Fourier Decomposition (G, pp. 130-132)

It turns out that:

For a given $V_0(y)$, there is a unique set, $\{C_n\}$, that solves this problem.

Writing $V_0(y)=\sum_{n=1}^{\infty} C_n \sin(k_n y)$ is called "Fourier decomposing" V_0.

In general, you will need $\cos(k_n y)$ terms as well. They don't appear here because of the BC's.

Fourier decomposition works because sin() and cos() are orthogonal functions.
Here's what orthogonality means: (n and n' are positive integers.)

$$\int_0^a \sin\left(\tfrac{2\pi nx}{a}\right)\sin\left(\tfrac{2\pi n'x}{a}\right)dx = \begin{cases} 0 \text{ if } n' \neq n \\ \tfrac{a}{2} \text{ if } n' = n \end{cases}$$

Integrate over one cycle.

$$\int_0^a \cos\left(\tfrac{2\pi nx}{a}\right)\cos\left(\tfrac{2\pi n'x}{a}\right)dx = \begin{cases} 0 \text{ if } n' \neq n \\ \tfrac{a}{2} \text{ if } n' = n \end{cases}$$

$$\int_0^a \sin\left(\tfrac{2\pi nx}{a}\right)\cos\left(\tfrac{2\pi n'x}{a}\right)dx = 0 \;\forall n',n$$

Aside, and a preview:
This property is called orthogonality, because one can think of the functions sin() and cos() as vectors and the integrals as dot products. In that sense, the functions are like orthogonal vectors.

This idea is (I think) discussed in Math 415, and is one of the essential mathematical tools in quantum mechanics (P486).

Let’s see how it works.

We want to know what $\{C_n\}$ will make $V_0\left(y\right)=\sum_{n=1}^{\infty}C_n\sin\left(k_n y\right)$

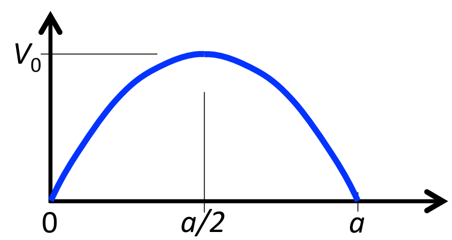
Multiply both sides by $\sin(k_{n'}y)$ and integrate: $k_{n'}\equiv\frac{n'\pi}{a}$

$$\int_0^a V_o\left(y\right)\sin\left(\tfrac{\pi n' y}{a}\right)dy=\sum_{n=1}^{\infty}C_n\underbrace{\int_0^a\sin\left(\tfrac{\pi n y}{a}\right)\sin\left(\tfrac{\pi n' y}{a}\right)dy}_{=\,0\text{ unless }n\,=\,n'}=\tfrac{a}{2}C_{n'}$$

Thus, we have a straightforward and unique way to calculate the “expansion coefficients” $\{C_n\}$, for a given $V(y)$.

Example: (not G’s example)

Suppose $V_0\left(y\right)=4V_0\left(\frac{y(a-y)}{a^2}\right)$



Then: $C_n=\frac{8V_0}{a^3}\int_0^a y\left(a-y\right)\sin\left(\tfrac{\pi n y}{a}\right)dy$

$$=\frac{32V_0}{\left(n\pi\right)^3}\quad \text{for odd } n \text{ only}$$

Note: (You can look it up!)

$$\int x\sin ax\,dx=\frac{1}{a^2}\left(\sin ax-ax\cos ax\right)$$

$$\int x^2\sin ax\,dx=\frac{1}{a^3}\left[2ax\sin ax+\left(2-\left(ax\right)^2\right)\cos ax\right]$$

Question: Why do we have odd *n* only?

Completeness & Orthogonality

These are very important properties of sets of functions.

The set $\{\sin(2\pi nx), \cos(2\pi nx)\}$, $\forall$ integer $n \geq 0$ (except sin(0)),
is complete on the domain $0 \leq x \leq 1$, because
any integrable function, $f(x)$, can be written as:

$$f(x) = c_0 + \sum_{n=1}^{\infty} \left(a_n \sin(2\pi nx) + b_n \cos(2\pi nx) \right), \text{ where}$$

Cos(0) = 1, so we could call c_0 b_0.

$$c_0 = \int_0^1 f(x)\,dx \quad a_n = 2\int_0^1 f(x)\sin(2\pi nx)\,dx, \text{ and} \quad b_n = 2\int_0^1 f(x)\cos(2\pi nx)\,dx$$

The Fourier decomposition is just one example of a general method for satisfying the boundary conditions. Here, sin() and cos() are useful, because they are solutions to Laplace's equation in Cartesian coordinates. Other linear differential equations will have other solution sets – these sets can also be used to decompose arbitrary functions.

This technique is very widely used, especially in quantum mechanics.

One more example, to show that $f(x)$ doesn't need to be symmetrical, and to illustrate a trap.

Suppose $f(x) = x$, for 0 < x < 1.

