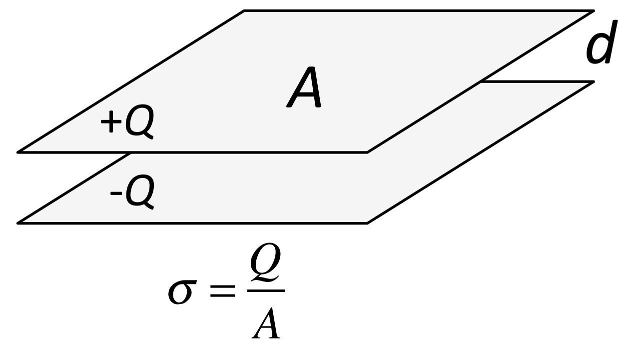
Capacitors (2.5.4)

We all know and love parallel plate capacitors:

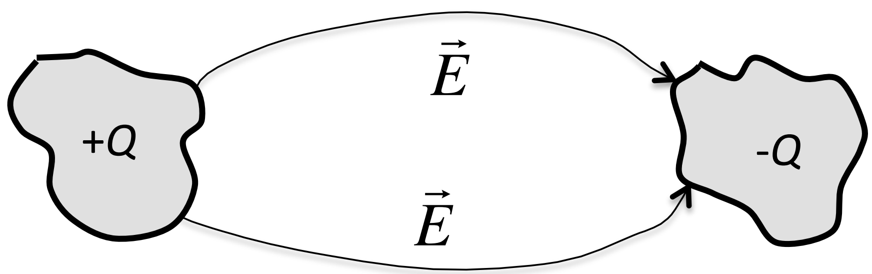
Ignoring the fringe fields, $C = \frac{\varepsilon_0 A}{d}$.

This expresses the fact that if you put +*Q* on one plate and –*Q* on the other, then:



$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}; \qquad V = Ed = \frac{Qd}{\varepsilon_0 A}$$

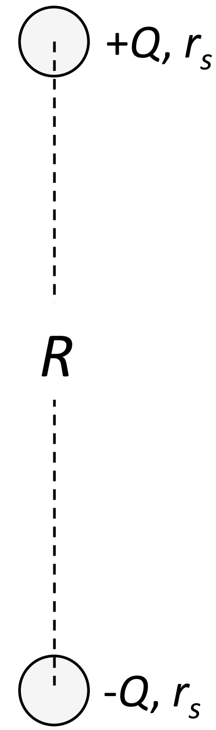
Capacitance is defined by $Q = CV$, so we have $C = \varepsilon_0 A/d$.
Capacitance occurs in any situation where there are multiple charge-carrying objects. The voltage depends on $\int_+^- \vec{E} \cdot d\vec{l}$.



Note:
We almost always consider ±*Q*.
Why?

This is, in general, a complicated problem. However, in every case $E \propto Q \Rightarrow V \propto Q$.
The constant of proportionality, 1/C, depends only on geometry.

Example:



Calculate the capacitance of two small, spherical conductors (radius r_s).
If $R >> r_s$, then the charge is distributed uniformly around each sphere.
(We'll learn how to solve the more general problem later.)

The potential due to each sphere separately is:

$$V_{\pm}(\vec{r}) = \frac{\pm Q}{4\pi\varepsilon_0 |\vec{r} - \vec{r}_{\pm}|}, \quad \text{where } \vec{r}_{\pm} \text{ are the centers of the spheres.}$$

Start at the surface of –Q and end at +Q. The potentials are:

$$V_{\pm s} = \frac{\pm Q}{4\pi\varepsilon_0}\left(\frac{1}{r_s} - \frac{1}{R - r_s}\right) \text{, so for } R >> r_s \quad \Delta V \approx \frac{Q}{2\pi\varepsilon_0 r_s}.$$

Therefore, $C \approx 2\pi\varepsilon_0 r_s$. Note the units of ε_0: $\varepsilon_0 = 8.85\times10^{-12}$ F/m.

Q/C = V

Small r_s gives small C. Even a little charge on a small object gives a large voltage.

Also note that a Farad is a huge capacitance. It is hard to make C that big.

