A Very Important Equation

We have: $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$ Divergence

$\vec{\nabla}\times\vec{E} = 0$

$\vec{E} = -\vec{\nabla}V$ Curl

Combine: $\vec{\nabla}\cdot\left(\vec{\nabla}V\right) = -\frac{\rho}{\varepsilon_0}$

Write this more compactly: $$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$ Poisson's equation (fishy?)

If $\rho = 0$, it's called Laplace's equation

∇^2 is called the Laplacian operator. In Cartesian coordinates, it is:

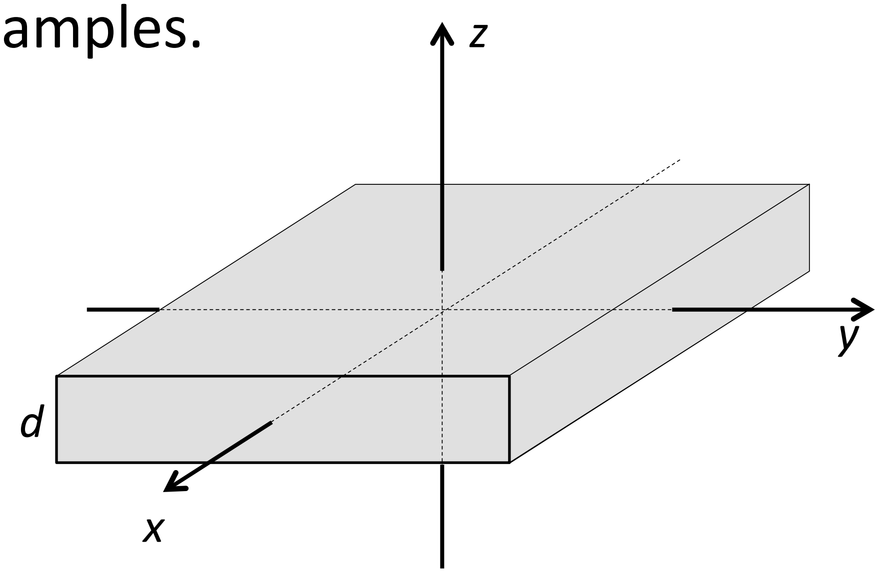
$$\nabla^2 = \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} + \frac{\partial^2}{\partial^2 z}$$

It's messier in cylindrical or spherical coordinates.
(See the inner front cover of Griffiths.)

This is useful because, given $\rho(\boldsymbol{r})$,
we have a linear, 2nd order differential equation for $V(\boldsymbol{r})$.
This is a solvable problem.

The Laplacian appears in Schrödinger's equation. Special relativity uses a 4-D version, called the d'Almbertian

Let's start with two 1-dimensional examples.



Example:

Consider a uniformly charged slab, charge density ρ, infinite in *x* and *y*, and thickness *d* ($\frac{-d}{2} < z < \frac{+d}{2}$).

By symmetry, *V* is a function only of *z*, so only the *z* term of the Laplacian contributes.

The problem is symmetric about $z = 0$, so pick $V = 0$ at the midpoint of the slab.

Inside the slab: $$\frac{d^2V}{dz^2} = -\frac{\rho}{\varepsilon_0}\ .$$

The solution is: $$V(z) = -\frac{\rho}{2\varepsilon_0}z^2 + C_1 z + C_2$$

$C_2 = 0$, due to our choice of reference point.

What about C_1? I'll pick it to be zero, but this requires more discussion (at the end).

Outside the slab: $\frac{d^2V}{dz^2} = 0$

So, V(z) = *az+b* .

We require that both *V* and $\frac{dV}{dz}$ be continuous at *z* = ±*d*/2. (Why?)

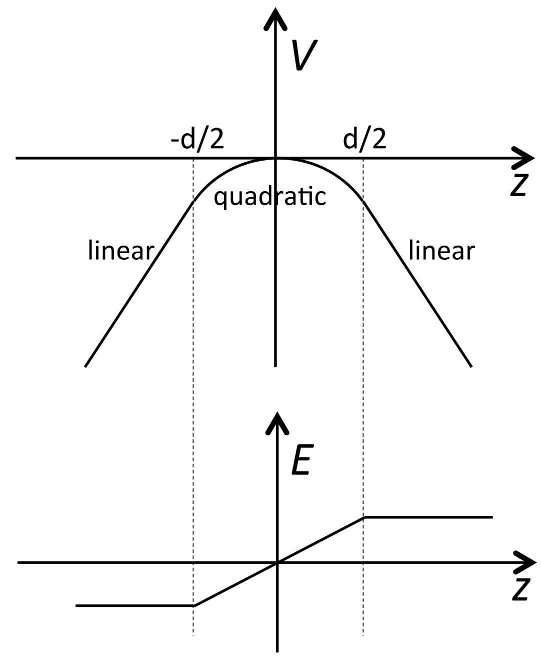
This tells us:

At *z* = +*d*/2: $V = -\frac{\rho d^2}{8\varepsilon_0}$, and $\frac{dV}{dz} = -\frac{\rho d}{2\varepsilon_0}$

So, for z > d/2: $a = -\frac{\rho d}{2\varepsilon_0}$, and $b = +\frac{\rho d^2}{8\varepsilon_0}$

Note: $E_z = -a = \frac{\rho d}{2\varepsilon_0} = \frac{\sigma}{2\varepsilon_0}$, what you expected.

