Symmetry Principles and Conservation Laws

Newtonian physics is characterized by the existence of various *conservation laws*, that is, principles that tell us that however complicated the interactions among the components making up the system, certain quantities never change. Perhaps the most familiar such law is the conservation of the total mass of the system, or "amount of matter" in it: as described in Feynman, §3, we believe that in principle, we can always account for all the mass – if it has disappeared from one point, it must turn up somewhere else. Other important conservation laws that operate under appropriate conditions are these for the total energy, momentum and angular momentum (on which see below). We also sometimes find conservation laws for non-mechanical quantities, such as the total electric charge.

An important aspect of conservation laws (or at least the ones for mechanical quantities) that gradually emerged in the nineteenth century is that, generally speaking, they are associated with some *symmetry* or *invariance* property of the physical situation (this is explained below). This connection between symmetry properties and conservation laws is, in classical physics, common to "particle" and "wave" phenomena; moreover, it has survived the violent conceptual upheavals of relativity and quantum mechanics, and I imagine that most physicists would guess that it would survive possible future upheavals. It is interesting that the idea that (for example) all positions in space are in principle equivalent, and that this equivalence leads to interesting physical consequences, is perhaps one of the most radical ways in which post-Newtonian physics breaks with Aristotle (for whom, we recall, everything had its unique "proper place").

Let us start with the idea of "symmetry" (or invariance). Consider, for example, an experiment done using billiard balls on a smooth table. It is clear that the progress and outcome of the experiment does not depend, first of all, on the *time* at which we do it: in simple mechanics, all times are equivalent. It does not depend, either, on *where* we do it (assuming that the billiard table is, if necessary, shifted appropriately); all positions (on Earth's surface) are equivalent. Also, it does not depend on the *direction* on Earth's surface (again, provided we rotate the table appropriately). We would like to express these obvious, intuitive ideas, and extensions of them, a bit more precisely.

To describe a physical experiment, we need to be able to specify, in the form of numbers, the *times* and *places* at which different events took place.* In the case of time, this means that we must specify (a) the "origin" from which we are reckoning time, and (b) the unit of time. In most industrialized countries nowadays, the unit of time is the second, minute or hour, and the "origin" of time is local midnight; thus 8:47 a.m. is the time that is 8 hours and 47 minutes after midnight, which (at least originally!) is approximately the middle of the night at the location in question. In the case of the

*This paragraph essentially repeats, for convenience, some considerations already discussed in lecture 7.

mechanics of space coordinates, things are a little more complicated, because in addition to the origin and the unit of distance, one must specify the *orientation* of the grid. As an example, in Urbana, we might take our origin as the intersection of Race and University streets, and the unit of distance as the meter (or foot). But we would still have to specify the choice of *direction* of the axes along which we measure. The conventional choice in such cases is to choose one axis vertical (i.e., parallel to the local gravitational field) and the others to be N–S and E–W. Note that it is usually assumed without explicit comment that we take our three axes to be mutually perpendicular. Thus, we can specify the "spacetime location" of any event by four coordinates (t, x, y, z). For example, an event whose coordinates are (5194, -470, -1200, 20) would take place at 5194 seconds after midnight at a location 470 m south and 1200 m west of the University–Race intersection (i.e., roughly at the position of Loomis Lab), at 20 m above Earth's surface.†

Once we have defined our coordinate system, we can start formulating the laws of mechanics and applying them to specific experiments. For example, one of the simplest such laws is N1, which states that a body on which no forces act remains at rest or moving with uniform velocity. On Earth's surface, we should, of course, apply this only to the *horizontal* motion, so let us assume the vertical coordinate z is fixed (e.g., by considering a billiard ball moving on the surface of a billiard table). Then we are describing a chain of "events" j corresponding to "presence of the billiard ball" that are characterized by spacetime coordinates (x_j, y_j, t_j), and the interesting question for most purposes is how the space coordinates x_j and y_j are related to the time coordinates t_j of these events. In the limit where the t_j are very closely spaced, this information amounts to giving the *trajectory* $x(t)$, $y(t)$ of the billiard ball. Clearly, in the case considered (free motion), this is of the form

$$x(t) = x_i + v_x(t - t_i), \quad y(t) = y_i + v_y(t - t_i) \tag{$*$}$$

where i stands for "initial". We do not know ahead of time the values of the constants t_i, x_i, y_i, v_x, v_y (i.e., when, where, how fast, and in what direction the motion in question took place); however, we *do* know that every horizontal motion that is genuinely force-free must be expressible in the above form.

