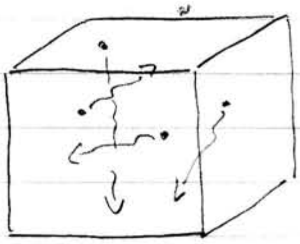
Lecture 9

Last time we showed that photon gas at temperature T inside a cavity of volume V has spectrum:



$$u(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\beta\hbar\omega} - 1}$$

and $$u_{tot} = \frac{U}{V} = \int_0^\infty d\omega\, u(\omega) = \frac{\pi^2}{15} \frac{(k_B T)^4}{\hbar^3 c^3}$$

$U = \alpha V T^4$ (notice $U \sim V$ since its an extensive quantity)

$C_V = \left(\frac{\partial U}{\partial T}\right)_V = 4\alpha V T^3$ heat capacity $\sim T^3$

For fixed cavity V $dU = T\,dS$ so

$$dS = \frac{dU}{T} = 4\alpha V T^2\, dT$$

$$S = \int \frac{dU}{T} = \frac{4}{3}\alpha V T^3$$ (const. of integration $= 0$ so $S(0) = 0$)

$$\therefore\ F = U - TS = V\left(\alpha T^4 - \frac{4}{3}\alpha T^4\right) = -\frac{1}{3}\alpha V T^4$$

pressure $$p = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{1}{3}\alpha T^4 = \frac{1}{3}\frac{U}{V}$$

(Notice that for ideal gas of particles with mass, we found $p = \frac{2}{3}\frac{U}{V}$ PS # 3)

So far these are properties of photon gas *inside* cavity
We want to know properties of photons *emitted* by a hot object

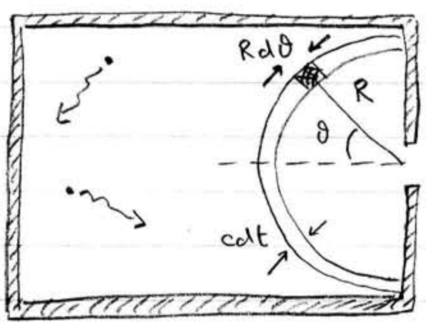
Imagine a cavity with a *small hole* of size A
photons emitted have same spectrum as photons inside

Define the "energy flux density" J_u as rate energy is emitted per unit area (units $\frac{W}{m^2}$). Also called *intensity*

By dimensional analysis, expect

$$J_u = c\, u_{tot} \times (\text{geometrical factor})$$

Now derive this exactly:

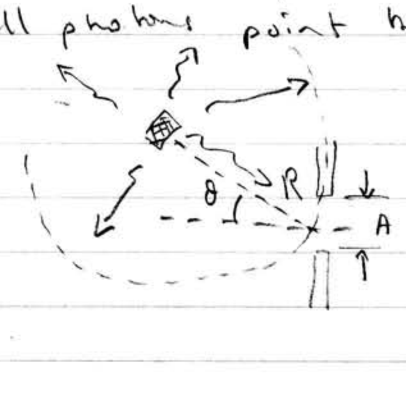


Photons that escape cavity *now* in time interval dt came from hemispherical shell some time before

Energy in a chunk of shell is:

$$\frac{U}{V} R^2 \sin\theta \, d\theta \, d\phi \, c\,dt$$

Not all energy from chunk of volume escapes, because not all photons point the right way



Photons emitted in every direction
Equal chance to land anywhere on sphere of area $4\pi R^2$
Area subtended by hole is $A\cos\theta$
so prob. escape $= \dfrac{A\cos\theta}{4\pi R^2}$

