

Physics 419:

Derivation of $E = mc^2$

$$F = \frac{dP}{dt} \text{ (Newton's second law),}$$

$$= \frac{d}{dt}(mv) = v \frac{dm}{dt} + m \frac{dv}{dt}.$$

$$m = \frac{m_0}{\sqrt{1-(v/c)^2}} \Rightarrow \frac{dm}{dt} \neq 0.$$

$$m^2 = \frac{m_0^2}{1-(v/c)^2}$$

$$\Rightarrow (mc)^2 - m^2 v^2 - m_0^2 c^2 = 0.$$

$$\Rightarrow 2c^2 m \frac{dm}{dt} - 2m^2 v \frac{dv}{dt} - 2v^2 m \frac{dm}{dt} = 0.$$

$$\begin{aligned} \Rightarrow c^2 \frac{dm}{dt} &= mv \frac{dv}{dt} + v^2 \frac{dm}{dt} \\ &= v \underbrace{\left(m \frac{dv}{dt} + v \frac{dm}{dt} \right)}_{dP/dt}. \end{aligned}$$

$$\Rightarrow F = \frac{c^2}{v} \frac{dm}{dt}$$

$$= c^2 \frac{dx}{dx} \frac{dm}{dt} = c^2 \frac{dm}{dx}.$$

$$\Rightarrow W = \int_{x_i}^{x_f} F dx = c^2 (m_f - m_i)$$

$$\text{let } m_i = m_0 \\ m_f = m.$$

$$W = mc^2 - m_0c^2.$$

Now use the W-K.E. theorem.

$$W = \Delta K.E.$$

$$K.E. = mc^2 - m_0c^2$$

Let's rewrite this eq.

$$mc^2 = K.E. + m_0c^2.$$

the energy is of the form $E = K.E. + U.$

\Rightarrow by analogy $E = mc^2.$

$$E = mc^2 = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow E^2 = (m_0c^2)^2 \left(1 - \frac{v^2}{c^2} \right)^{-2} = (m_0c^2)^2 \left(1 - \frac{v^2}{c^2} \right)^{-2}$$

$$\Rightarrow E^2 - (mv)^2c^2 = (m_0c^2)^2$$

$$\boxed{\Rightarrow E^2 - (pc)^2 = (m_0c^2)^2}$$

\nearrow
new invariant in S.R.