Announcement:

I want to encourage discussion in lecture. If you email me a question or topic you want me to discuss, before 9 PM the day before a lecture, I will spend 5-10 minutes the next day covering those topics that are of general interest.

Reminder:

The first exam will be in class on October 2. (two weeks from this Wednesday)

All exams are open notes (your handwriting only), not open book.
I will supply the essential formulas (*e.g.*, Laplacian in cylindrical coordinates).

Boundary Conditions

By email request (yesterday):

Explain the physics behind the boundary conditions.

The integration constants in Laplace's equation (and other differential equations) ar determined by boundary conditions. These come from three sources:

- The specific configuration of the problem:
  - Where the objects are and what their properties are.
  - Imposed conditions, such as an applied external field.
- Constraints from physical reality ($\boldsymbol{E} \neq \infty$).
  - We often violate these, for sake of simplicity.

Configuration constraints:

- *j* = 0:
  V = constant everywhere in a conductor, particularly on the surface.
  Corollary: $\vec{E} \parallel \hat{n}$ at the surface of a conductor. (The tangential component = 0.)

- Gauss's Law:
  $E_{\text{n}} = -\frac{\partial V}{\partial n} = \frac{\sigma}{\varepsilon_0}$ at the surface of a conductor. (Normal component of ***E***)

- Surface (2D) charge distributions: (+ and – denote the two sides of the surface)
  $E_{\text{n+}} - E_{\text{n-}} = \frac{\sigma}{\varepsilon_0}$ across the surface. This is also Gauss's law. It is a generalization of the conductor situation, where the charge all resides on the surface.
  This is an exception to the requirement that ***E*** be continuous (see below).
  Realistically, all objects have some thickness, and the situation is that of a thin slab (discussed last Wednesday), for which there is no discontinuity. However, the problem is simpler to solve in the surface density approximation.

- Symmetry:
  The solution must have the same symmetry (*e.g.*, spherical) as the configuration.

Physics constraints:

- Conservation of energy:
  $\nabla \times E = 0$ is not enough. Laplace's equation already guarantees this.
  We must make sure that $\oint \vec{E} \cdot d\vec{l} = 0$. Math is not always equivalent to physics.
  Example: The "1/s" circulating field in the homework.

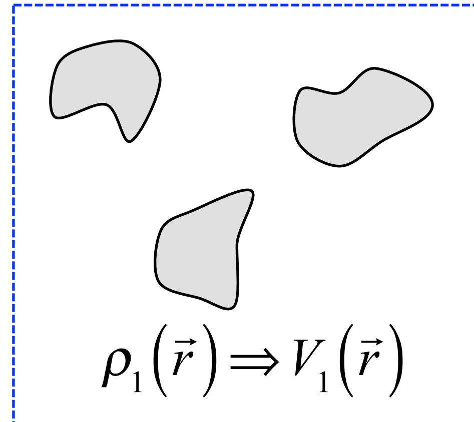
- Gauss's law (again):
  E is continuous in any region with a finite charge density, so that the divergence (a derivative) is finite.

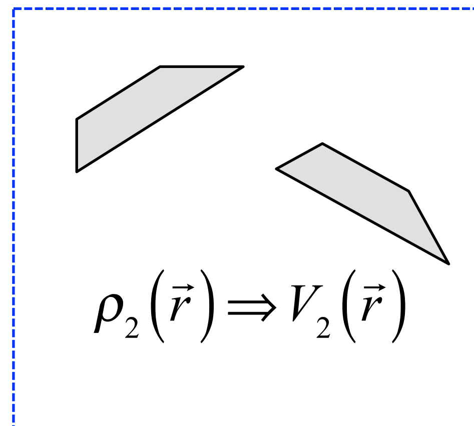
- ***E*** is finite. Equivalently, *V* is continuous.
  … except at specified point or line charges.
  The important application of this constraint is at $\vec{r} \to \infty$.
  Example: For localized charge distributions, solutions of the form $e^{\pm ax}$ are only allowed in confined spaces.

Green's Reciprocity Theorem

Consider two unrelated situations. Both are localized, so V goes to 0 at ∞.