Image Charge (3.2)

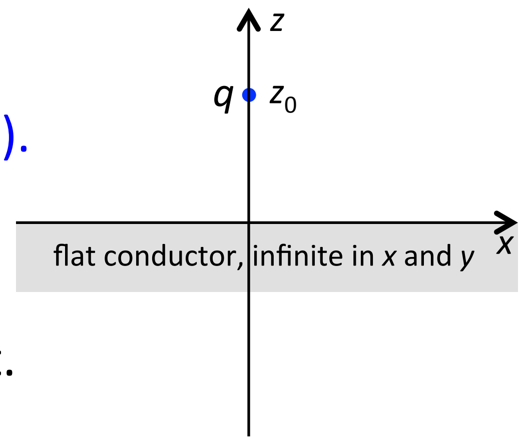
I'm doing this a bit out of "book order", because it fits in with charges and conductors. This is a method of solving the problem without actually solving the differential equation.

Consider this simple (?) problem:

A point charge near a conducting surface (the $z = 0$ plane).

What is V(***r***) in the region above the surface ($z > 0$)?

Because $z < 0$ is inside the conductor, we can't know anything about that region. We'll take advantage of that.



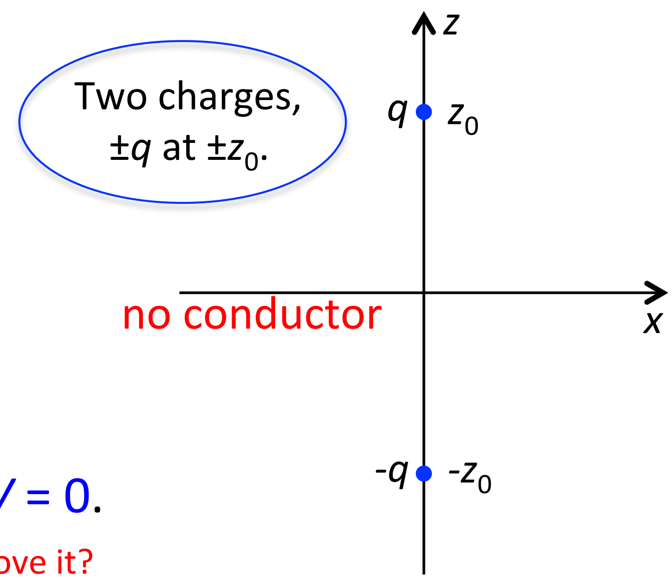
V(***r***) due to q only is: $V_q\left(\vec{r}\right) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{\left|\vec{r}-\vec{r}_0\right|}$, where $\boldsymbol{r}_0 = (0, 0, z_0)$.

This can't be the right answer, because the Coulomb field is not perpendicular to the surface. Surface charge must be induced to satisfy the boundary condition: $\vec{E} \,||\, \hat{n}$ at $z = 0$. Note that $\hat{n} = \hat{z}$.

Can we solve this problem? Yes, by using a trick. (It's not the only way ...)

Consider this (related ?) problem:

In this case, $V(\vec{r}) = \dfrac{q}{4\pi\varepsilon_0}\left(\dfrac{1}{\left|\vec{r}-\vec{r}_+\right|} - \dfrac{1}{\left|\vec{r}-\vec{r}_-\right|}\right)$.



The field looks like the bottom figure.
By symmetry, ***E*** points along –***n*** in the z = 0 plane.
V is constant in that plane (for all x and y). In fact, V = 0.

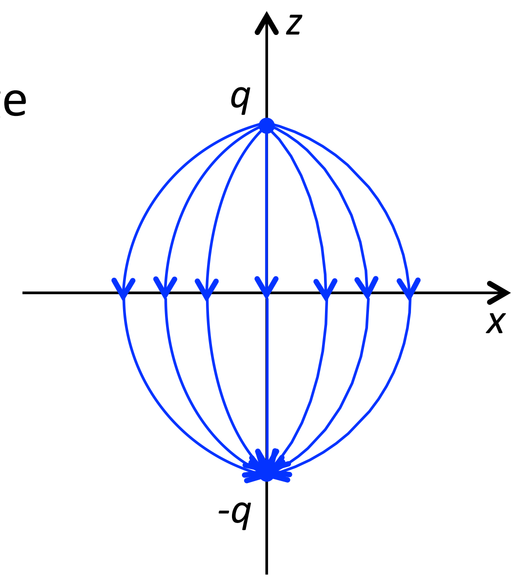
Can you prove it?

