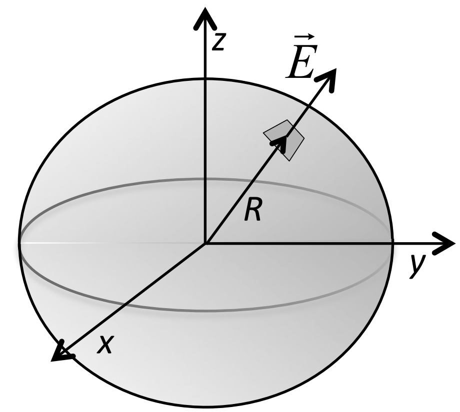
## An often-confusing point:

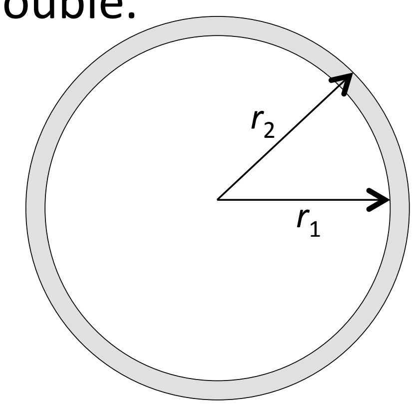



Recall this example from last lecture:
***E*** due to a uniform spherical surface charge, density = σ.

Let's calculate the pressure on the surface. Due to the repulsive forces, there is an outward pressure.

The force per unit area on a charged surface is ***F*** = σ***E***.
However, $E = \frac{Q}{4\pi\varepsilon_o r^2}$ outside, and $E = 0$ inside. What value of *E* should we use?

The way to deal with this is to eliminate the discontinuity by considering a very thin shell, analyze the problem, and then take the limit as the thickness goes to zero. If we get a plausible result, then we're good; otherwise, we're in trouble.




So, consider a spherical shell between $r_1$ and $r_2$, with an arbitrary (but spherically symmetric) charge density, ρ(*r*).

The force on an infinitesimal shell, *dr*, at radius *r* ($r_1 < r < r_2$) is:

$$dF = E(r)\cdot\left(4\pi r_1^2\right)\rho\left(r\right)dr$$

I write $r_1$ (*i.e.*, a constant), because, for a thin shell, the surface area does not change much between $r_1$ and $r_2$.

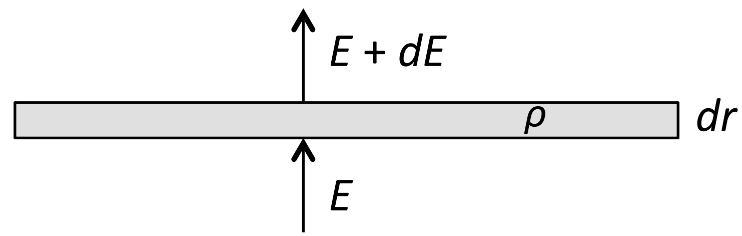
This is discussed in section 1.14 of Purcell and Morin, *Electricity and Magnetism*.

We have: $dF = E(r)\cdot\left(4\pi r_1^2\right)\rho\left(r\right)dr$ .

To integrate this, (to find the total force) we first need to relate $E(r)$ and ρ($r$).

Remember Gauss's law. It tells us that as one passes through a slab of charge, the electric field changes:

$$dE = \frac{\sigma}{\varepsilon_0} = \frac{\rho dr}{\varepsilon_0}$$




Thus, $dF = E\left(4\pi r_1^2\right)\varepsilon_0 dE$

Now we can do the integral:

$$F = 4\pi\varepsilon_0 r_1^2 \int_{E(r_1)}^{E(r_2)} E\,dE = \left[4\pi\varepsilon_0 r_1^2\right]\left[\tfrac{1}{2}\left(E^2(r_2) - E^2(r_1)\right)\right]$$

$$= 4\pi\varepsilon_0 r_1^2\left(E(r_2) - E(r_1)\right)\tfrac{E(r_2)+E(r_1)}{2}$$

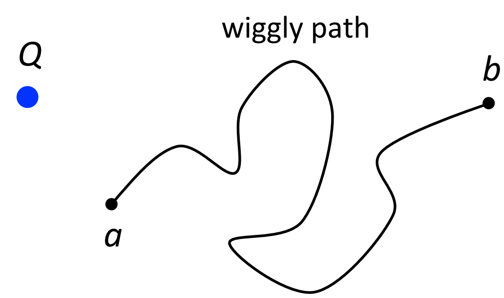
$$= 4\pi r_1^2\sigma\,\tfrac{E(r_2)+E(r_1)}{2} = Q_{tot}\,\tfrac{E(r_2)+E(r_1)}{2}$$

So, the right answer is to use the average of the fields on the inside and outside. This is a reliable result as the shell approaches zero thickness, because it does not depend on the thickness (as long as the shell is reasonably thin).

