Example: (continued)

Pick a specific $V_0(\theta)$: On the surface (at $r = R$): $V_0(\theta) = V_0(1+\cos\theta)$.

Remember, we are assuming that all the charge lies on the surface of the sphere.

a) Calculate $V(r,\theta)$ everywhere.

b) What is the charge density on the surface?

a) We don't need to do any integrals, because $1+\cos\theta = P_0 + P_1$.

Inside: $V(r,\theta) = a_0 r^0 + a_1 r^1 \cos\theta = a_0 + a_1 z$

The boundary condition at $r = R$ tells us: $a_0 = V_0$, and $a_1 = V_0/R$.

$$V(r,\theta) = V_0\left(1 + \frac{r}{R}\cos\theta\right) = V_0 + \frac{V_0}{R}z$$

Note that the mean value theorem is satisfied: $V(r=0) = V_0 = V_{av}$.

There is a uniform electric field inside the sphere: $E_z = -V_0/R$.

Outside: $V(r,\theta) = \frac{b_0}{r} + \frac{b_1}{r^2}\cos\theta$, where $b_0 = RV_0$, and $b_1 = R^2 V_0$.

$$V(r,\theta) = V_0\left(\frac{R}{r} + \left(\frac{R}{r}\right)^2 \cos\theta\right)$$

I'm calling the coefficients b_0 and b_1 to distinguish them from a_0 and a_1.

b) To find the surface charge density, use the fact that E_r is discontinuous at $r = R$:

$$E_r(\text{outside}) - E_r(\text{inside}) = \frac{\sigma(\theta)}{\varepsilon_0}$$

$$E_r(\text{outside}) = -\left.\frac{\partial V}{\partial r}\right|_{R^+} \qquad E_r(\text{inside}) = -\left.\frac{\partial V}{\partial r}\right|_{R^-}$$

$$\frac{\sigma(\theta)}{\varepsilon_0} = V_0\left(\frac{R}{r^2} + \frac{2R^2}{r^3}\cos\theta\right)\Bigg|_{r=R} - V_0\left(\frac{\cos\theta}{R}\right)\Bigg|_{r=R}$$

$$= \frac{V_0}{R}(1+\cos\theta)$$

Note: $Q_{\text{tot}} = (4\pi R^2)\dfrac{\varepsilon_0 V_0}{R} = 4\pi\varepsilon_0 V_0 R \neq 0$

We could predict this, because $V_{av} = V_0 \neq 0$.

Multipole Expansion (G: 3.4 and F: 6-2,4,5)

Legendre polynomials are not merely mathematical tools.

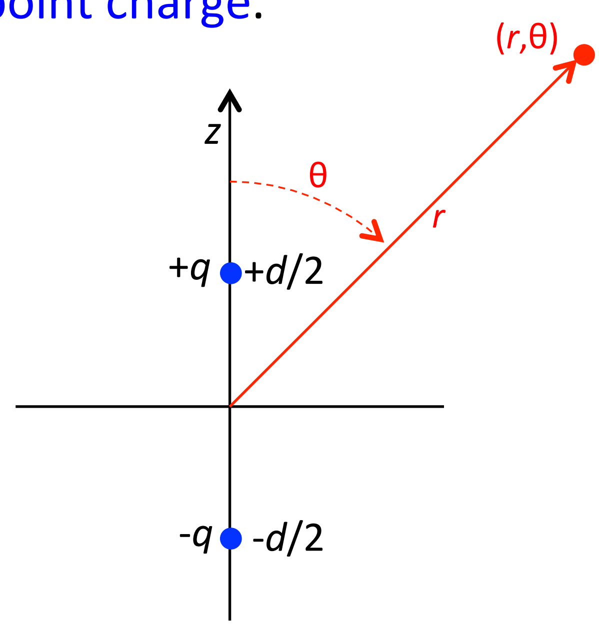
- Each $r^{-(l+1)}P_l(\cos\theta)$ term can be produced by a very simple configuration of point charges. We only consider the negative powers (*i.e.*, the behavior at large *r*).
- Because P_l is associated with $r^{-(l+1)}$ radial dependence, if there are several l terms the term with smallest l dominates at large distances.
  This makes for a useful approximation method.