Claim: $\int V_1(\vec{r})\rho_2(\vec{r})\,dVol = \int V_2(\vec{r})\rho_1(\vec{r})\,dVol$





Proof: Consider this integral:

$$\int \vec{E}_1 \cdot \vec{E}_2\,dVol = \int \left(\vec{\nabla} V_1\right)\cdot\left(\vec{\nabla} V_2\right) dVol$$

Use vector identity #5 again, letting: $f = V_1$ and $\vec{A} = \vec{\nabla} V_2$

$$\nabla \cdot \left(V_1 \vec{\nabla} V_2\right) = V_1\left(\vec{\nabla}\cdot\vec{\nabla} V_2\right) + \vec{\nabla} V_2 \cdot \vec{\nabla} V_1$$

So: $\int \left(\vec{\nabla} V_1\right)\cdot\left(\vec{\nabla} V_2\right) dVol = \int \vec{\nabla}\cdot\left(V_1 \vec{\nabla} V_2\right) dVol - \int V_1 \nabla^2 V_2\,dVol$

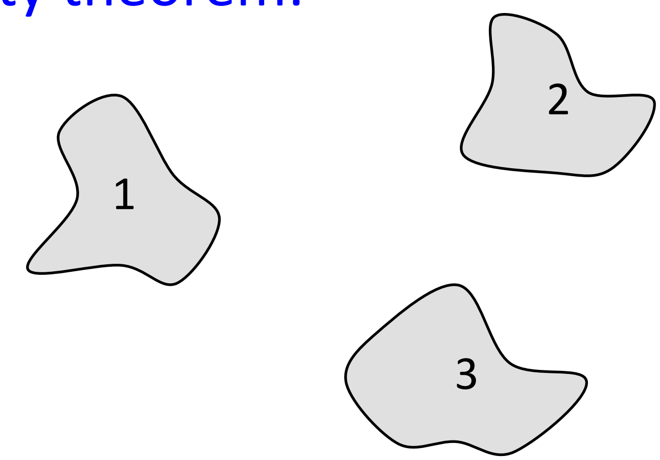
$$= \oint_{\text{surface}} V_1 \vec{\nabla} V_2 \cdot d\vec{a} - \frac{1}{\varepsilon_0}\int V_1 \rho_2\,dVol$$

= 0, by letting the surface go to ∞.

But we can also let $f = V_2$ and $\vec{A} = \vec{\nabla} V_1$ Therefore: $\int V_1 \rho_2\,dVol = \int V_2 \rho_1\,dVol$

An interesting implication of the reciprocity theorem:

Consider some uncharged conductors:



Put some charge, Q, on conductor 1. It will produce a voltage V_2 on conductor 2.

Instead, put the same charge, Q, on conductor 2. It will produce a voltage V_1 on conductor 1.

Amazingly, $V_2 = V_1$ for any geometry.

It follows immediately from the fact that the conductors are equipotentials.

$$\int \rho_1 V_2 \, dVol = \int \rho_2 V_1 \, dVol$$

$$V_2 \int \rho_1 \, dVol = V_1 \int \rho_2 \, dVol$$

$$QV_2 = QV_1$$

Solving Laplace's Equation by Separation of Variables (3.3)

You saw in discussion how to approach this problem in Cartesian coordinates:

$$\frac{\partial^2 V(x,y,z)}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

Assume that a solution of this form exists: $V(x,y,z)=X(x)Y(y)Z(z)$

Then, because the derivatives each act only on one function:

$$YZ\frac{d^2X}{dx^2}+XZ\frac{d^2Y}{dy^2}+XY\frac{d^2Z}{dz^2}=0 \Rightarrow \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{Function of x only}}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0$$

Each term is a function of a different variable. Therefore, Each term must be constant:

c_x, c_y, and c_z are called separation constants.

$$\frac{1}{X}\frac{d^2X}{dx^2}=c_x$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2}=c_y$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2}=c_z$$

where:

$$c_x+c_y+c_z=0$$

You know how to solve these equations.