# The Electric Potential: Curl of *E*

Consider our trusty point charge: $\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^2}$

Let's calculate the work done on another charge as it moves from *a* to *b*.




We need to integrate: $W = \frac{qQ}{4\pi\varepsilon_0}\int_a^b \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot d\vec{l}$

This integral is most easily done in spherical coordinates.
It simplifies from 3-D to 1-D, because $\hat{\mathscr{r}} = (1,0,0)$.
Therefore we only need $dl_{\mathscr{r}} = d\mathscr{r}$.

For completeness:
$dl_\vartheta$ = r dθ
$dl_\Phi$ = r sinθ dϕ

Thus, $W = \frac{qQ}{4\pi\varepsilon_0}\int_{\mathscr{r}_a}^{\mathscr{r}_b}\frac{d\mathscr{r}}{\mathscr{r}^2} = \frac{qQ}{\varepsilon_0}\left(\frac{1}{\mathscr{r}_a} - \frac{1}{\mathscr{r}_b}\right).$

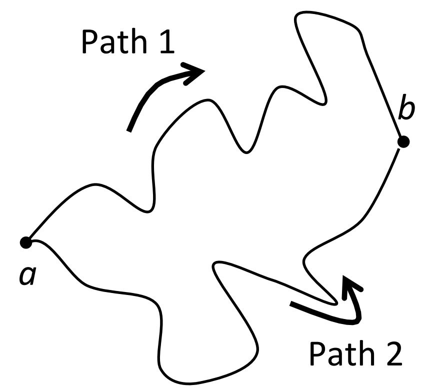
This is nice: $V = \frac{Q}{4\pi\varepsilon_0 r}$. But that's not my main point.

The important point is that W does not depend on the path taken!

## On to the curl! (all you surfer dudes)

- Superposition tells us that path independence holds for any distribution of charge.
- Path independence tells us that the loop integral of ***E*** is zero for any loop: $\oint_{loop} \vec{E}\cdot d\vec{l} = 0$

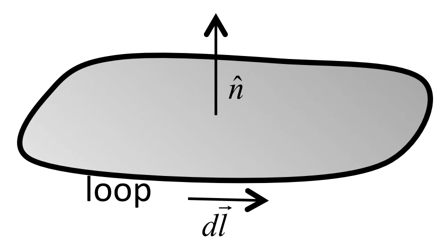
  Consider two paths between *a* and *b*. The two path integrals are equal. Thus, the loop integral is zero, because one path will be traversed backwards.




- Now, put in a little math: Stokes' theorem (discovered by Kelvin).

$$\oint_{loop} \vec{E}\cdot d\vec{l} = \int_{surface} \left(\vec{\nabla}\times\vec{E}\right)\cdot\vec{n}\, dA \quad \Leftarrow \text{No physics here!}$$

  The loop is the boundary of the surface. Every surface has a boundary, and the loop integral on every boundary is zero. Therefore, we have $\int_{surface} \left(\vec{\nabla}\times\vec{E}\right)\cdot\vec{n}\, dA = 0$ for every surface.



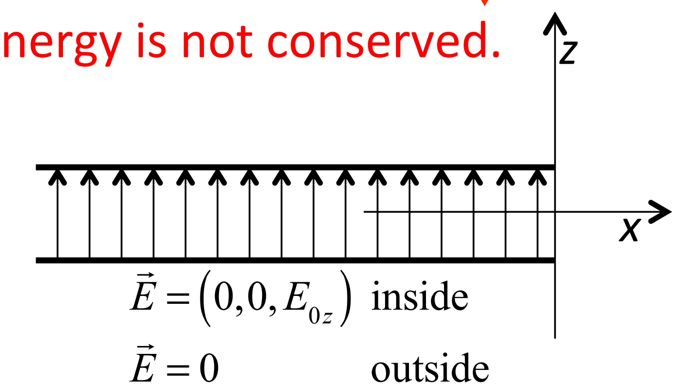

Therefore, $\vec{\nabla}\times\vec{E} = 0$ everywhere.

This is another part of Maxwell's equations.

For static fields. We'll deal with time varying fields later.

## Comment:

There is an intimate connection between $\vec{\nabla}\times\vec{E}=0$ and the existence of the electric potential. Basically, if $\vec{\nabla}\times\vec{E}\neq 0$, then energy is not conserved.