So energy that escapes is:

$$dE_{escape} = \int_{\text{hemisphere}} \frac{A\cos\vartheta}{4\pi \cancel{R^2}} \frac{U}{V} \cancel{R^2} \sin\vartheta \, d\vartheta \, d\phi \, c\, dt$$

$$= \frac{A\, c\, dt}{4\pi} \frac{U}{V} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\pi/2} \sin\vartheta \cos\vartheta \, d\vartheta}_{\frac{1}{2}}$$

$$= \frac{1}{4} A c\, dt \frac{U}{V}$$

$$\Rightarrow \quad \text{Power} = \frac{dE_{escape}}{dt} = \frac{1}{4} A c \frac{U}{V}$$

$$J_u = \frac{\text{Power}}{A} = \frac{1}{4} c \frac{U}{V} = \frac{1}{4} c\, u_{tot}$$

$$\boxed{J_u = \frac{\pi^2}{60} \frac{(k_B T)^4}{\hbar^3 c^2} = \sigma_B T^4}$$

$$\sigma_B = 5.67 \times 10^{-8} \frac{W}{m^2 K^4}$$

$\uparrow$ Stefan-Boltzmann constant

This described flow of energy out of cavity, but what about an object in space radiating? (the sun is <u>not</u> a cavity with a hole in it!)

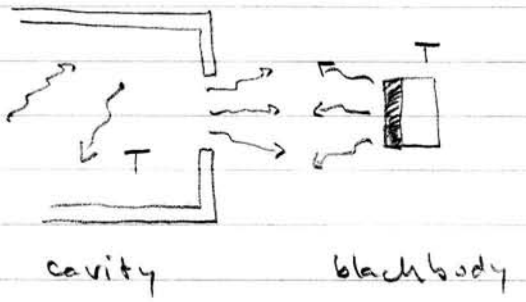
Equations turn out to be <u>exactly the same</u>.

Consider a black object
- absorbs all EM radiation
- also emits EM radiation

object is a "<u>blackbody</u>"

Proof that blackbody emits radiation the same as a cavity with a hole

Thought experiment – hole in a cavity, blackbody face each other. Both at same T same size A



- Each emits photons and absorbs each other's photons
- In fact, each absorbs same fraction of other's radiation

What power is emitted by the blackbody?
If less than cavity, then energy flows to blackbody
If more " then " out of blackbody

Neither can happen because T is the same. (At thermal equilibrium, not net energy flow)

$$\therefore \quad J_{cavity} = J_{blackbody} = \frac{\text{Power}}{\text{Area}}$$

Not only that, but spectra must be the same.
Use same argument, but now place a filter between cavity hole and blackbody that let's through frequencies $\omega, \omega + d\omega$

Blackbody radiation
- absorbs all light (no reflections hence "black")
- emits radiation according to its temperature
  spectrum given by Planck law
  total energy flux by Stefan Boltzmann law
- $P_{emit} = P_{absorbed}$ blackbody radiates as much energy as it absorbs

What about bodies that reflect light (i.e. objects that are not black)?

Go back to <u>thought experiment</u>
Let's say that for every 3 photons emitted by cavity that hit object, 1 is reflected 2 absorbed (at one λ)

To remain at equilibrium with cavity at T, the object <u>must</u> emit 2 photons — no net energy flow

$$P_{abs}(\omega) = a(\omega)\, P_{inc}(\omega)$$

absorption coefficient at ω (here $\frac{2}{3}$)

$$P_{emit}(\omega) = P_{abs}(\omega)$$
$$e(\omega)\, P_{inc}(\omega) = a(\omega)\, P_{inc}(\omega)$$

