the backward light cone of $E_2$. Note that all observers will agree on the sign of $\Delta t$ $\left( \Delta t' = \frac{(1-v/c)\Delta t}{\sqrt{1-v^2/c^2}} \text{ and } v < c \right)$ (and also, assuming one dimension, on the sign of $\Delta x$).

(b) Now suppose $|\Delta x| < c|\Delta t|$ (see figure 5(b)). In this case, $\Delta s^2 > 0$ for all observers. It is straightforward to show that we can find a reference system in which $\Delta x = 0$, i.e., the events occur *at the same point* at *different times*. Also $\Delta x$ can have either sign depending on the observer. The sign of $\Delta t$ is still unique: if one observer sees $E_1$ occurring before $E_2$, so will any other. $E_1$ and $E_2$ are said to be "timelike separated".

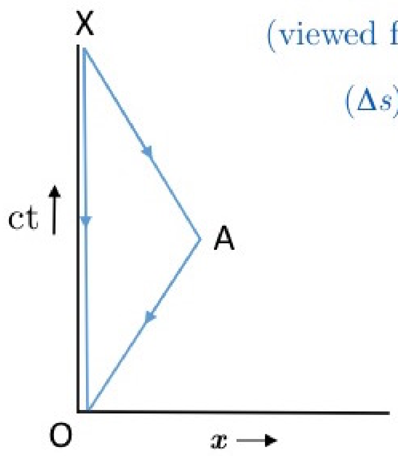
Analogy in ordinary (2D) space (trivial!):




$$(\Delta s)^2{}_{2D} = (\Delta x)^2 + (\Delta y)^2$$

$$\Delta s_{O \to A \to X} = \Delta s_{OA} + \Delta s_{AX} \underset{(>)}{\neq} \Delta s_{OX}$$

Minkowski space:




(viewed from Alice's [inertial] frame)

$$(\Delta s)^2{}_{Min} = c^2(\Delta t)^2 - (\Delta x)^2$$

$$\text{Proper time elapsed} = (\Delta s_{total})/c$$

$$(\Delta s_{total})_{Barbon} = \Delta s_{OA} + \Delta s_{AX} \underset{(<)}{\neq} \Delta s_{OX} = ct$$

Figure 4: The twins paradox in Minkowski space
*(Based largely on A. P. French,* Special Relativity, *pp. 154-9.)*

(c) Finally, suppose $|\Delta x| > \Delta t$. Since now $\Delta s^2 < 0$ for all observers, there is no Lorentz transformation which will put us in a frame in which $\Delta x > 0$. The two events are said to be "spacelike separated". For such events, *different observers* *may disagree about the time order.* (Cf. Zbinden et al. experiment.)

