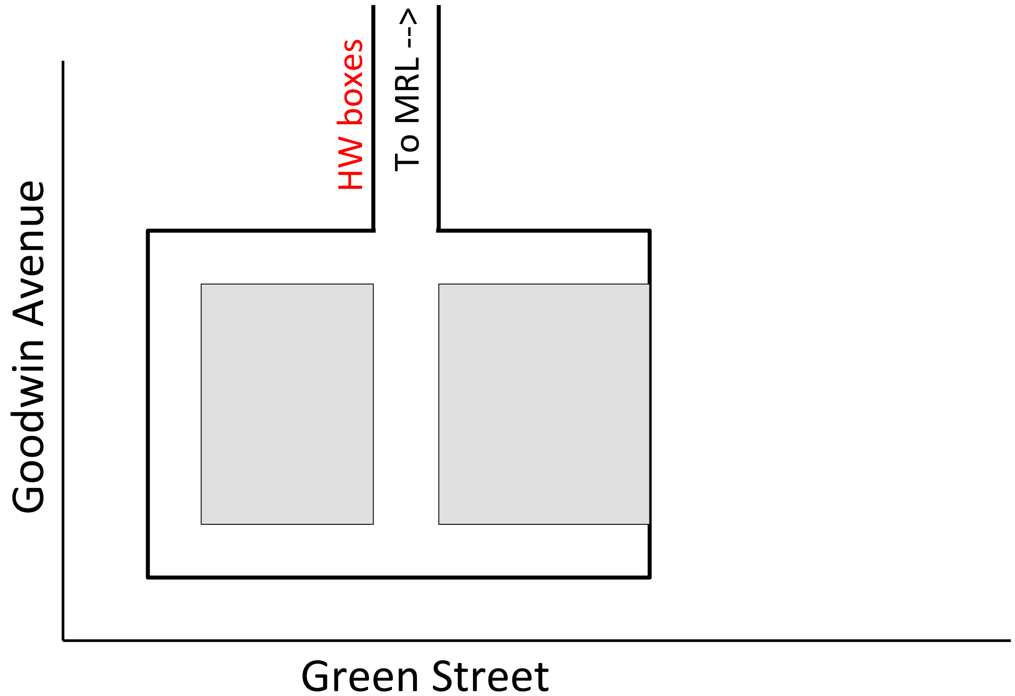
Announcement

Homework:

- Homework is due at 5 PM on Friday.
- Put your homework in the P435 HW box,
  which is in the 2nd floor passage between Loomis and MRL:



Today's Topic: The Electric Field

Griffiths, 2.1 and 2.2

Empirically, we have Coulomb's law: $\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$, where $\vec{\mathscr{r}} = \vec{r} - \vec{r}_Q$

The force on a charge, *q*, at ***r*** is: $\vec{F} = q\vec{E}(\vec{r}) = \frac{qQ}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\left(\text{or } \frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right)$

Notes:

- $\vec{F} \propto q$ If that weren't true, it would not be possible to talk about an electric field independent of *q*.
- $\vec{E} \propto Q$ This is superposition. More generally, if we have several charges, the total ***E*** is the sum of each charge's field. $\vec{E}_{\text{tot}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{Q_i \hat{\boldsymbol{\mathscr{r}}}_i}{\mathscr{r}_i^2}$

  If we have a continuous distribution of charge, the sum becomes an integral, Q_i becomes $\rho(\boldsymbol{r}')$, and $\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{r}'$.

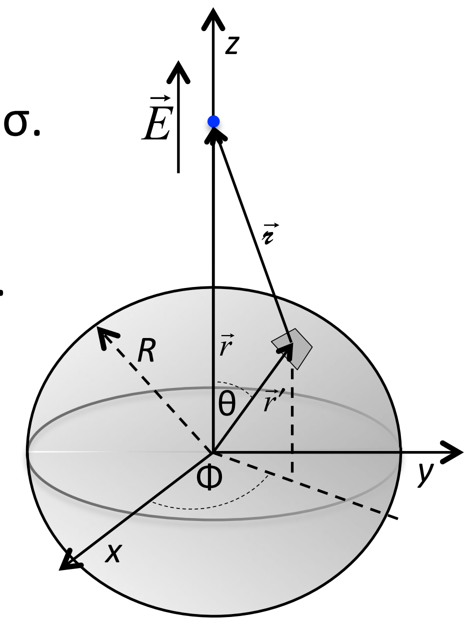
These aren't really independent. Swap the roles of q and Q, and remember Newton's 3rd law.

