Issues with question 3 of homework 3 (the curl):

Both fields have only a ϕ component, which only depends on s.

So the only term in the curl that survives is $\left(\vec{\nabla}\times\vec{v}\right)_z = \frac{1}{s}\frac{\partial}{\partial s}\left(s v_\varphi\right)$.

i) $\vec{v} = (0, s, 0) \rightarrow \frac{1}{s}\frac{\partial}{\partial s}\left(s^2\right) = 2$

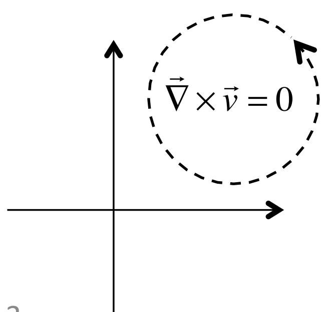
ii) $\vec{v} = (0, \frac{1}{s}, 0) \rightarrow \frac{1}{s}\frac{\partial}{\partial s}\left(1\right) = 0 \leftarrow$ How can we reconcile this with the fact that $\oint \vec{E}\cdot d\vec{l} \neq 0$?

The answer is the same as the resolution of the divergence paradox. (slides 25-26, August 26)

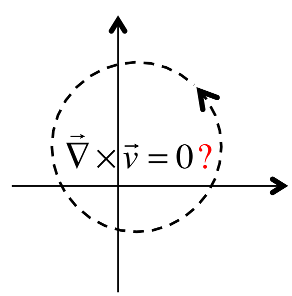
The 1/s function diverges (goes to ∞) at the origin.

$\Rightarrow$ Our formula for the curl fails there.

This loop satisfies Stokes' theorem



This loop appears to violate Stokes' theorem



The resolution of the paradox is that the curl is infinite at the origin (a Dirac delta function). The integral of $\vec{\nabla}\times\vec{v}$ over any circle that encloses this delta function is 2π, saving Stokes.

The Θ equation also has polynomial (powers of cosθ) solutions but they are more complicated.

Legendre polynomials:

Define $x \equiv \cos\theta$

l must be an integer.

l	$P_l(x)$
0	1
1	x
2	$(3x^2-1)/2$
etc.	

$P_l(x)$ is an even function of x for even l, odd function of x for odd l.

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Comments:

- By convention, every P_l is defined so that $P_l(1) = 1$ (*i.e.*, at θ=0).
- There is a second set of solutions, but they involve $\ln\left(\frac{1+x}{1-x}\right)$ (not useful for us).

Diverges at x = ±1 (θ = 0 and π).

Physical significance of solutions:

$l = 0$:

$P_0 = 1$ This is the isotropic solution.

$V(r) = A + B/r$

A: $E = 0$

B/r: Point charge

$l = 1$:

$P_1 = x = \cos\theta$

$V(r,\theta) = A\, r\cos\theta + B\cos\theta/r^2$

$r\cos\theta = z$ — Uniform E in z-direction.

$B\cos\theta/r^2$: Electric dipole. We'll discuss this later.

The Legendre polynomials are orthogonal and complete:

The region of interest is -1 < x < 1 (0 < θ < π).

$$\int_{-1}^{1} P_l(x) P_{l'}(x)\,dx = \int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta\, d\theta$$
$$= 0 \quad \text{if } l' \neq l$$
$$= \frac{2}{2l+1} \quad \text{if } l' = l$$

Note the θ part of spherical integration.

We can write any integrable $V_0(\theta)$ as a superposition:

$$V_0(\theta) = \sum_{l=0}^{\infty} a_l P_l(\cos\theta)$$

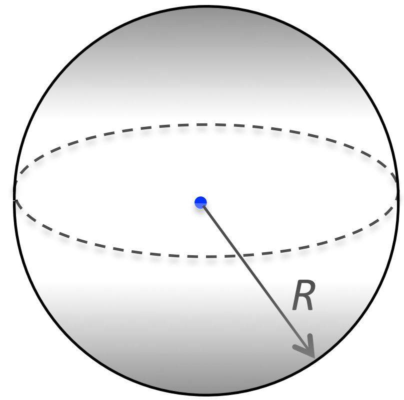
where, $$a_l = \frac{2l+1}{2} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta\, d\theta$$

Beware:
This is slightly different from G, p. 140.

The derivation is exactly the same as for the Fourier decomposition.

This is our second example of the decomposition of a function into an orthogonal (and complete) set of functions. Each linear differential equation has a natural set of orthogonal, complete functions.

Example:



Suppose $V(\theta) = V_0 \cos^2(\theta)$ at $r = R$.

We don't need to do any integrals, because:
By inspection, $\cos^2\theta = \frac{2}{3}P_2(\theta) + \frac{1}{3}P_0(\theta)$.

So: $a_0 = V_0/3$ and $a_2 = 2V_0/3$. All others are zero.

We need more information (radial boundary conditions) to determine $V(r,\theta)$.

Example (G, 3.6-7):

Given $V_0(\theta)$ on the surface of a sphere of radius *R*, calculate $V(r,\theta)$ everywhere.

Assume there is charge only on the surface, so that Laplace's equation is valid everywhere else.

Inside: $(r < R)$ $$V(r,\theta) = \sum_{l=0}^{\infty} \left(\frac{r}{R}\right)^{l} a_l P_l(\cos\theta)$$

$r^{-(l+1)}$ is not allowed, because it blows up at $r = 0$.

Outside: $(r > R)$ $$V(r,\theta) = \sum_{l=0}^{\infty} \left(\frac{r}{R}\right)^{-(l+1)} a_l P_l(\cos\theta)$$

r^{l} is not allowed, because it blows up at $r = \infty$.

$\frac{r}{R}$ = 1 at $r = R$. This makes $V(r,\theta) = V_0$ at $r = R$.

Comments:

- *V* is a continuous function of *r*. $V = V_0(\theta)$ ar $r = R$. dV/dr is discontinuous at $r = R$, because there is surface charge at $r = R$.
- For large *r*, only the smallest non-zero a_l is important.
- We used these radial BCs: 1: $V = V_0$ at $r = R$. 2: *V* is finite at $r = 0$. 3: $V = 0$ at $r = \infty$.
- You should read and understand the logic of Griffiths' examples 3.6-9 (pp. 139-144).

End 9/20/13