Now let us assume that we decide to change our choice of spacetime coordinate system. As a result, the numbers that describe a given event will in general change: call the new numbers t', x', y', z'. For example, if we were to decide to change our "origin of time" to 1 a.m. and our origin of space coordinates to Goodwin and University, without changing the units or the "NSEW" grid orientation, then the event mentioned above would have the new coordinates $t' = 1594$, $x' = -470$, $y' = 0$, $z' = 20$. In general, we can express the new coordinates in terms of the old ones; for all the kinds of coordinate

†There is obviously a slight approximation made here in that it is implicitly assumed that the surface of the Earth in Champaign-Urbana is exactly flat.

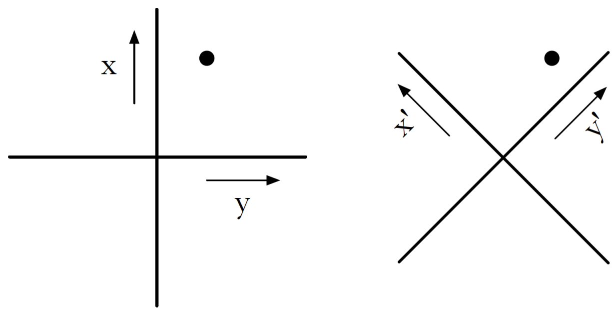

Figure 1: Rotation of the coordinate system by 45°.

change considered below, the relation is *linear*, e.g., $x' = \alpha x + \beta y$ (not, e.g., $x' = \gamma x^2$).‡ Here are some examples of possible changes.

1. Shift of the "origin" of time (e.g., reckoning from 1:00 a.m. rather than midnight). This is sometimes called "time translation". The formal expression of such a transformation is
$$t' = t - t_0, \quad x' = x, \quad y' = y, \quad z' = z$$
where t_0 is the new origin expressed in the old coordinates (in the example considered, $t_0 = 3600$).

2. Shift of the origin of the space coordinate system ("space translation"). If the new origin expressed in the old coordinate system is (x_0, y_0, z_0) (e.g, for the example considered: (0, - 1200, 0)) then the formal transformation is:
$$t' = t, \quad x' = x - x_0, \quad y' = y - y_0, \quad z' = z - z_0$$

3. Rotation of the space coordinate system. We may consider, without loss of generality, the case of a rotation around the existing origin (since for a rotation around an arbitrary point, it is always possible to make the latter the origin by a transformation of type 2). For simplicity, we will consider here only a special example, namely rotation in the xy-plane through 45° (Fig. 1). In this case, the transformation may be verified by a little trigonometry to be of the form:
$$x' = \frac{1}{\sqrt{2}}(x - y), \quad y' = \frac{1}{\sqrt{2}}(x + y)$$
In the more general case of an arbitrary 3D rotation we have $x' = \alpha x + \beta y + \gamma z$, etc., where α, β, γ are coefficients that depend on the axis and angle of the rotation.

‡It is, of course, possible to consider more complicated (nonlinear) modifications, but in the present context, these are not particularly relevant.

4. Time reversal. This simply consists in reckoning time backwards, so that:

$$t' = -t, \quad x' = x, \quad y' = y, \quad z' = z$$

5. Space inversion. This corresponds to reversing the signs of *all* the space coordinates:

$$x' = -x, \quad y' = -y, \quad z' = -z$$

   Note that if we were to consider only a *two*-dimensional system – e.g., forget altogether about the z-axis – then inversion is exactly the same as a rotation through 180°. In three dimensions, however, this is no longer true: inversion changes a right-handed glove into a left-handed one! Note also that a transformation that changes the sign of one space coordinate only is equivalent to a 180° rotation *plus* an inversion, while as just noted, one that changes the sign of two out of three is equivalent to a rotation.

