Two common difficulties with HW 2:

- Problem 1c:
  For what values of n does the divergence of $\vec{v} = r^n \hat{r}$ diverge at the origin?

  In this context, "diverge" means becomes arbitrarily large ("goes to infinity").
  We have $\nabla \cdot \vec{v} = \left(n+2\right) r^{n-1}$, except at n = -2, where that expression gives 0/0.
  So, it diverges (negative exponent) for all n < 1, except, possibly, n = -2.
  What about n = -2? That's the situation that gives the Dirac delta function at the origin.
  So, the answer is, the divergence diverges (!) for all n < -1.

- Problem 2c:
  What is the field at the center of a regular n-sided polygon, when n is odd?

  It is not sufficient to simply say, "It's zero by symmetry." More detail is needed.
  I suggest either of two arguments:
  ° Suppose the field is non-zero. Then, it must point in some direction. Now, rotate the problem by 360°/n. We have the same configuration of charges, but the field vector has rotated by 360°/n. This is only possible if the field = 0.
  ° Draw a line through one vertex and the center. The problem has mirror symmetry about this line. So, the field cannot have a non-zero component perpendicular to that line. This is true for any similar line (through a different vertex), so the field must have two distinct zero components. Therefore the field = 0.

Laplace's Equation (3.1)

Let's go through a more systematic treatment of the problem.

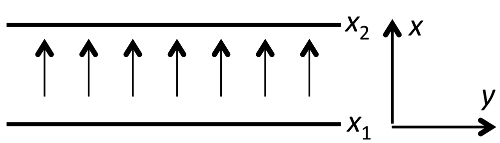
In general, $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$. Let's start with a simpler problem, a charge-free region.

In that case, we have Laplace's equation, $\nabla^2 V = 0$.

In Cartesian coordinates, this is: $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

Start with the 1-dimensional case (no *y* or *z* dependence): $\frac{d^2 V}{dx^2} = 0$

This describes a parallel-plate capacitor, or similar geometry.



Solution: *V*(*x*) is a linear function: $V(x) = ax + b$

How do we determine the constants, *a* and *b*?

We need to specify two boundary conditions.

For example: $V(x_1) = V_1$ and $V(x_2) = V_2$

Alternatively: $V(x_1) = V_1$ and $\left.\frac{dV}{dx}\right|_{x_1} = a$

Nomenclature:

- Specification of *V* is called a Dirichlet boundary condition.
- Specification of the derivative is called a Neumann boundary condition.
- This is mixed boundary conditions.

1D in spherical coordinates:

Suppose there is no angular (θ or ϕ) dependence. Then:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV(r)}{dr}\right)=0 \quad\Rightarrow\quad r^2\frac{dV(r)}{dr}=C_1 \quad\Rightarrow\quad V(r)=-\frac{C_1}{r}+C_2$$

We have either the potential due to a point charge, or a constant (which has no physical significance). That's the only spherically symmetric possibility.

1D in cylindrical coordinates:

Suppose there is no angular (ϕ or *z*) dependence. Then:

$$\frac{1}{s}\frac{d}{ds}\left(s\frac{dV(s)}{ds}\right)=0 \quad\Rightarrow\quad s\frac{dV(s)}{ds}=C_1 \quad\Rightarrow\quad V(s)=C_1\ln(s)+C_2$$

In G's notation, *s* is the radial coordinate.

We have either the potential due to an infinite line charge, or a constant.

Note: The only possible 1D angular dependence is $V(\theta)$ or $V(\phi) = C_1$. Try it.

Now, 2-dimensions:

Things are not so simple in 2-D. We now have a partial differential equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

One important difference between ordinary and partial differential equations:

To obtain a unique solution (to determine all the arbitrary constants) to a p.d.e., one must specify $V(x,y)$ or dV/dn on the boundary of the region of interest (perhaps at ∞), or some equivalent amount of information.

We'll defer examples until we learn how to solve Laplace's equation by separation of variables.

The near term plan:

- Mean Value theorem (briefly)
- Uniqueness theorems
- Solution techniques in 2D and 3D.

It's useful to understand some of the general features before plunging into specific problems.

A Short Diversion (Feynman 7-2)

This is an advertisement for Math 446 (Complex Variables) :

Consider functions of complex numbers: $z = F(w) = F(x+iy)$ $w = x + iy$

Complex numbers (z, w); Real numbers (x, y)

Here's an example: $F(w) \equiv w^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

We can write *F* as the sum of two functions, one for the real part and one for the imaginary part. $F(w) = U(w) + iV(w)$, where $U = x^2 - y^2$ and $V = 2xy$ are both real functions of *x* and *y*.