They are all the same: $\dfrac{d^2 f(x)}{dx^2} = cf(x)$

For $c > 0$: $f(x) = ae^{kx} + be^{-kx}$ where: $k = \sqrt{c}$

For $c < 0$: $f(x) = a\sin(kx) + b\cos(kx)$ where: $k = \sqrt{-c}$

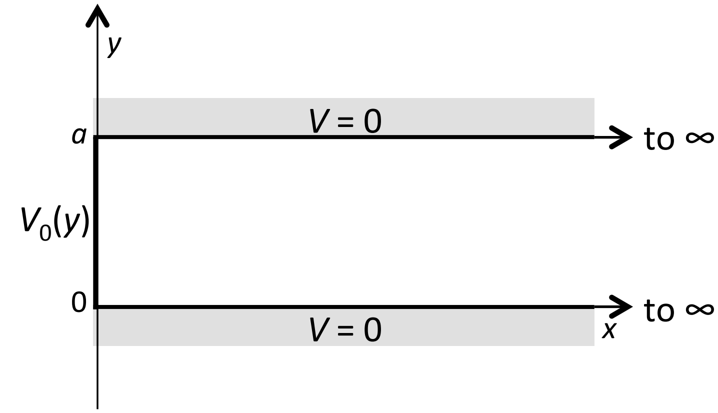
For $c = 0$: $f(x) = ax + b$

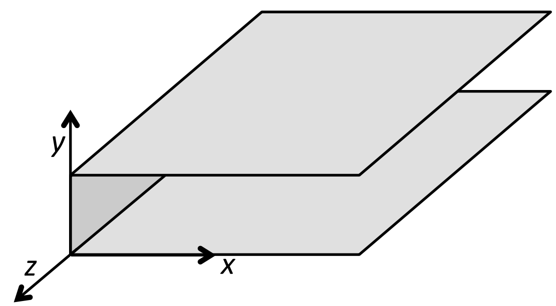
In each case, the constants *a* and *b* are determined by the boundary conditions.

BEWARE !!! The separation constants, c_x, c_y, and c_z are also constrained by the boundary conditions.

Example: (G, pp 127-132).

Consider a 2-D problem (no *z* dependence):
It's a box that extends to $x = \infty$ (to the right)
and also to $z = \pm\infty$ (in and out of the picture).





Boundary conditions (I'll call them bc's):

At $x = 0$ (left): $V = V_0(y)$, a known function.
At $y = 0$ (bottom): $V = 0$
At $y = a$ (top): $V = 0$
At $x = \infty$ (right): V is finite (That's all we require.)

We have $c_x = -c_y = c$. One is positive, and one is negative, so (defining $k \equiv \sqrt{|c|}$):

$$V(x,y) = X(x)Y(y) = \left(Ae^{kx} + Be^{-kx}\right)\left(C\sin(ky) + D\cos(ky)\right)$$ Exponential in *x*

$$\text{or } = \left(A\sin(kx) + B\cos(kx)\right)\left(Ce^{ky} + De^{-ky}\right)$$ Exponential in *y*

$$\text{or } = \left(Ax + B\right)\left(Cy + D\right)$$ Linear in *x* and *y*

The second solution is ruled out by the bc's at $y = 0$ and $y = a$:

At $y = 0$: $C + D = 0$

At $y = a$: $Ce^{ka} + De^{-ka} = 0$

➔ $C\sinh(ka) = 0$

$D\sinh(ka) = 0$

C and D are both zero unless $k = 0$, in which case $B = 0$, so $V = 0$ everywhere.

The third solution is also ruled out:

At $y = 0$: $D = 0$

At $y = a$: $Ca = 0$

The first solution:

At $y = 0$: $D = 0$ because $\sin(0) = 0$

At $y = a$: $C\sin(ka) = 0$

$\Rightarrow \sin(ka) = 0$

$\Rightarrow k_n = \frac{n\pi}{a}$ An ∞ number of discrete possibilities: $n = 1,2,3,\ldots$

Note: Does this look like the quantum "particle in a box" from P214?
I hope so. It's the same equation.

Now, consider the *x* dependence: $X(x) = Ae^{kx} + Be^{-kx}$ (the same k)

In order for $X(x)$ remain finite as x goes to ∞, $A = 0$.

So, the 2D solution is: $V(x,y) = \underbrace{BC}_{\text{Call it } C}\sin(k_n y)e^{-k_n x}$

Note that the bc's have have not yet constrained the value of *C*.

Unfortunately, this can't be correct, because $V(0,y) = C\sin(ky) \neq V_0(y)$

What to do?

We'll take advantage of the fact that Laplace's equation is linear. This means that

any superposition of solutions is also a solution.

Thus, the general solution is: $V(x,y) = \sum_{n=1}^{\infty} C_n \sin(k_n y)e^{-k_n x}$

We now have: $$V(0,y) = \sum_{n=1}^{\infty} C_n \sin(k_n y)$$

The big question: Can we pick the C_n's so that this equals $V_0(y)$?

End 9/16/13