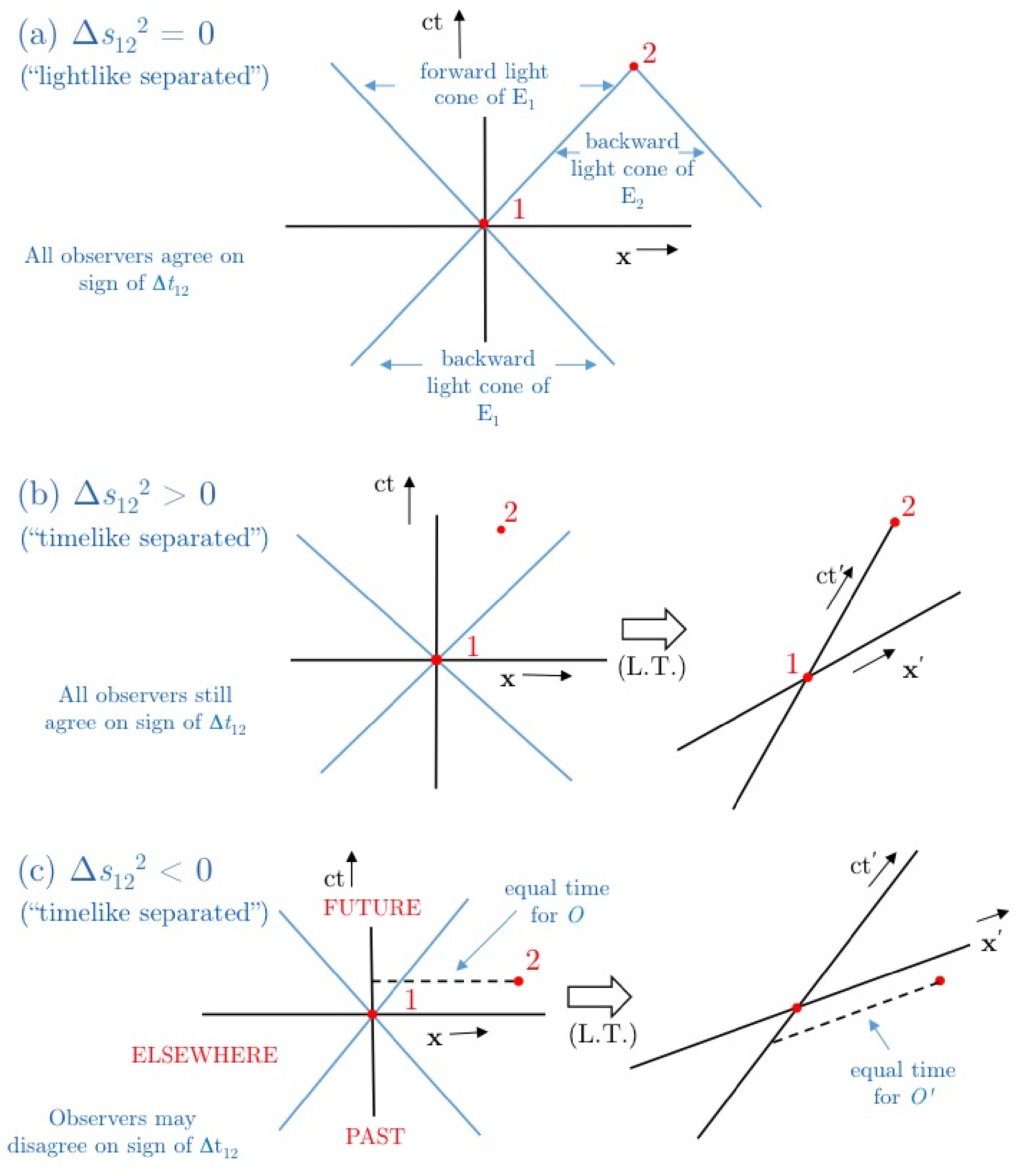



Figure 5

It is clear that for observer $O$, event $E_2$ is later than for $E_1$, but for $O'$, the reverse is true.

## Causality in special relativity

Already before the advent of SR, Maxwell's electromagnetic theory had provided a possible solution to at least part of the problem of action at a distance: in that theory, it is entirely possible to ascribe the interaction of electric charges and currents to the propagation of EM waves between them. Any such interaction would not of course be truly "instantaneous", but would be limited by the speed of EM waves in free space (the speed of light $c$); since this is enormous by ordinary terrestrial standards, it is not surprising that these interactions "look" instantaneous.

However, there is nothing in Maxwell's electrodynamics as such that excludes there being other kinds of influence that propagate faster than the speed of light; indeed, in the late 19th century, it seems to have been widely believed that gravity was such an effect. But this is where special relativity really introduces something new: if we allow for such effects, then by definition we are allowing causal effects to propagate between pairs of events that are spacelike-separated, and then we must automatically allow that by performing a Lorentz transformation we can invert the time order of these events. Thus, at least in some reference frames, the future would appear to "cause" the present! Whether this is or is not a tolerable state of affairs may depend on one's basic notion of the meaning of "cause", something we are going to have to return to.

However, we should be careful to distinguish three notions involving causation:

(a) superluminal ("faster-than-light") causation ($Y$ causes $X$ when the two events are spacelike-separated, and in the relevant inertial frame $Y$ is earlier than $X$);

(b) backward causation outside the light cone (not only are $X$ and $Y$ spacelike-separated, but in the relevant inertial frame, $Y$ is later than $X$);

(c) backward causation within the light cone ($X$ and $Y$ timelike-separated and $Y$ later than $X$).

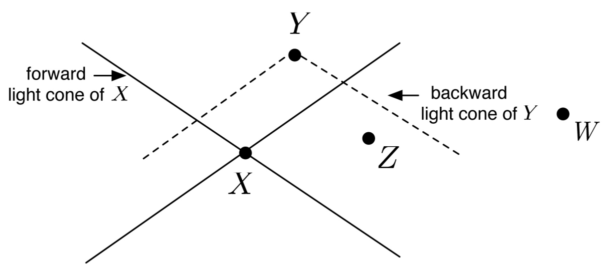



Figure 6

As we have just seen, (a) implies (b) (and, reciprocally, (b) implies (a)). On the other hand, it is straightforward to convince oneself (cf. the figure) that *provided* that we allow also the standard "forward" causation within the light cone, and make the usual assumption that causal relations are transitive, then (c) implies (a) (since given (c), $Y$ can cause $Z$, and hence $X$ can cause $Z$). Moreover, given the transitivity condition, (a) and (b) (and hence, from the above, (a) alone) implies (c), since we can always find a point (call it $W$) in the "elsewhere" of both $X$ and $Y$ such that $Y$ causes $W$ and $W$ causes $X$; thus, $X$ is caused by $W$, an event in its absolute future. Hence, given the general framework of SR, allowing superluminal transmission of causal effects has rather catastrophic consequences.

These considerations will become important when we discuss the so-called "EPR-Bell" experiments in the context of quantum mechanics (lecture 20).