We now steal a theorem from Math 446:

For any differentiable (holomorphic) function, *F*(*w*), *U* and *V* satisfy:

$$\frac{\partial U}{\partial x} = +\frac{\partial V}{\partial y}$$

$$\frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}$$

Cauchy-Riemann equations

Well, you might say, so what?

It's easy to show that both *U* and *V* satisfy Laplace's equation:

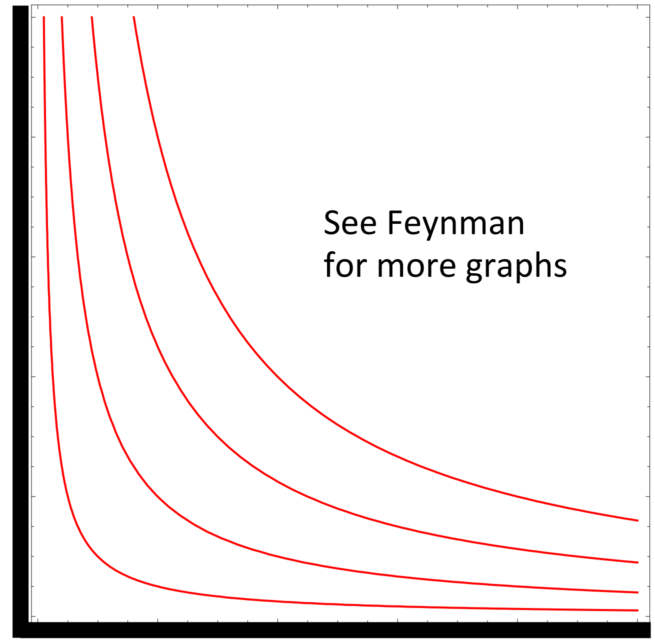
$$\nabla^2 U = 0 \text{ and } \nabla^2 V = 0$$

So, from one function, *F*,

we obtain two solutions of Laplace's equation.

Caveat: This only works in 2-D.

Example: Consider $V = xy$ (from previous slide).
Here's a graph of curves of constant *V*:

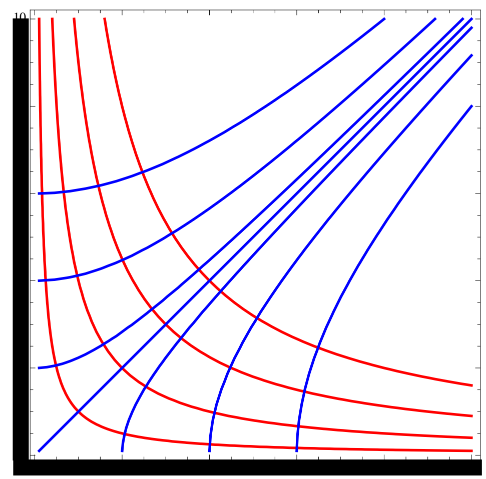


Because $V = 0$ if $x = 0$ or $y = 0$, this function is the solution for the potential when the situation is two grounded metal plates on the *x* and *y* axes.

Relation between *U* and *V*:

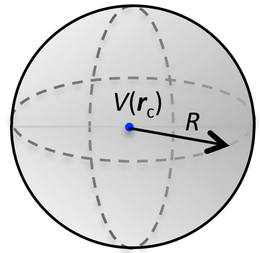
- Curves of constant *U* are perpendicular to curves of constant *V*. Look at the Cauchy-Riemann equations. Remember, derivatives are slopes.
- ***E*** is everywhere perpendicular to constant *V*. (why?)
- So, the *U* curves show us the field lines.

Note: The roles of *U* and *V* can be reversed.

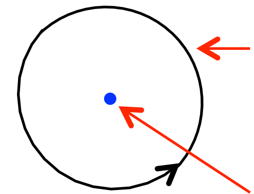


Proof of the Mean Value Theorem in 3D:

$$V\left(\vec{r}_c\right) = V_{\text{av}} = \frac{1}{4\pi R^2}\oint_{\text{Surface}} V\, dA$$



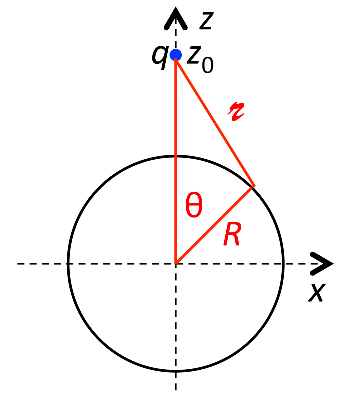
The theorem also works in 1D and 2D (if the geometry really is 1D or 2D):

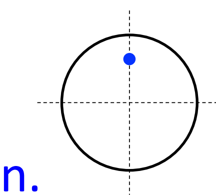


Prove it for a single point charge outside the sphere.
The general situation follows by superposition.
Put the center of the sphere at the origin and the charge at z_0.
Do the integral in spherical coordinates:

$$V_{\text{av}} = \frac{1}{4\pi R^2}\frac{q}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\pi}\frac{R^2\sin\theta\, d\theta\, d\varphi}{\underbrace{\left(z_0^2 + R^2 - 2z_0 R\cos\theta\right)^{\frac{1}{2}}}_{=\,\mathscr{r}}} = \frac{q}{4\pi\varepsilon_0 z_0} = V\left(\vec{r}_c\right)$$

For $z_0 > R$.
G does the details.