Comments:

- -q is inside the conductor. It can't be the source of the observed field, because we aren't sensitive to any charge behind the conducting surface.

  -q is called an "image charge", in analogy with optical images

- This gives us a solution to the problem if we can find a surface charge density that produces the same V in the observable region (z > 0).



What distribution, $\sigma(x,y)$, of surface charge (*i.e.*, at z = 0) will give the same V(***r***) in the observable region?

Use the fact that $\vec{E} = -\vec{\nabla} V$.

$$E_x = -\frac{\partial V}{\partial x} = \frac{-qx}{4\pi\varepsilon_0}\left[-\frac{1}{\mathscr{r}_+^3} + \frac{1}{\mathscr{r}_-^3}\right] = 0, \qquad \text{when } z = 0 \left(\mathscr{r}_+ = \mathscr{r}_- \equiv \mathscr{r}\right)$$

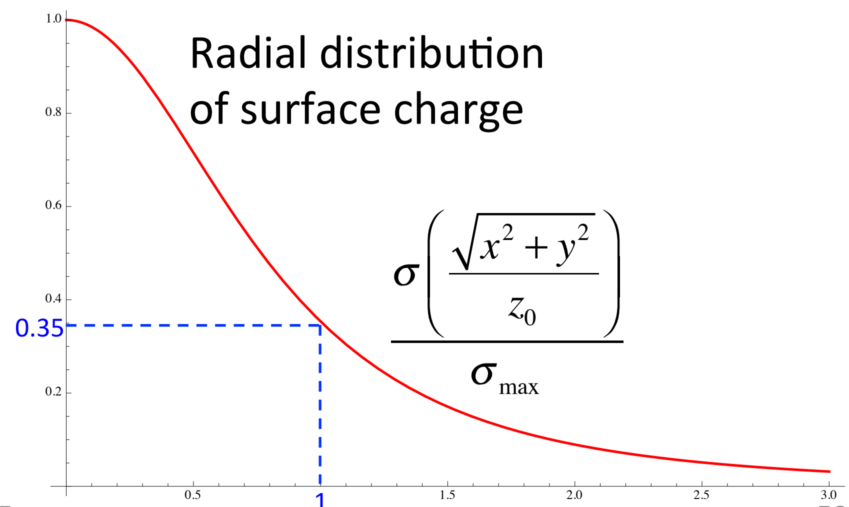
$$E_y = -\frac{\partial V}{\partial y} = \frac{-qy}{4\pi\varepsilon_0}[\text{same}] = 0, \qquad \text{when } z = 0$$

$$E_z = -\frac{\partial V}{\partial z} = \frac{-q}{4\pi\varepsilon_0}\left[-\frac{z - z_0}{\mathscr{r}_+^3} + \frac{z + z_0}{\mathscr{r}_-^3}\right] = \frac{-qz_0}{2\pi\varepsilon_0\mathscr{r}^3}, \qquad \text{when } z = 0$$

In the z = 0 plane, $\mathscr{r}^2 = x^2 + y^2 + z_0^2$.

So, $\sigma(x,y) = \varepsilon_0 E_z = \dfrac{-qz_0}{2\pi\left(x^2 + y^2 + z_0^2\right)^{\frac{3}{2}}}$

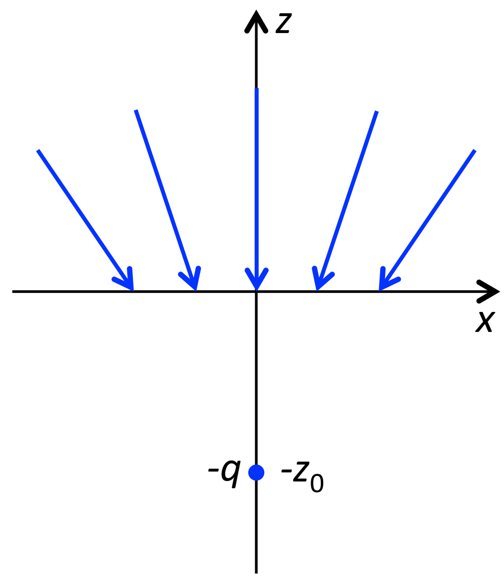
This charge distribution yields the same field outside the conductor as the (+q, -q) pair.