emissivity at ω (here $\frac{2}{3}$) $\therefore$ $a(\omega) = e(\omega)$

Also $P_{ref}(\omega) = r(\omega)\, P_{inc}$

reflectance at ω

By energy balance $P_{emit} + P_{ref} = P_{inc}$ so

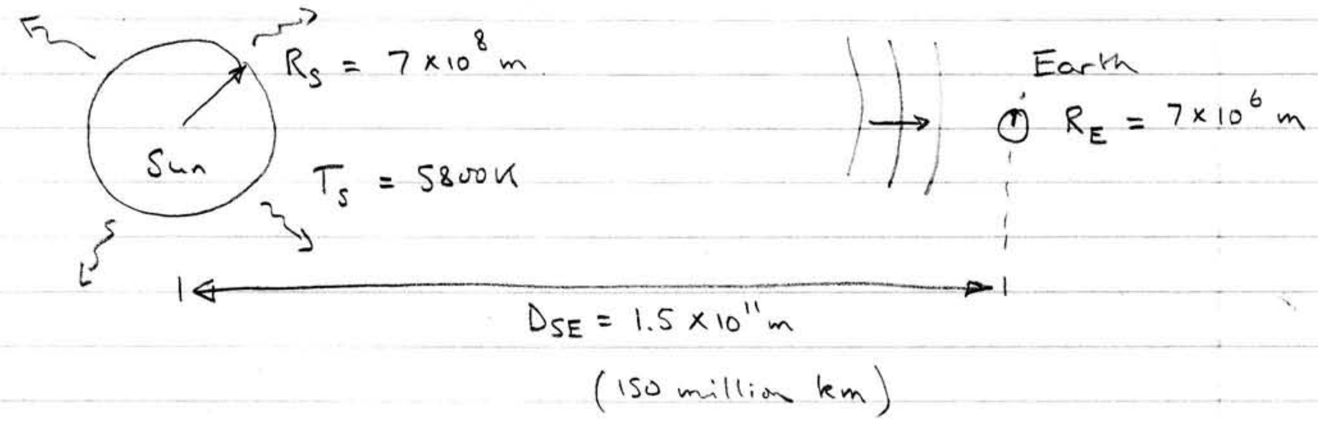
$$\boxed{a(\omega) = e(\omega) = 1 - r(\omega)}$$

- A good absorber is a good emitter (Ex: blackbody $a = e = 1$)
- A good reflector is a poor absorber <u>and</u> emitter

Assuming e is constant with ω : $J_u = \sigma e T^4$

For many things $e = e(\omega)$

Ex: The Sun and the Earth



$$P_S = \sigma A_S T_S^4 = \sigma 4\pi R_S^2 T_S^4$$
$$= (5.67 \times 10^{-8}) 4\pi (7 \times 10^8)^2 (5800)^4$$
$$= 3.9 \times 10^{26} \text{ W}$$

sun's luminosity

What is the energy flux at the earth?

$$J_S = \frac{P_S}{4\pi D_{SE}^2} = \frac{3.9 \times 10^{26} \text{ W}}{4\pi (1.5 \times 10^{11} \text{ m})^2} = 1370 \frac{\text{W}}{\text{m}^2}$$

solar constant

So the incident power at the earth is

$$J_S \times \text{Cross section of Earth} = J_S (\pi R_E^2) = P_{inc}$$

Assuming the earth is an ideal blackbody $(a = e = 1)$

$$P_{abs} = J_S \pi R_E^2 = P_{emit} = \sigma 4\pi R_E^2 T_E^4$$

so ...

$$T_E = \left(\frac{J_s}{4\sigma}\right)^{1/4} = \left(\frac{1370}{4\times 5.67\times10^{-8}}\right)^{1/4} = 279\,K \quad !!$$ (pretty close to 15°C)

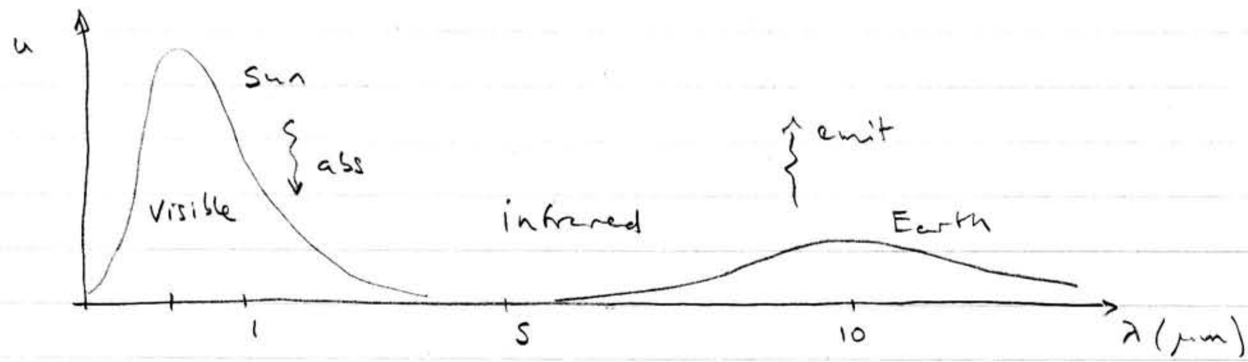
$$T_E = \left(\frac{\cancel{\sigma}\, \cancel{4\pi} R_s^2 T_s^4}{4\cancel{\sigma}\, \cancel{4\pi} D_{SE}^2}\right)^{1/4} = \left(\frac{R_s}{2D_{SE}}\right)^{1/2} T_S$$