$l = 0$: $V(r,\theta) = a_0/r$. This is the potential due to a single point charge.
It is called a monopole.

$l = 1$: $V(r,\theta) = a_1\cos\theta/r^2$.
The physical situation:
Two point charges, $\pm q$, on the *z*-axis at $z = \pm d/2$.

We are interested in potential far away ($r >> d$).

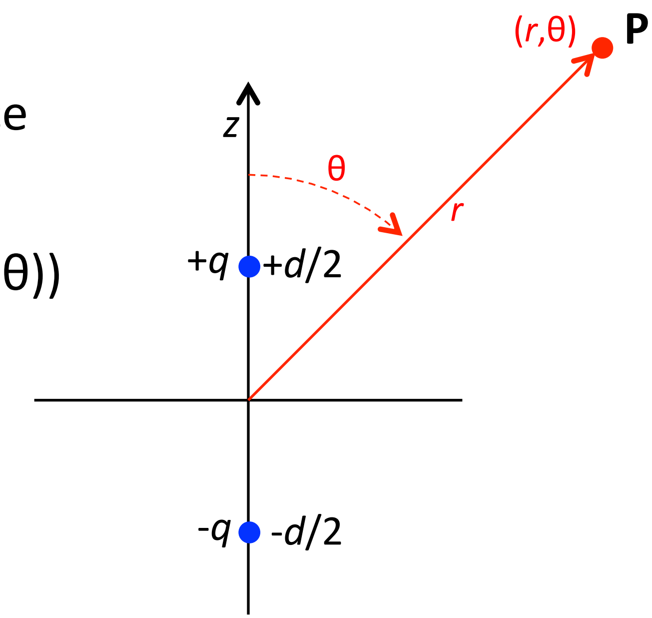
The analysis is on the next slide.



$l = 1$: $V(r,\theta) = a_1\cos\theta/r^2$.

If the charges were at the origin, they would produce $V = \pm q/(4\pi\varepsilon_0 r)$, and they would cancel.

However, for $r >> d$, the +q is closer to point **P** (at (r,θ)) by $(d/2)\cos\theta$, and $-q$ is farther by the same amount.



So the potential at (r,θ) is:

$$V(r,\theta) \approx \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r - \frac{d}{2}\cos\theta} - \frac{1}{r + \frac{d}{2}\cos\theta}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{d\cos\theta}{r^2 - \left(\frac{d}{2}\cos\theta\right)^2}\right) \approx \frac{qd}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2} \equiv \frac{p}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}$$

p is called the electric dipole moment.

The orientation of the dipole is important. To deal with this, we can write p as a vector, $\vec{p}$, that points from $-q$ to $+q$. Then, $V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$ ← $qd\cos\theta$

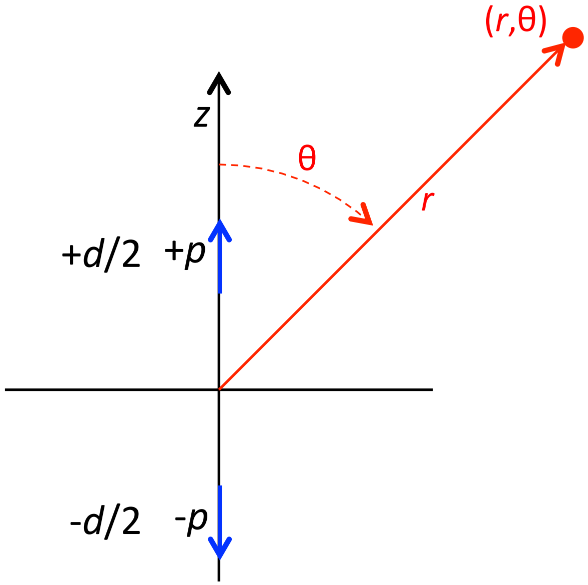
Note that in our Legendre decomposition, $a_1 = \frac{p}{4\pi\varepsilon_0}$

At large r, the dipole potential falls as $1/r^2$.

l = 2: This will give us the quadrupole moment

One can take the next step by considering two slightly displaced dipoles (rather than charges).

I won't go through it. One obtains the a_2 term.



This leads to the main value of multipoles:

Multipoles are a useful expansion when we are looking at the field far from a localized distribution of charge.

End 9/23/13