Similarly, for z < -d/2: $a = +\frac{\rho d}{2\varepsilon_0}$, and $b = +\frac{\rho d^2}{8\varepsilon_0}$



OK, so what about C_1?

Inside the slab, we have: $V(z) = -\frac{\rho}{2\varepsilon_0} z^2 + C_1 z$

We can pick $C_1 \neq 0$, without affecting the reference point at z = 0. However, it adds a constant slope to $V(z)$. This Is OK as long as we add the same slope to the exterior solutions as well. What does that mean?

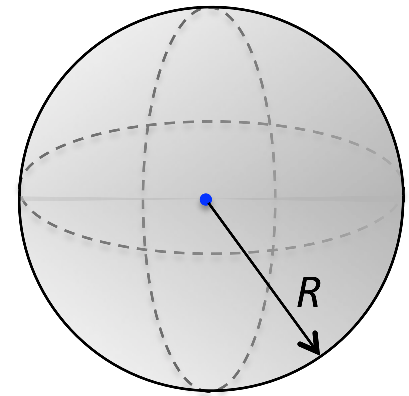
Adding a constant slope, means that there is now an extra (constant) electric field: E_z(extra) = $-C_1$.

When $C_1 \neq 0$, the slab is immersed in a uniform ***E*** field that fills all space.
This would have been disallowed if we had been able to set $V = 0$ at $z = \infty$.
Unfortunately, we can't do that, because our slab has infinite extent, and E does not fall off at large distances.

Infinite objects (slabs, long wires, *etc*.) are not physically realizable, and one must be careful when dealing with them.

Example:

Consider a sphere of radius, *R*, and uniform charge density, *ρ*. We want to calculate *V* and ***E*** inside the sphere, where $\rho \neq 0$.



By spherical symmetry, there is only *r*-dependence, not θ or *ϕ*: *V* = *V*(*r*).

So, our diff. eq. is the radial part of the spherical Laplacian:

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV(r)}{dr}\right) = \frac{-\rho}{\varepsilon_0}$$

See front cover of Griffiths.
Total derivatives because r-dependence only.

Integrate: $$r^2 \frac{dV}{dr} = -\frac{r^3}{3}\frac{\rho}{\varepsilon_0} + C_1$$

Integrate again: $$V = -\frac{r^2}{6}\frac{\rho}{\varepsilon_0} - \frac{C_1}{r} + C_2$$

C_1 and C_2 are integration constants

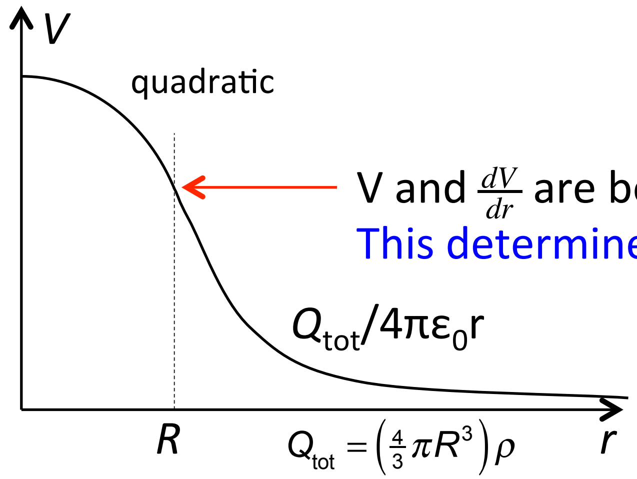
The $\vec{E}$ field: $$E_r = -\left(\vec{\nabla} V\right)_r = -\frac{dV}{dr} = \frac{\rho}{3\varepsilon_0} r - \frac{C_1}{r^2}$$

C_1 is non-zero only if there is a point charge at the origin.

C_2 is determined by the choice of r = ∞ as the reference point.

This is the solution inside the sphere.
What is the solution outside?

Graphs of V and E:

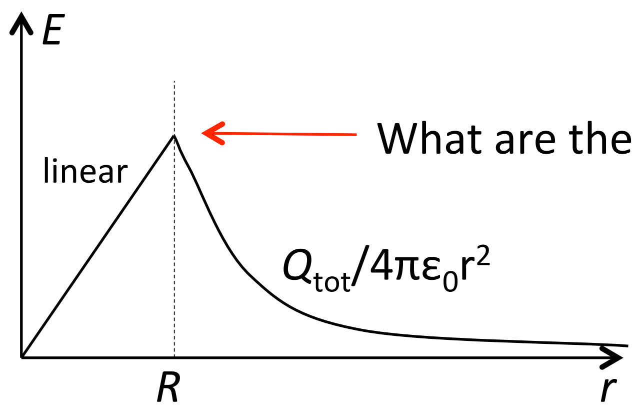


V and $\frac{dV}{dr}$ are both continuous here.

This determined C_2:

$$\frac{Q_{tot}}{4\pi\varepsilon_0 R} = -\frac{\rho R^2}{6\varepsilon_0} + C_2$$

$$C_2 = \frac{\frac{4}{3}\pi R^3 \rho}{4\pi\varepsilon_0 R} + \frac{\rho R^2}{6\varepsilon_0} = \frac{\rho R^2}{2\varepsilon_0}$$



What are the conditions on E here?

End 9/11/13