6. Change of unit of time (new unit is α in terms of old one):

$$t' = t/\alpha, \quad x' = x \quad \text{etc.}$$

7. Change of unit of distance (new unit is β in terms of old one):

$$t' = t, \quad x' = x/\beta, \quad y' = y/\beta, \quad z' = z/\beta$$

   (It is of course also possible to adopt *different* units for the three axes, but this is of no great interest in the present context.)

8. Galilean transformation. For definiteness, we assume that the new frame is moving with respect to the old one along the x-axis (this can always be ensured by a suitable operation of type 3), with velocity v. Then, as already noted in earlier lectures, we have:

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t$$

   Note that this is the only one of the transformations considered that "mixes" the space and time coordinates.

In the above, we assumed that we wished to describe the *same* event by two *different* coordinate systems. An alternative approach is to consider two *different* events described in the *same* coordinate systems, the two events in question being related by operations corresponding to 1-8 above. For example, in the case of operation 1, we could consider an event that is identical in nature to our original one, but now occurs at 12:25 a.m. rather than 1:25 a.m. As regards arguments concerning commutation, conservation laws, etc., this so-called "active" picture is entirely equivalent to the "passive" one used above, and which to use is just a matter of taste and convenience.

An important point to notice is that not all of the above operations "commute" with one another. Two operations are said to "commute" (with one another) if the final result is independent of the order in which they are performed. For example, the operations of putting on a left sock and a right shoe commute, but those of putting on a right sock and a right shoe clearly do not. In the above list, it is clear that, for example, operations 2 and 3 do not, in general, commute; this is perhaps easier to see using the "active" picture – e.g., displacing a point through 1 m along the x-axis and then carrying it through 45° around the origin does not in general give the same result as first carrying it through 45° around the origin and then displacing it! A very important point is that while two rotations around the *same* axis commute, rotations around different axes in general do not. (You can easily verify this by taking an asymmetrical 3-dimensional object such as a textbook and rotating it [a] first 90° around a vertical axis and then 90° around a particular horizontal one, and [b] vice versa. It is immediately clear that the final orientation of the book achieved by process [a] is not the same as that in process [b]!)

An interesting question, for any given physical system, is now: under which of the above transformations is the form of the governing laws invariant? Let us consider, as an example, the especially simple case of free 2-D motion on Earth's surface. We found earlier that, in the old coordinate system, the most general form of allowed motion was

$$x(t) = x_i + v_x(t - t_i), \quad y(t) = y_i + v_y(t - t_i) \qquad (**)$$

where x_i and y_i are the x- and y-coordinates at some initial time t_i. Recall that the quantities t_i, x_i, y_i, v_x and v_y are characteristic of the particular motion considered, but that the above *form* of equations is quite general for a body moving under no external force. Consider now what the particular motion just described would look like in a new coordinate system related to the old one by operation 1 (shift in the "origin" of time). For any given value of t, since $x' \equiv x$, we have

$$x' = x_i + v_x(t - t_i)$$

and since $t' = t - t_0$, this can be rewritten

$$x' = x'_i + v_x(t' + t_0 - t_i) \equiv x'_i + v_x(t' - t'_i)$$

where t'_i is just $t_i - t_0$ and x'_i is identical to x_i. Thus, writing x' explicitly as a function of t', we have:

$$x'(t') = x'_i + v_x(t' - t'_i) \qquad (\text{and similarly for } y'(t'))$$

Although the *number* t'_i is, for any specific motion, different from the number t_i, it is clear that the *form* of the equation is exactly the same as the original one in the old reference system. Thus, if we are told that the form of the equation of motion of a free body is $(**)$, we can tell nothing about where the origin of time has been set.