Standard notation:

- 1-dimensional (line) density: $\lambda(x)$
- 2-dimensional (surface) density: $\sigma(x,y)$
- 3-dimensional (volume) density: $\rho(x,y,z)$

Example: (G 2.7, p. 64)

Calculate E due to a uniform spherical surface charge, density = σ.



Spherical symmetry lets us evaluate ***E*** on the *z*-axis, where it points in the ***z*** direction. We must consider both $z > R$ and $z < R$.

$$\vec{r} = (0,0,z)$$

$$\vec{r}\,' = R(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \qquad |r'| = R$$

$$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta \qquad \text{Law of cosines}$$

$$\mathscr{r}_z = z - R\cos\theta$$

So, $$E_z = \frac{\sigma}{4\pi\varepsilon_0}\oint \frac{\mathscr{r}_z}{\mathscr{r}^3}dA = \sigma R^2 \int_0^{2\pi}\int_0^{\pi} \frac{z - R\cos\theta}{\left(R^2 + z^2 - 2Rz\cos\theta\right)^{\frac{3}{2}}}\sin\theta\, d\theta\, d\varphi$$

$$= \frac{2\pi R^2\sigma}{4\pi\varepsilon_0 z^2}\underbrace{\left[\frac{z-R}{|z-R|} + \frac{z+R}{|z+R|}\right]}_{\substack{=0 \text{ if } z<R \\ =2 \text{ if } z>R}}$$

This is obtained using a trig substitution <u>and</u> partial fractions.

Note: $4\pi R^2\sigma = Q_{tot}$.

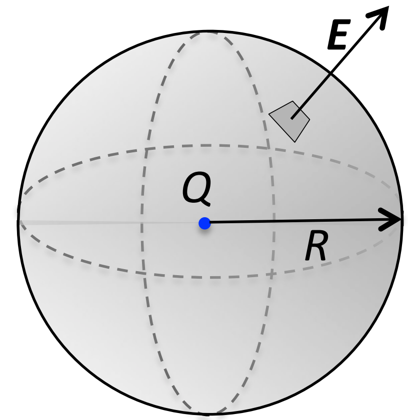
I suggest:
Do Griffiths problems 2.4, 2.5, and 2.6.
(2.3 will be a HW problem.)

Note: This problem, for gravity, led Newton to develop calculus.
He needed to verify that the force falls off as $1/r^2$ for a spherical (not point) mass.

Divergence and Curl of $\boldsymbol{E}$

Griffiths, 2.2

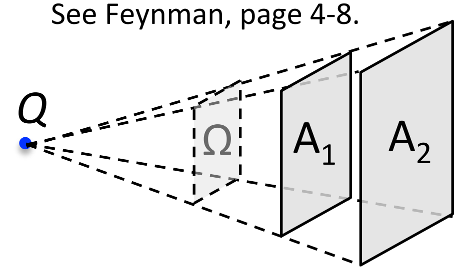
Coulomb's law: $\vec{E} \propto 1/r^2$ is the reason that we can draw electric field lines that start on a + charge and end on a - charge.



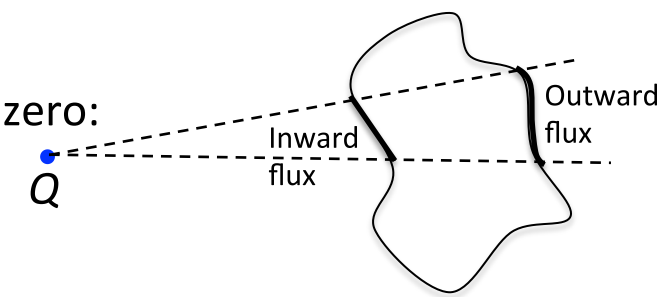
Consider a point charge surrounded by a spherical surface of radius R: Because $A \propto R^2$ the total flux through the sphere is independent of R. Flux = Q/ε_0. The electric field lines don't end in empty space.