Do the integrals:

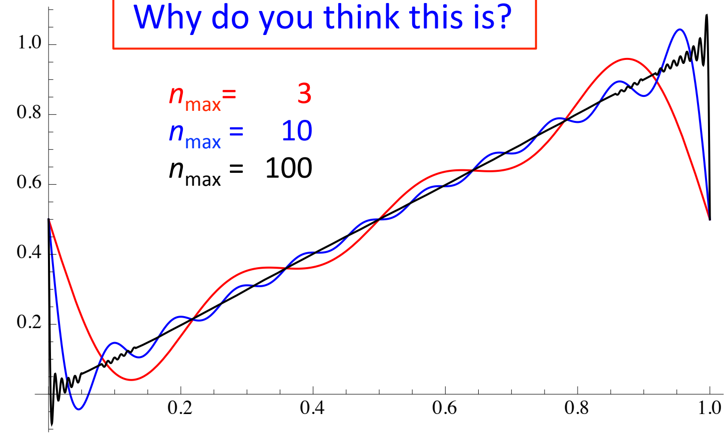
$$c_0 = \int_0^1 x\,dx = \tfrac{1}{2}$$

$$a_n = 2\int_0^1 x\sin(2\pi n x)\,dx = \tfrac{-1}{n\pi}$$

$$b_n = 2\int_0^1 x\cos(2\pi n x)\,dx = 0$$

Here's the result:

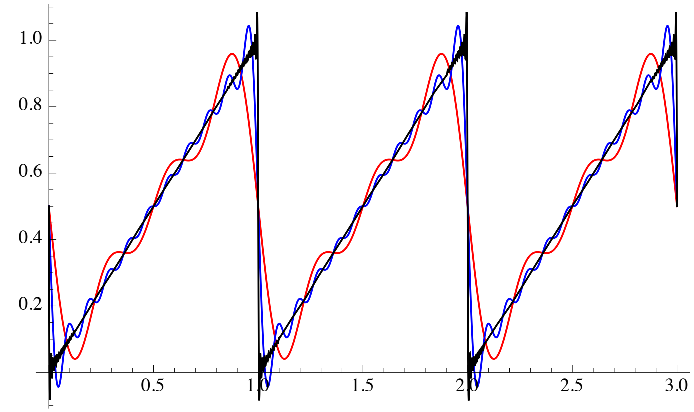
It converges very slowly.
Why do you think this is?



It looks nice. However, beware. Here's $f(x)$ outside the region of interest:

It's periodic! This may not be what we want, but that behavior is guaranteed by our use of periodic expansion functions.

Don't apply a Fourier decomposition outside the region of interest.



Making and using graphs:

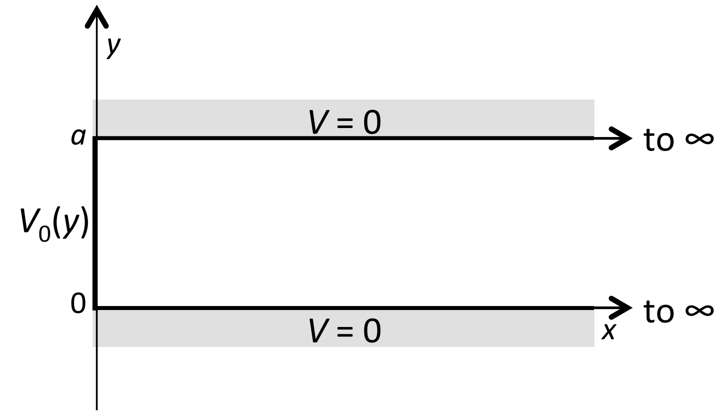
I made the graphs on the previous slide using Mathematica. You can do the same using MatLab, Python, or other scripting languages. It is very instructive to play around with the graphs (*e.g.*, "What happens if I change the charge?"). It is also a useful way do verify your solutions.

Here is my Mathematica code for the graphs:

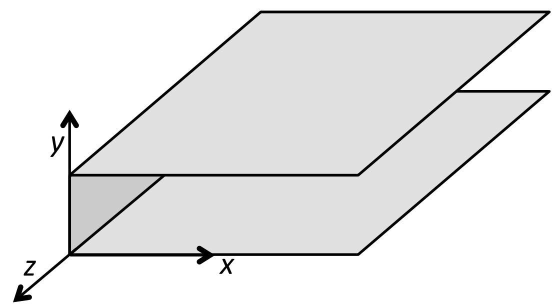
```
(* Define a Fourier series approximation to f(x) = x, for 0 < x < 1. *)
aLine[x_, nMax_] := 1/2 - Sum[1/(n π)*Sin[2 π n x], {n, 1, nMax}]

(* Plot this function for three values of nMax.  The plots are overlayed. *)
nMax1 = 3
nMax2 = 10
nMax3 = 100
(* Plot the region of interest: 0 < x < 1. *)
Plot[{aLine[x, nMax1], aLine[x, nMax2], aLine[x, nMax3]}, {x, 0, 1},
  PlotStyle -> {Red, Blue, Black}]
(* Plot more than the region of interest: 0 < x < 3. *)
Plot[{aLine[x, nMax1], aLine[x, nMax2], aLine[x, nMax3]}, {x, 0, 3},
  PlotStyle -> {Red, Blue, Black}]
```