A similar argument shows that if we were, instead of 1, to carry out operation 2 (say, for definiteness, with $y_0 = 0$), the result is

$$x'(t') = x'_i - v_x(t' - t'_0) \quad x'_i \equiv x_i - x_0 \quad t'_0 \equiv t'$$

Again, the *number* x'_i is different from x_i, but the general form of the equation of motion is unchanged.

Continuing along these lines, we actually find that for the special case of free motion (no external forces) the laws of motion (i.e., N1–N3) are invariant under *all* of the operations 1-8. However, this is no longer true when we introduce external forces or interactions between the particles composing the system. In particular, the "scale" transformations 6 and 7 practically never leave the laws of motion invariant. Consider, for example, N2, which says (provided mass is constant) that the acceleration of a body is equal to the force on it divided by its mass, and suppose the "force" in question is gravitational. Then, according to Newton's law of universal gravitation, it is a function only of the distance from the attracting object and is quite independent of the unit we use to measure time. On the other hand, the acceleration is a rate of change of velocity (i.e., distance/ unit time) with time, and *does* depend on the unit used: the number that gives the gravitational acceleration at Earth's surface is approximately 10 m/s^2, but it is approximately 36,000 m/minute^2! Thus, there is no invariance under operation 6[§] – in the sense that if we consider, for example, a planet orbiting the Sun, while we can always write

$$\text{acceleration} = g/r^2$$

the actual number g depends on the unit of time chosen. In the same sense, there is, in general, no invariance in operation 7.

As regards the other operations listed above, it depends somewhat on the exact physical situation considered. The operation of space inversion 5 is almost invariably a "good" symmetry (i.e., does not change the form of the laws of motion); it is impossible, by inspecting a movie of particle motion, to tell whether the coordinate system used is left- or right-handed[¶] (roughly speaking, there is no natural "front" or "back"). Similarly, 4 (time reversal) is a good symmetry, so long as we can neglect friction, air resistance and other so-called "dissipative" processes; under these conditions, it is impossible to tell, by inspection of the movie, whether it is being run backward or forward. However, time reversal and space inversion are not *continuous* symmetries, and therefore, in classical physics, do not give rise to conservation laws (it is interesting that they do in quantum mechanics).

[Inversion (a) symmetry in biology.]

Thus, we are left with the four operations 1, 2, 3 and 8 – time translation, space translation, rotation and Galilean transformation. In pre-relativistic physics, the role of Galilean transformation is somewhat anomalous – it gives rise to no new conservation law as such, but rather relates two existing ones (see end of this lecture). We therefore

[§]For the special case of the simple harmonic oscillator (force proportional to displacement), we do have invariance under 7 (but not 6), and there is in fact a conservation law associated with this, but it is rather subtle.

[¶]The so-called "left-hand" rule, etc., of textbook electromagnetic theory is illusory in this respect; it merely defines the convention for the sense of the magnetic field.

focus first on 1-3.

Let us start with operation 2, space translation. We can state that the dynamics of any body, or system of bodies, on which no external forces act is invariant under space translation. For a single body we have, indeed, already seen this: such a body performs uniform motion with constant velocity, and we saw above that this means that the form of the dynamics is invariant under a shift in the origin of space coordinates. For a system of bodies subject to no *external* force, we already saw (lecture 7) that N2 plus N3 implies conservation of the *total momentum*, i.e., the quantity[‖] $\sum_i m_i v_{ix}$ (and similar quantities involving the y- and z-components of velocity). Let us reformulate this. We introduce the "center of mass x-coordinate" X by the prescription

$$X \equiv \sum_i m_i x_i / \sum_i m_i$$

(If we think for a moment of the different masses as all in the same line and joined by light rigid bars, the center of mass is just the point at which we would have to support the array so that it would not tilt either way under the influence of gravity.) Now if we consider the "velocity" of the center of mass (i.e, the rate of change of X), since $\sum_i m_i$ is just the total mass M of the system, we have:

$$M \times (\text{rate of change of } X) = \sum_i m_i \times (\text{rate of change of } x_i)$$

$$\equiv \sum_i m_i v_{ix} \equiv \text{total momentum (in } x \text{ direction)}$$