Gauss' law is a generalization of this to the situation where the charge is not at the center and where the surface is not a sphere. Because the field lines are radial, the flux through surfaces A_1 and A_2 depends only on the solid angle, Ω, that they subtend. This works even if the surfaces are tilted. So, Flux = Q/ε_0 is universally true.

See Feynman, page 4-8.



If Q is outside the surface, the total flux is zero:



Superposition lets us generalize Gauss' law to continuous charge distributions.

Gauss' law follows from Coulomb's law and superposition:

$$\oint_{\text{Surface}} \vec{E}\cdot\hat{n}\,dA = \frac{Q_{in}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_{\text{Volume}} \rho(\vec{r})\,dV$$

However, this is an integral equation, not so easy to solve, except in special cases. We can convert it into a differential equation using a math identity, the divergence theorem: (***v*** is any differentiable vector function)

$$\oint_{\text{Surface}} \vec{v}\cdot\hat{n}\,dA = \int_{\text{Volume}} \vec{\nabla}\cdot\vec{v}\,dV$$

← No physics here !!

Letting ***v*** be ***E***, we immediately see that (because the volume is arbitrary), the two volume integrands must be equal:

$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$$

Gauss' law in differential form.

This tells us the local behavior of the electric field due to the charge density at a given point.

Note: The divergence theorem is the fundamental theorem of calculus in 3-D. If you Google "divergence theorem proof" (no quotes), you'll find innumerable proofs on the web.

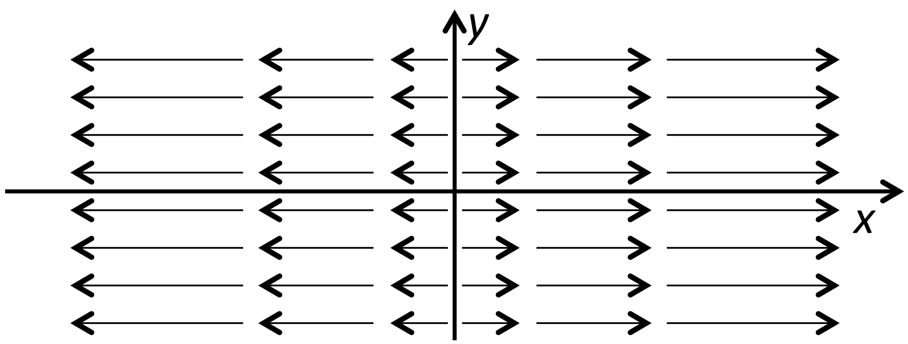
Calculation (and Use) of Divergence

Example:

Suppose we want to create an electric field with the behavior shown. What charge density is required?

$$\vec{E} = ax\hat{x} = (ax, 0, 0)$$

In Cartesian coordinates:



$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = a$$

So, $\rho = a\varepsilon_0$. Is that reasonable?

How can a uniform charge density pick out a preferred direction and position?

Look at the differential equation more closely. For $\rho = a\varepsilon_0$, an electric field of the form, $\vec{E} = a_1 x\hat{x} + a_2 y\hat{y} + a_3 z\hat{z} + \vec{E}_0$, is a solution for any $a_1 + a_2 + a_3 = a$, and for any $\boldsymbol{E_0}$.

The parameters, a_1, a_2, a_3, and $\boldsymbol{E_0}$ are determined by the boundary conditions that we have imposed on the problem.

Question: What does $\boldsymbol{E}$ look like if $a_1 = a_2 = a_3 = a/3$?

