Announcements

Office hours:

• Xueda Wen	Wednesday	10-11 AM	4111 ESB
• Peter Sahanggamu	Wednesday	3-4 PM	390C L-S Interpass
• Jon Thaler	Thursday	5-7 PM	158 Loomis
• Chun Kit Chan	Friday	2-3 PM	279 Loomis

Lectures:

Starting with Friday's lecture, I will post the lecture slides on the web page the night before (early, I hope).

Caveat: I don't guarantee that the slides will be exactly what I show.
(I hope they'll be close.)

Work and Energy (2.4)

We have been talking about the electric potential, and we'll spend a lot of time calculating it.

However, there is another aspect of the problem that we also need to deal with, the amount of energy (Joules, not Volts) needed to assemble a collection of charges.

Consider a collection of point charges: $q_1, \mathbf{r}_1$ $\quad q_2, \mathbf{r}_2$ $\quad q_3, \mathbf{r}_3$

> **Note:**
> It is tacitly assumed that the charges are brought in from infinity, where there is no initial energy.

How much work does it take to assemble this configuration?
A tempting, but incorrect, answer: (inspired by $U = qV$)

We can easily calculate the potential at every charge (due to the others):

$$V(\vec{r}_i) = \frac{1}{4\pi\varepsilon_0} \sum_{j \neq i} \frac{q_j}{\mathscr{r}_j} \quad \text{where } \vec{\mathscr{r}}_j = \vec{r}_i - \vec{r}_j.$$

So, is $W = \sum_i q_i V(\vec{r}_i)$??? No !!!

> **Note:**
> $j \neq i$, because a charge does not exert a force on itself.

The problem with this:

We are wrong by a factor of two. Two see why, consider two point charges:

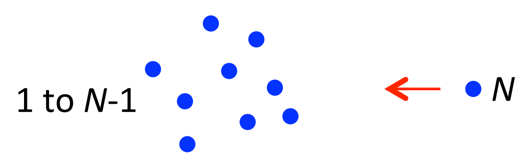
Put it together one charge at a time. The first one takes no energy. The second one takes this much work:

$$W = \left(\frac{1}{4\pi\varepsilon_0} \frac{q_1}{\mathscr{r}_{12}} \right) q_2 = q_1 V_1 \left(\vec{r}_2 \right) = q_2 V_2 \left(\vec{r}_1 \right)$$

So, if we sum over both charges, we'll be double counting.

By superposition, this argument can be extended to any collection of charges.

Another way to see this. Bring in the N^{th} charge:



The additional energy (work) is only due to the force that 1 to N-1 exert on N, not the force that N exerts on 1 to N-1, because they are not moving.

So, the correct answer is:

$$W = \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \sum_{j \neq i} \frac{q_i q_j}{\mathscr{r}_{ij}}$$

Continuous charge distributions:

For a continuous charge distribution, the sums become a single volume integral:

$$W = \frac{1}{2}\int \rho(\vec{r})\,V(\vec{r})\,d\text{Vol}$$

Was: $\sum_{i}$

Was: q_i

Was: $\sum_{j} \frac{q_j}{\mathscr{r}_{ij}}$

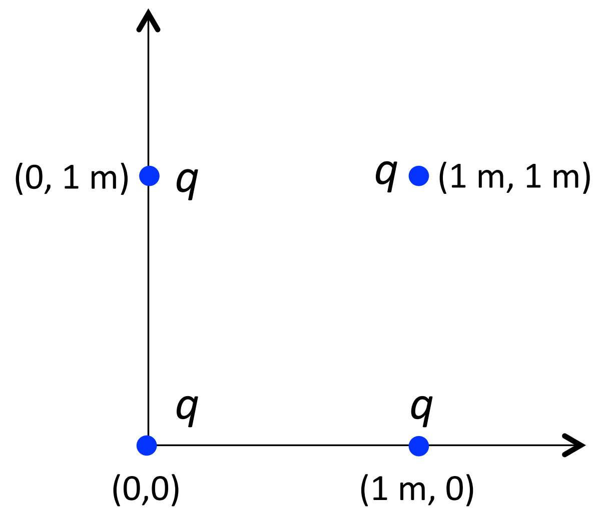
There is only one integral, because the second sum is implicit in $V(\boldsymbol{r})$.
Also, we no longer need to worry about $i \neq j$.

Example:

Four equal charges at the corners of a square:
$q = 10^{-6}$ C.