But we know that the total momentum is constant, and the total mass M is also constant – and thus that, for a system of bodies subject to zero external force, *the center of mass moves at constant velocity.* The form of this motion, like that of a single body, is invariant under space translation. What about the *relative* motion of the constituent bodies? In general, this will be very complicated, and we cannot predict it without knowing the details of the forces acting between them. However, the crucial point is that this *relative* motion is *automatically* invariant under space translation of the coordinate system: if $x' = x - x_0$ (where x_0, remember, is a definite number, namely that x-coordinate of the new origin in the old coordinate system), then obviously we have:

$$x_i' - x_j' = x_i - x_j$$

Thus, we can say that the conservation of total momentum implies invariance against space translation. Actually, the modern point of view turns this around and says that *invariance under space translation implies conservation of total momentum.* Obviously, this is not a logical consequence of the former statement, and to make it work one

[‖]The notation "$\sum_i p_i$" simply means the sum of the quantities p for the different particles i involved. Thus, for example, for two particles 1 and 2, the expression "$\sum_i m_i v_{xi}$" means $m_1 v_{x1} + m_2 v_{x2}$.

actually needs to specify the properties which have to be invariant under translation a little more carefully than done above.**

Next we turn to operation 3, rotation of the space coordinate system. This time, we shall proceed in the opposite direction, from invariance to conservation. Consider first a body that is subject to a force exerted by another (much more massive) body, e.g., a planet in the gravitational field of the Sun. It is natural to take as the origin of coordinates the center of the Sun, and the rotations we shall consider are around this origin. Suppose the force in question has the property of being "central," that is, it is directed along the line connecting the body in question to the origin and depends *only on distance, not on direction*. This is certainly the case for gravitation, according to Newton's universal law, and it also turns out to be true for a number of other important forces, e.g., the electrostatic force (lecture 10). To illustrate this point, let us ask the question: are the dynamics of a body moving in Earth's gravitational field invariant under translation in the vertical (z-) direction? To the extent that we can regard the field as constant (a good approximation near Earth's surface), the form of the equation of motion is indeed invariant under the transformation $t' = t, z' = z - z_0$; however, the z-component of momentum is clearly not conserved! To get around this apparent violation of the invariance-conservation connection, we need to perform a maneuver that at the present stage looks like cheating: we invent the concept of *potential energy*, which is defined below and in this case is just mgz, and note that quantity is not invariant under the above transformation; thus, the theorem is not violated. More generally, we define a system to be *invariant* under a given transformation if not only the form of the equations of motion, but also the total potential energy, are invariant under it.

If this is true, then it is intuitively clear that all directions in space are, regarded from the origin, in some sense "equivalent", and our problem will possess the property of invariance under rotation (around the origin). (This statement can be proved formally, but the proof is a bit messy if one does not use vector notation, so I will rely at this point on intuition.)

Now we already saw (lecture 6) that the "central" property of the force between the Sun and the planets (or more precisely, that part of it that states that the force is along the line joining them) implies Kepler's second law, namely that the "rate of areal sweeping" by the planetary orbits is constant. (For a nice discussion, see Feynman pp. 41-3.) It is convenient (we shall see below why) to multiply this rate by twice the mass and define:

$$2 \times \text{mass} \times \text{rate of areal sweeping} = \text{“angular momentum”}$$

Since the mass is constant, we can thus state that for any central force, the angular momentum is conserved. An alternative (and equivalent) definition is that "angular

**Technical note (for cognoscenti only): The required addendum is that not only the form of the equation of motion, but also the actual value of the potential energy (see below) should be invariant. (To see why this is necessary, consider, e.g., a body falling in the *uniform* gravitational field at Earth's surface.)

momentum" = radius $\times$ transverse component of momentum.[††]