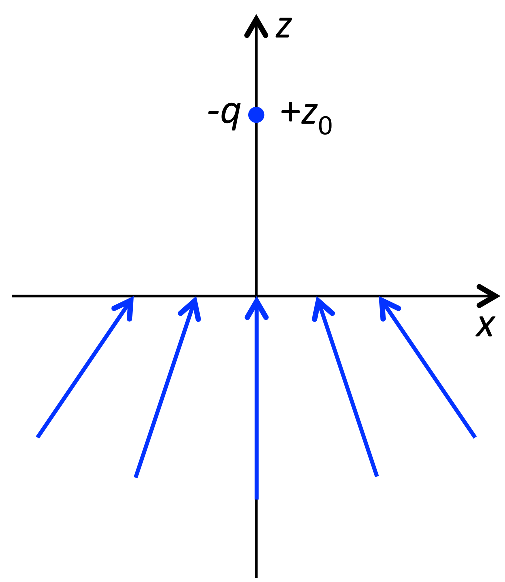
What about ***E*** inside the conductor?

Let's look at the field produced by the surface charge.

The field outside the conductor is the same as that produced by the image charge, –q:



By symmetry (because the charge lies in the z = 0 plane), the field produced at z < 0 (inside the conductor) looks like one produced by a charge $-q$ at $+z_0$, *i.e.*, at the same place as the +q charge.



The ***E*** fields produced by $+q$ and by the surface charge cancel exactly, leaving E = 0 inside the conductor.

So, we have a solution. Is it the only solution? Yes.
We will prove (soon) that the solutions are unique.

Comments:

- If you integrate σ(x,y) over the whole surface (See G, p. 123), you obtain $-q$. This is a general result (a HW problem).
- The $+q$ charge feels a force due to the induced surface charge. It is exactly the same as the force that the image charge would produce (with no conductor):

$$\vec{F} = \frac{-q^2}{4\pi\varepsilon_0 (2z_0)^2}\hat{z}$$

- However, the energy is not the same as that of a (+q, -q) pair, because the electric field only fills the $z > 0$ half of the universe:

$$W = \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\frac{-q^2}{2z_0}\right)$$

You can also show this by integrating $W = \int_{\infty}^{z_0} \vec{F}\cdot d\vec{l}$

Where does the ½ come from? Think about the difference between the behavior of a real $-q$ and an image charge as the $+q$ is brought in from infinity.

Note:
Rearranging charge on the conductor takes no energy, because it's an equipotential.

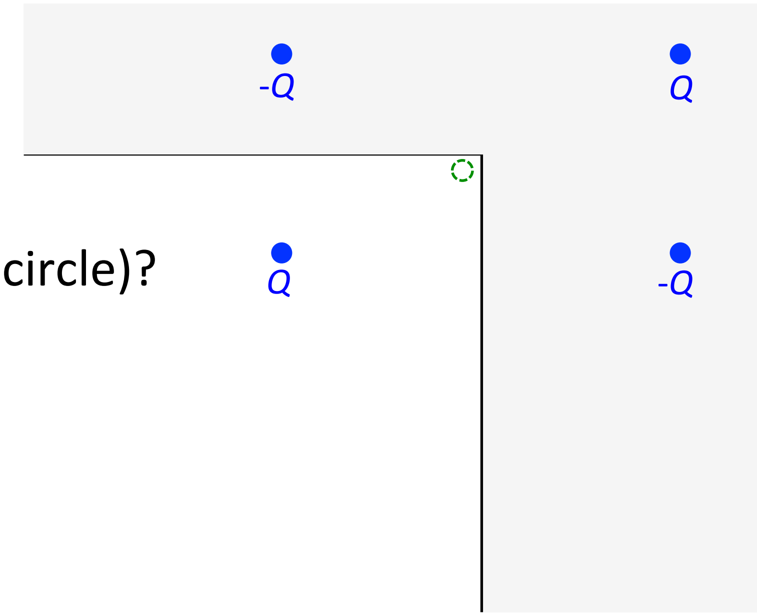
Conclusion:

If you put image charges inside a conductor such that the field configuration (ignoring the conductor) satisfies the boundary conditions, then this gives the correct solution outside the conductor.

Note: If the surface is curved, the magnitude of the image charge ≠ Q. (a HW problem)

Multiple images:

The solution to this problem requires three images, as shown.



Puzzle:

What is ***E*** near the corner (the dashed green circle)?
This is a general result for corners.

End 9/9/13