Unfortunately, earth is not an ideal blackbody, it reflects ~30% of sunlight, so we should replace 1370 with $1370 \times (1-0.3)$

$$\Rightarrow \quad T_E = \left(\frac{1370\times 0.7}{4\times 5.67\times10^{-8}}\right)^{1/4} = 255\,K \quad !!$$ (frigid!)

This is also incorrect: solution is that $e(\omega)$ is a strong function of ω

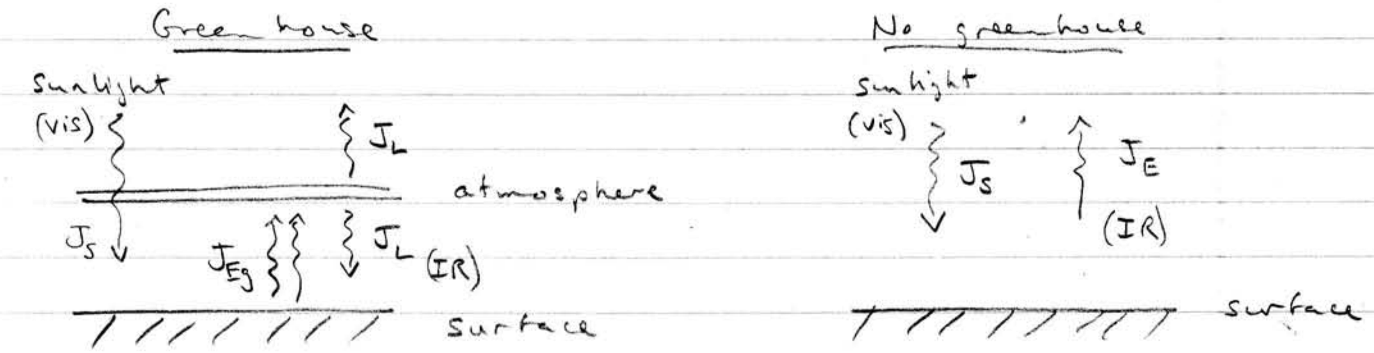
Solar radiation peaked in near infrared – most light in visible
($\hbar\omega_{max} = 2.82\, k_B T_s = 1.41\,eV \rightarrow 880\,nm$)
Earth radiation peaked in far infrared $\rightarrow \sim 10\,\mu m$
because $T_E \sim 300\,K$



Earth absorbs visible light from sun (some reflected by clouds) <u>but</u> has to re-emit it as IR light (because $T_E \ll T_s$)
Problem is that atmosphere absorbs IR light
$\Rightarrow$ this leads to <u>green house effect</u>

Simple model for Greenhouse effect (see K+K)

Treat atmosphere as single layer above surface.
Transparent to visible, absorbs IR light



with greehouse effect incident energy flux is J_S
emitted flux from Earth is J_{Eg}, from atmosphere layer J_L. Atmosphere emits to space and back to earth

By energy balance $J_S = J_L$ and $J_S + J_L = J_{Eg}$

$\therefore \quad J_{Eg} = 2 J_S$

In contrast with no greenhouse $J_S = J_E$

$$\therefore \quad \frac{J_{Eg}}{J_E} = 2 = \left(\frac{T_{Eg}}{T_E}\right)^4$$

$$T_{Eg} = 2^{1/4}\, T_E = 2^{1/4} \cdot 255\,K = 303\,K \quad \text{(a bit better)}$$