Consider next two (point) bodies interacting in the free space (i.e., no external force), with forces that are "central" in the sense that they act along the line between the bodies and are direction-independent. We already defined the "center of mass" ("COM") of such a system, and showed that it moves with constant velocity. We can therefore choose an inertial frame in which the COM is at rest. Now, since the forces are directed along the line between the bodies and the center of mass lies on this line, we can take the COM as origin and apply Kepler's second law to each of the two bodies separately. As a result, the angular momentum of each, and hence the total angular momentum, relative to the COM is conserved. Clearly, this statement must hold for *any* inertial observer, since the *relative* velocity of the bodies around their COM, and hence the rate of areal sweeping, is the same for any such. Thus, for any system of bodies subject to no external force and interacting only by central forces:

$$\text{total angular momentum around COM} = \text{constant}$$

So far, so good. What if we choose, however, to reckon the angular momentum around some point other than the COM? It turns out (this is not obvious) that the so-defined "angular momentum" is then the sum of two terms: the angular momentum around the COM, which we have just seen is constant, and the "angular momentum of the COM itself" – i.e., the total mass $\times$ the rate of areal sweeping *by* the COM relative to our chosen origin. But since, as we have seen, the velocity of the COM is constant, this term is also constant! Thus, we can finally make the following statement: irrespective of the choice of origin,

For a system subject only to internal central forces (i.e., invariant under total rotation), the total angular momentum is conserved.

Note that this statement is true even though with different choices of origin the actual *value* of the angular momentum is different.

Finally, we consider the consequences of invariance under time translation. In a way which is not at all obvious prima facie (and took a couple of centuries to work out in full detail!) this is associated with what is perhaps the most famous of all the laws of physics: conservation of energy.

In his work on pendulums and related systems, Galileo made two very important observations concerning bodies for which friction, air resistance, etc., is negligible:

1. A body dropped from rest at a certain height always rises again to exactly that height; and

[††] Technically, the angular momentum is a vector (which, for a planet, is in the direction normal to the plane of the orbit), but we do not need to worry about that for present purposes.

2. If a body starts from rest ($v_i = 0$) and falls through a height Δh, its final velocity v_f is given by $\sqrt{2g\Delta h}$; more generally, $v_f^2 - v_i^2 = 2g \times \Delta h$.

Using a certain amount of hindsight, we can combine these statements (actually with some degree of generalization) into an equation that has the form of a conservation law; for any body moving in Earth's gravitational field in the absence of friction, etc.:

$$\boxed{\frac{1}{2}mv^2 + mgh = \text{const}} \qquad (\alpha)$$

The first term, which depends on the velocity but not the position, is called the "kinetic energy" (originally, "vis-viva" ["living force"]) and often denoted T. The second term, which depends on the position (height) but not the velocity, is called the "potential energy". The total energy of the system is the sum of the kinetic and potential energies, and according to the above statement is *conserved.*

The great advantage of this formulation is that it can be readily generalized to many types of force other than gravitational. Consider a small change in the h and v in equation α. Indicating "change of" by the symbol Δ, we can write:

$$\Delta\left(\frac{1}{2}mv^2\right) + mg\Delta h = 0$$

Now, if we consider only a small region of space, any force F can be regarded as "equivalent" to a gravitational force mg_{eff}, with the provisos that (a) g_{eff} not in general be independent of m, and (b) the direction of the force not in general be vertical. It thus seems very natural to generalize the above equation to this case to read

$$\Delta\left(\frac{1}{2}mv^2\right) - F\Delta x = 0 \qquad (\beta)$$

where Δx is a small displacement in the displacement in the direction *parallel* to that of the force (hence the "–" sign).[‡‡] Suppose now that we can find a quantity V that depends on position in space in such a way that we always have:

$$\Delta V = -F\Delta x \qquad (\gamma)$$

It is not obvious that such a quantity must exist, and indeed it will only do so if the force in question has a certain property (technically called "conservative"); fortunately almost all of the (non-frictional) forces of interest do possess this property. If so, then, devoting the kinetic energy as above by T, and the quantity $T + V$ by E, we have

$$\boxed{T + V \equiv E = \text{const}}$$

i.e., the law of conservation of (mechanical) energy. This law may be generalized to systems of many bodies; in this case, the potential energy will in general depend on all their

[‡‡]Note that increasing "height" (h) is conventionally taken in the direction opposite to Earth's gravitational force.

coordinates. However, if no external forces act on the system then it is straightforward to show that the potential energy can depend only on the *relative* coordinates.