Look at the first example a little more:

The *y* dependence of the solution can be written as a sum of functions: sin($n\pi y/a$). The entire solution $V(x,y)$ is a sum of pairs of functions:



$$V(x,y)=\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi y}{a}\right) e^{-\frac{n\pi x}{a}}$$



Each *n* term decays exponentially with *x*. Larger *n*'s decay more rapidly, so at large *x*, only the smallest non-zero c_n is significant.

Knowing V, we can calculate $\vec{E}=-\vec{\nabla}V$:

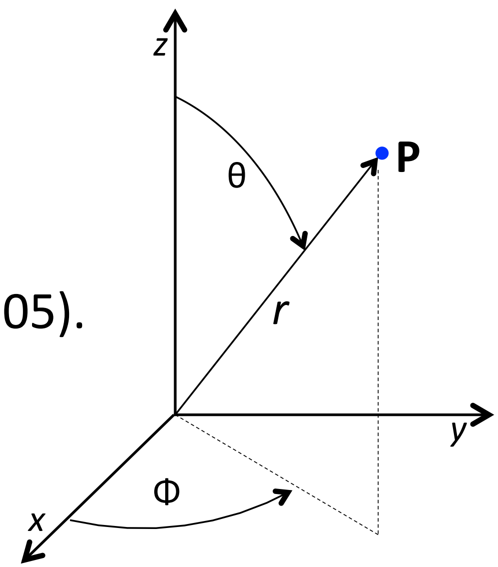
$$E_x=\frac{n\pi}{a}\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi y}{a}\right) e^{-\frac{n\pi x}{a}} \quad \Leftarrow \quad =0 \text{ at } y=0 \text{ and } y=a \text{, as expected.}$$

$$E_y=-\frac{n\pi}{a}\sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi y}{a}\right) e^{-\frac{n\pi x}{a}}$$

Spherical Coordinates (3.3.2)

Many problems have spherical symmetry, either due to the imposed BCs (*e.g.*, spherical conductors) or because the problem is inherently spherical (*e.g.*, atoms).

In this situation, it is usually more convenient to use spherical coordinates. This is not a requirement. We could solve the problem in Cartesian coordinates, but we'd usually find the solution to be very messy.



In general, the potential at point **P** is given by $V(r,\theta,\phi)$. However, to keep the math simple, Griffiths assumes that there is no ϕ dependence. (I will, as well.) You will deal with the more general problem in QM (P486) and graduate EM (P505).

Here's Laplace's equation in spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}}_{\text{Ignore this term}}=0$$

This diff. eq. is different from the Cartesian one, so the solution functions will be different. This means that the "Fourier" decomposition will also be different.

Let's use separation of variables to solve this problem.

Write: $V(r,\theta) = R(r)\Theta(\theta)$. Substitute in and divide by $R\Theta$:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0$$

As with Cartesian coordinates, we have a separation constant (only one, because this is a 2D problem). Here (for reasons that will become clear) we'll call the constant $l(l+1)$.

Note: You know the physical significance of l in QM. Here, it's just math.

As before, we have two ordinary diff. eqs:

$$\frac{1}{R}\frac{d}{dR}\left(r^2\frac{dR}{dr}\right) = l(l+1)$$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)$$

The R equation has two simple solutions:

$$R(r) = r^l \quad \text{or} \quad r^{-(l+1)}$$

Thus, the general radial solution is: $R(r) = A_l r^l + B_l r^{-(l+1)}$

For example ($l = 0$): $R(r) = A_0 + B_0/r$

The Θ equation also has polynomial (powers of cosθ) solutions
but they are more complicated.

Legendre polynomials:

Define $x \equiv \cos\theta$

l must be an integer.

l	$P_l(x)$
0	1
1	x
2	$(3x^2-1)/2$
etc.	

$P_l(x)$ is an even function of x for even l,
odd function of x for odd l.

Comments:

- By convention, every P_l is defined so that $P_l(1) = 1$ (*i.e.*, at $\theta=0$).
- There is a second set of solutions, but they involve $\ln\left(\frac{1+x}{1-x}\right)$ (not useful for us).
  - Diverges at x = 1 (θ = 0).

Physical significance of solutions:

$l = 0$:

$P_0 = 1$ This is the isotropic solution.

$V(r) = A + B/r$

- A: E = 0
- B/r: Point charge

$l = 1$:

$P_1 = x = \cos\theta$

$V(r,\theta) = A\, r\cos\theta + B \cos\theta/r^2$

- $r\cos\theta$ = z: Uniform E in z-direction.
- $B\cos\theta/r^2$: Electric dipole. We'll discuss this later.

End 9/18/13