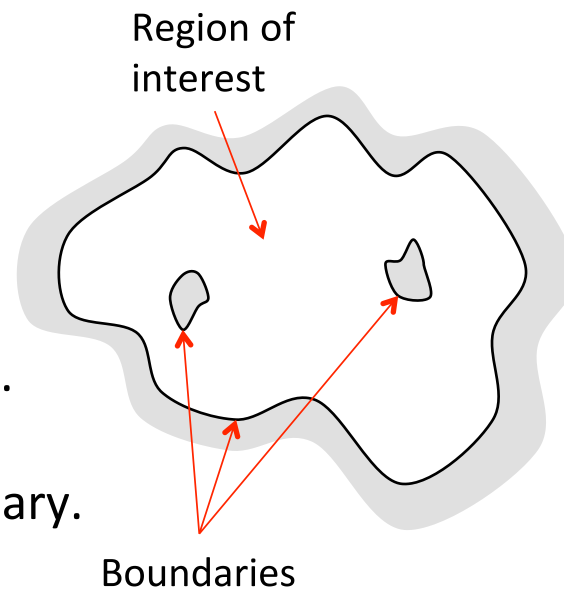
For $z_0 < R$, the answer is $\frac{q}{4\pi\varepsilon_0 R}$ (HW).

The most important consequence is Earnshaw's Theorem:
V cannot have a maximum or minimum inside a charge-free region.
$\Rightarrow$ You cannot confine a charge with electrostatic fields only.

Comment:
G says that the MVT suggests the **method of relaxation** as a numerical solution technique.
This technique is very inefficient, approaching the correct solution logarithmically.

Uniqueness of Solutions (3.1.5-6)

You may recall: I claimed that uniqueness of solutions requires specifying V or dV/dn (or some combination) at the boundaries of the region of interest. Let's prove it.



Comments:

- In 2D, the boundaries are lines. In 3D, they are surfaces.
- The outer boundary might be at ∞. Set $V = 0$ there.
- I'll prove that these conditions are sufficient, not necessary.
- Why are we bothering with this??
  Don't be proud … If you can find a solution by any nefarious means (*e.g.*, images), you're done. There aren't any other solutions.

There are three theorems:

- When V is specified,
- When dV/dn ($= E_{perp}$) is specified, and
- When charge (on conductors) is specified.

G proves the first and third. I'll do the first and second.

Proofs by contradiction:

1: Suppose there are two distinct solutions, V_1 and V_2, satisfying the same boundary conditions. Then, because the equations are linear:

$V_3 = V_1 - V_2$ is also a solution to Laplace's equation, with different boundary conditions: $V_3 = 0$ on all boundaries. However, the mean value theorem tells us that $V_3 = 0$ everywhere in the region. Therefore, $V_1 = V_2$.

Note that this proof also applies to Poisson's equation, where $\rho \neq 0$ in the region

2: The boundary condition is on E_{perp}, so compare two distinct solutions, $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$. As above, consider $\boldsymbol{E}_3 = \boldsymbol{E}_1 - \boldsymbol{E}_2$. $\vec{\nabla} \cdot \vec{E}_3 = 0$ everywhere in the region of interest, even if $\rho \neq 0$. Also, $E_{3perp} = 0$ on every surface.

Use G's vector identity #5: $\vec{\nabla} \cdot \left(f\vec{A} \right) = f\left(\vec{\nabla} \cdot \vec{A} \right) + \vec{A} \cdot \left(\vec{\nabla} f \right)$. Let f be V_3 and $\boldsymbol{A}$ be $\boldsymbol{E}_3$. Then, $\vec{\nabla} \cdot \left(V_3 \vec{E}_3 \right) = V_3 \left(\vec{\nabla} \cdot \vec{E}_3 \right) + \vec{E}_3 \cdot \left(\vec{\nabla} V_3 \right) = 0 - E_3^2$

Now, integrate over the region of interest and use the divergence theorem:

$$\int_{\text{volume}} \vec{\nabla} \cdot \left(V_3 \vec{E}_3 \right) dVol = -\int_{\text{volume}} E_3^2 \, dVol$$

$$= \oint_{\text{surfaces}} V_3 \left(\vec{E}_3 \cdot \hat{n} \right) da = 0$$

Therefore, $\boldsymbol{E}_3 = 0$ everywhere.

End 9/13/13