One might well ask what on Earth all this has to do with invariance under time translation. The answer is the following: in the above, it was implicitly assumed that the force F is *time-independent* (and therefore, does not depend on the way in which we choose the "origin of time"). Suppose F, while still "conservative", is allowed to depend on time, then while eqn. β will still be valid, we can no longer write eqn. γ: V will in general now also have to be time-dependent, and thus ΔV can be nonzero even if Δx is zero. So, in general, we will no longer be able to derive the conservation of energy. A nice example of this is a child working herself up on a swing; if we regard the "system" as simply the mechanical system formed by the child's body and the swing,* then it is clear that when she reaches maximum height on successive swings, the kinetic energy is the same (zero) but her potential energy has increased. The reason is that, by the child's volition, the forces between the component parts of the system (e.g., legs and torso) are time-dependent, so the conservation of (mechanical) energy is violated.

In the course of the eighteenth and nineteenth centuries, the concept of "energy" was gradually generalized to include, besides mechanical energy, energy, for example, associated with static or propagating electric and magnetic fields (lecture 10), energy associated with chemical reactions, and, most importantly, the "heat" energy associated with the random motion of atoms and molecules. Taking the latter into account enables us to apply the principles of conservation of energy even to cases involving friction; friction (and other dissipative processes) *is* nothing but the conversion of macroscopic mechanical energy into "random" kinetic (or potential) energy of atoms and molecules. We will return to this topic later in the course in connection with irreversibility.

Finally, a brief word concerning the consequences of invariance under Galilean transformation. Let us study how the momentum and energy transform under a GT from one inertial frame to another moving with velocity u relative to it.

Since velocities transform as $v' = v - u$, we have for a single free particle (no potential energy!):

$$p' \equiv mv' = m(v-u) = p - mu$$

$$E' \equiv \frac{1}{2}mu'^2 = \frac{1}{2}m(v-u)^2 = \frac{1}{2}mv^2 - muv + \frac{1}{2}mu^2 \equiv E - up + \frac{1}{2}mu^2$$

In the case of a compound system, the result is simplified by the fact that, provided no *external* forces act, the potential energy is a function only of the *relative* coordinates, and therefore is invariant under GT. Since the total momentum P and kinetic energy E is just a sum of the contributions of the individual bodies, we have simply (M = total mass):

$$P' = P - Mu, \qquad E' = E - uP + \frac{1}{2}Mu^2$$

*Thereby ignoring the "energy" that we would now associate with biochemical changes in her muscles, etc.

From this, we derive the following conclusion: if there exists one inertial frame in which *both* momentum and energy are conserved, then they are conserved in any inertial frame. Moreover (Galileo!), if momentum alone is conserved in any one inertial frame, it is conserved in any. However, from the fact that energy is conserved in one frame, it does not necessarily follow that it is conserved in any. As an example, consider a ball bouncing against the wall of a carriage in a uniformly moving train.[†] As viewed from the train, the energy of the ball is conserved, but its momentum is not. However, an observer on Earth will see the magnitude of velocity relative to him *changed* on the bounce, and will therefore reckon that energy is *not* conserved. The reason is that the force exerted by the (moving) wall (admittedly a very short-range one!) depends strongly, for given position relative to the ground, on time, and according to the argument given above, this permits violation of energy conservation.

We will see later that the special theory of relativity allows us to view the relationship between energy and momentum conservation is a new and instructive light.

[†]The train is *not* counted as part of the "system", but as a source of "external" forces.