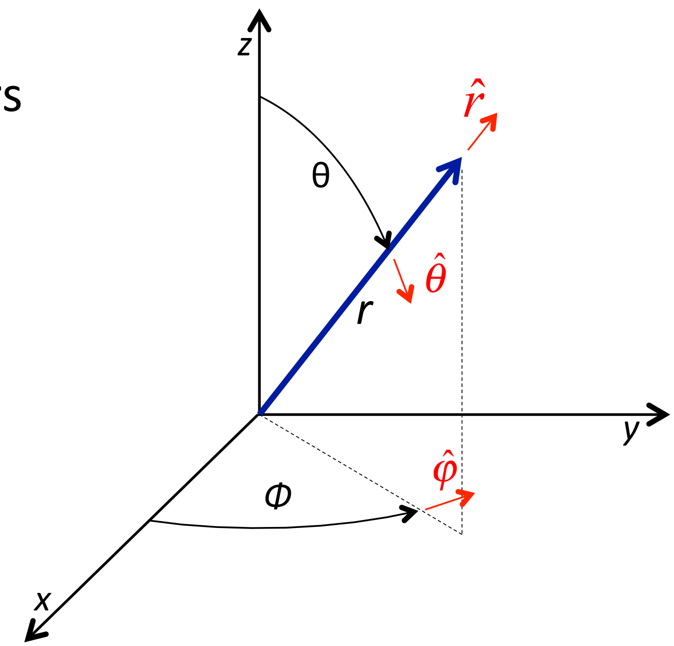
Spherical Coordinates

Sometimes, problems are simpler to understand and solve if we use other coordinate systems.

Spherical coordinates are most useful when there is spherical symmetry (*i.e.*, the problem looks the same in any orientation).

The Cartesian components of the spherical unit vectors are:

$$\hat{r} = \left(\sin\theta\cos\varphi,\ \ \sin\theta\sin\varphi,\ \ \cos\theta\right)$$
$$\hat{\theta} = \left(\cos\theta\cos\varphi,\ \ \cos\theta\sin\varphi,\ -\sin\theta\right)$$
$$\hat{\varphi} = \left(-\sin\varphi,\ \ \cos\varphi,\ \ 0\right)$$



Each tells us the (*x,y,z*) motion when only that particular component changes. One important feature of spherical (and other curvilinear) coordinates is that the directions of the unit vectors are not constant, so that taking derivatives in those coordinate systems is more complicated.

Divergence in Spherical Coordinates

Contrary to intuition, this expression is not correct:

No! $\vec{\nabla}\cdot\vec{E} = \frac{\partial E_r}{\partial r} + \frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_\varphi}{\partial \varphi}$ No!

It fails, because $\hat{r}$, $\hat{\theta}$, and $\hat{\varphi}$ are functions of position.

Instead, we have:

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta E_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial E_\varphi}{\partial\varphi}$$

This form is useful in problems with spherical symmetry.

Let's calculate the divergence of the Coulomb field, $\vec{E} = a\frac{\hat{r}}{r^2}$. Simple:

Beware of this word. It's usually a trap.

$E_\theta = 0$. $E_\Phi = 0$. $E_r = a/r^2$, so $\partial\left(r^2 E_r\right)/\partial r = 0$.

Therefore, $\vec{\nabla}\cdot\vec{E} = 0$ everywhere !! ??

We know this is wrong, because there is charge at the origin. The problem is that ***E*** is not well behaved at the origin – its derivatives are not defined at the singularity.

How can we deal with this?

Singularities and the Dirac δ-function

See Griffiths, 1.5

We have a point charge at the origin. ρ is zero everywhere except there, where it is infinite. That's the singularity.

We know how to deal with it, because the total charge is Q. That means that if we integrate $\int_{Volume} \rho \, dV$ over any volume that contains the origin, we obtain Q. If the volume does not contain the origin, we obtain 0.

> A function, in this case, $\rho(\boldsymbol{r})$, that is zero everywhere except at one point, but that has a finite (non-zero) integral, is called a Dirac δ-function.

Comments:

- δ-functions are defined in 1, 2, and 3 dimensions.
- I won't discuss them more now, but they appear now and again. In particular, they are important in the development of Green's functions solutions to differential equations.

Graphically, a δ-function is a "needle":

Disclaimer: Mathematically inclined students will complain that I'm being sloppy. That's true …