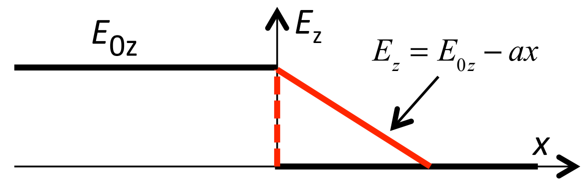

## Example: Parallel plate capacitor

We usually model the capacitor like this:

***E*** has only a z-component, and it is independent of *x*, except at the edge of the capacitor.

The discontinuity doesn't make sense, so let's suppose that $E_z$ goes linearly to zero.

(This is not the exact solution, but close.)

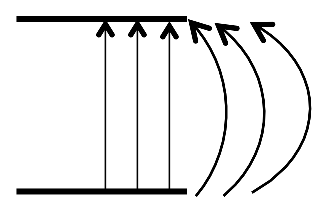



However, this implies that the y-component of $\vec{\nabla}\times\vec{E}$ is not zero:

$$\left(\vec{\nabla}\times\vec{E}\right)_y = \frac{\partial E_x}{\partial z} - \underbrace{\frac{\partial E_z}{\partial x}}_{-a}$$

To keep it zero, we need $E_x = -az$. So, the field looks like this:

This is called the fringe field of the capacitor.



Puzzle: Why do the field lines curve differently as one moves to the right?

**Note:** This can't be the right answer, because $E_x$ is discontinuous at x = 0 and z ≠ 0, changing from 0 inside to $-az$ outside. We'll learn how to solve this kind of problem soon.

# $\vec{\nabla} \times \vec{E} = 0 \Rightarrow$ Electric Potential

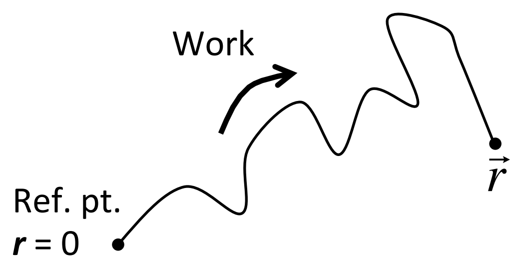
An important implication: We can define *V*(***r***), a function of position only, such that:

$E_{tot}$ = *KE* + *qV*(***r***) is a "constant of the motion". (Energy is conserved.)

Because $\vec{\nabla} \times \vec{E} = 0$, this math identity: $\vec{\nabla} \times \left(\vec{\nabla} V(\vec{r})\right) = 0$ lets us write $\vec{E} = -\vec{\nabla} V$ .

Calculate the work done on a charge in a field:

$$W = q\int_0^{\vec{r}} \vec{E} \cdot d\vec{l}$$
$$= -q\int_0^{\vec{r}} \vec{\nabla} V \cdot d\vec{l}$$
$$= -q\left(V(\vec{r}) - V(0)\right)$$




So, *W* + *q*Δ*V* = 0, or *KE* + *qV* is constant.

Note: *U* = *qV*.
*U* is the potential energy.
*V* is the electric potential.

$$\vec{F} = -\vec{\nabla} U$$
$$\vec{E} = -\vec{\nabla} V$$

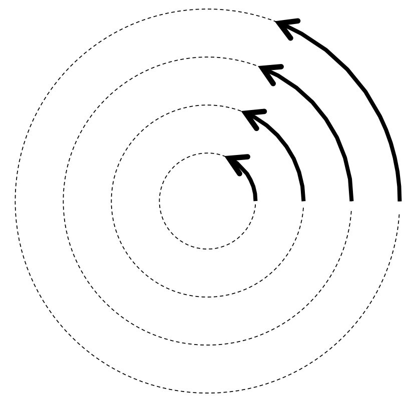
## A problem solving strategy:

If you have a choice between using ***E*** and *V* to solve a problem (*e.g.*, "What is the particle's energy when it is at ***r***?"), you are usually better off with *V*. If you use ***E***, you'll probably end up doing the ***E***·d***l*** integral anyway. Don't do unnecessary work.

### Curl example:

Consider this vector field (in cylindrical coordinates): $\vec{A} = r\hat{\varphi}$

It circulates around, and its magnitude is proportional to r.



The curl only has a *z*-component (out of the page):

$$\left(\nabla \times \vec{A}\right)_z = \frac{1}{r}\frac{\partial\left(rA_\varphi\right)}{\partial r} = \frac{1}{r}\frac{\partial r^2}{\partial r} = 2$$, *i.e.*, it's constant.

As a check, consider a circle of radius *R*.

- The integral of the curl over the area of the circle is $2(\pi R^2)$.
- The loop integral of A around the circle is $(2\pi R)R$, the same result.

Puzzle: What about this field: $\vec{B} = \hat{\varphi} / r$? (a magnetic field near a wire). It circulates, but has no curl. Does that make sense?

Note: In mechanics, field ***A*** describes rotation of a solid, and field ***B*** describes the vortex motion of a fluid (a whirlpool).

End 8/30/13