The potential at each corner is that due to the other three charges:

$$V = \frac{10^{-6}\ \mathrm{C}}{4\pi\varepsilon_0}\left(\frac{2}{1\ \mathrm{m}} + \frac{1}{\sqrt{2}\ \mathrm{m}}\right) = 24.3\ \mathrm{kV}$$



So, the energy required to assemble this configuration (what you could get back by taking it apart) is:

$$W = \frac{1}{2}\left(4 \times 24.3\ \mathrm{kV} \times 10^{-6}\ \mathrm{C}\right) = 0.486\ \mathrm{J}$$

Example:

Calculate the energy stored in a spherical shell of radius R = 1 m, that has a uniform surface charge density, $\sigma = 10^{-6}$ C/m^2.

This problem has spherical symmetry. At the surface: $V = \frac{Q_{\text{tot}}}{4\pi\varepsilon_o R} = \frac{\sigma R}{\varepsilon_o} = 113\ \text{kV}$

$$W = \frac{1}{2}\int \sigma V\, dA = \frac{1}{2}\left(\frac{\sigma^2 R}{\varepsilon_0}\right)\left(4\pi R^2\right) = 0.71\ \text{J}$$

Comments:

- A microCoulomb is a lot of charge. It gives significant energies.
- Because this is surface charge, $\int \rho\, dV \to \int \sigma\, dA$

Configuration Energy in Terms of *E*

This involves some math, but it's instructive.

$$W = \frac{1}{2}\int \rho V \, d\text{Vol}$$

Use: $\rho = \varepsilon_o \vec{\nabla} \cdot \vec{E}$

$$= \frac{\varepsilon_0}{2}\int \left(\vec{\nabla} \cdot \vec{E}\right) V \, d\text{Vol}$$

Integrate by parts: $u dv = d(uv) - v du$. Here, u is V and v is $\vec{E}$.

$$= \frac{\varepsilon_0}{2}\left[\oint_{\text{surface}} \left(\vec{E} V\right) \cdot d\vec{a} - \int_{\text{Vol}} \vec{E} \cdot \vec{\nabla} V \, d\text{Vol}\right]$$

To see how this works with vectors, write the *x*, *y*, and *z* components explicitly.

Now, note that: $\vec{E} = -\vec{\nabla} V$.

$W = \frac{\varepsilon_0}{2}\int_{\text{Vol}} E^2 \, d\text{Vol}$ plus the surface integral.

Commentary:

Integration by parts is a very common manipulation, because the surface term can often be ignored, for example by letting the volume of integration become very large (or infinite).
This works if the u and v quantities (here, $\mathbf{E}$ and V) go to zero sufficiently rapidly.

Example: A localized charge distribution
For large r, the surface area increases as r^2, $\mathbf{E}$ falls as $1/r^2$, and V falls as $1/r$.
EV falls as $1/r^3$, so it works.

BEWARE of situations where this is not the case, such as an infinite line of charge.

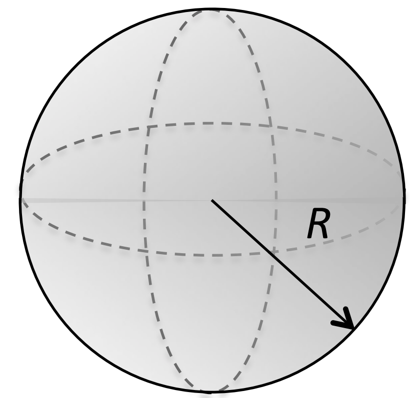
BEWARE also of using $\mathbf{E}$ to calculate the energy of point charges.
You'll get infinity (next slide).

Calculating the energy by integrating E^2 looks like a bookkeeping trick. However, it is possible to have $\mathbf{E}$ in the absence of any charges (EM radiation). In this situation, the $\mathbf{E}$ and $\mathbf{B}$ fields themselves carry energy – it's not merely a trick.

A puzzle:

In next week's homework, you'll calculate the energy of a uniform sphere of charge.

$$\text{Answer:} \quad \text{Energy} = \frac{3}{5}\frac{1}{4\pi\varepsilon_0}\frac{Q^2}{R}$$



An interesting question:

Suppose the electron is a small sphere of charge.

For what radius, *R*, does the electrostatic energy equal the rest energy, $m_e c^2$?

$m_e c^2$ = 511 eV = 8.2×10^{-14} J, so $R = 1.7\times10^{-15}$ m.

However, the electron is known to be smaller than 10^{-22} m !!

TL;DR: Classical E&M fails at these distance scales.

This question is addressed in quantum field theory (Physics 582-3).

Note: The 3/5 factor depends on the charge distribution.
What distribution do you think gives the smallest factor (minimizing *R* for a given *E*)?

End 9/4/13