Derivation: the volume of a D-dimensional hypershell

Phys 427
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In class we worked out the multiplicity of an ideal gas by summing over volume elements in *k*-space (each of which contains one quantum state) that have energy *E*. We showed that the surface of states that have energy *E* is a hypershell of radius *k* in D-dimensions where $k^2 = k_1^2 + k_2^2 \cdots + k_D^2$. Integrating in D-dimensions in Cartesian coordinates is easy, but how do you integrate over a hypershell in D-dimensions? We used dimensional arguments in class to relate a volume of *k*-space in Cartesian vs. hyperspherical coordinates:

$$dk_1 dk_2 \cdots dk_D = k^{D-1} dk d\Omega$$

where Ω is the solid angle subtended by the hypershell.

If we integrate over all D-dimensional space in Cartesian vs. hyperspherical coordinates, we should get the same answer:

$$\iint_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dk_1 dk_2 \cdots dk_D = \int_0^{\infty} g_D k^{D-1} dk$$

where we integrated over the solid angle to obtain the factor g_D. We would like to determine g_D. The above formula is not very useful in that respect because both sides are infinite. What we want is a function that can be integrated over all of *D*-dimensional space in both Cartesian and hyperspherical coordinates and give a finite answer. A Gaussian function over k_1 ... k_D will do the trick. We can write:

$$\iint_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\left(k_1^2 + k_2^2 \cdots + k_D^2\right)} dk_1 dk_2 \cdots dk_D = \int_0^{\infty} g_D e^{-k^2} k^{D-1} dk$$

where we wrote the Gaussian in terms of *k* on the right side of the equation.

Let's look at the left side of the equation first. The integral can be written in terms of the product of *D* identical 1-dimensional integrals:

$$\int_{-\infty}^{\infty} e^{-k_1^2} dk_1 \int_{-\infty}^{\infty} e^{-k_2^2} dk_2 \cdots \int_{-\infty}^{\infty} e^{-k_D^2} dk_D = \left(\int_{-\infty}^{\infty} e^{-k_1^2} dk_1 \right)^D = \pi^{D/2}$$

where we evaluated the Gaussian integral in the last step (look up Kittel & Kroemer Appendix A for help on evaluating Gaussian integrals). Now let's look at the right side of the equation. Switching variables from k to $u = k^2$ we get:

$$\int_0^\infty g_D e^{-k^2} k^{D-1} dk = \frac{1}{2} g_D \int_0^\infty e^{-u}\, u^{\frac{D}{2}-1} du = \frac{1}{2} g_D \Gamma\left(\frac{D}{2}\right)$$

where the Gamma function Γ is defined as (look up Appendix A if you're unfamiliar with Gamma functions):

$$\Gamma(n+1) = \int_0^\infty e^{-u}\, u^n du = n!$$

We're almost done. We can now set the two sides of the equation equal to each other:

$$\pi^{D/2} = \frac{1}{2} g_D \Gamma\left(\frac{D}{2}\right)$$

from which it follows that g_D is:

$$g_D = \frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)}$$

g_D is the total solid angle for a D-dimensional hypershell, and the volume of the hypershell is $g_D\, k^{D-1}\, dk$ (you can check that for D = 2, $g_2 = 2\pi$, and for D = 3, $g_3 = 4\pi$ as expected). In class the dimensionality of the problem was $D = 3N$ (N gas particles in 3-dimensional space), *and* we only had to consider the quadrant with $k > 0$ so we need to divide the expression above by 2^{3N}. The geometrical factor is then:

$$\frac{2\pi^{3N/2}}{2^{3N}\Gamma\left(\frac{3N}{2}\right)} = \frac{2\pi^{3N/2}}{2^{3N}\left(\frac{3N}{2}-1\right)!}$$

which is the expression given in class. Note that when N >> 1 you can usually ignore the -1 in the factorial.