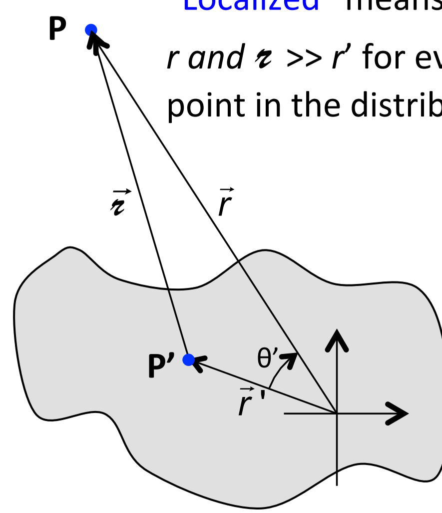
Now, consider an arbitrary localized distribution:

We are interested in the contribution to *V*(**P**) due to charge at **P’**. Then we’ll integrate over **P’** to determine *V*(**P**). (see G, p. 147)

“Localized” means r *and* $\mathscr{r} >> r'$ for every point in the distribution.



$$\mathscr{r}^2 = r^2 + r'^2 - 2rr'\cos\theta' \quad \text{(law of cosines)}$$

$$= r^2\left[1+\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'\right] \equiv r^2\left[1+\varepsilon\right]$$

$r' << r$ $\qquad$ $\varepsilon << 1$

$$V \propto \frac{1}{\mathscr{r}} = \frac{2}{r\left[1+\varepsilon\right]^{1/2}} = \frac{1}{r}\left[1-\frac{\varepsilon}{2}+\frac{3}{8}\varepsilon^2-\frac{5}{16}\varepsilon^3+\ldots\right]$$

← Power series expansion for $\frac{1}{\left[1+\varepsilon\right]^{1/2}}$

I’ll skip some steps (done in Griffiths).

Replacing ε with $\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'$ and rearranging the cosθ’ terms, one obtains:

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l\left(\cos\theta'\right)$$

The Legendre polynomials appear again.

Obtain the total *V* at **P** by integrating over the distribution:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \int_{\text{object}} r'^{l} P_l(\cos\theta') \rho(\vec{r}\,') \, d\text{Vol}$$

Let's look at the physical significance of the first two terms.

$l = 0$: The integral becomes $\int_{\text{object}} \underbrace{r'^{0}}_{=1} \underbrace{P_0}_{=1}(\cos\theta') \rho(\vec{r}\,') \, d\text{Vol} = Q_{\text{object}}$

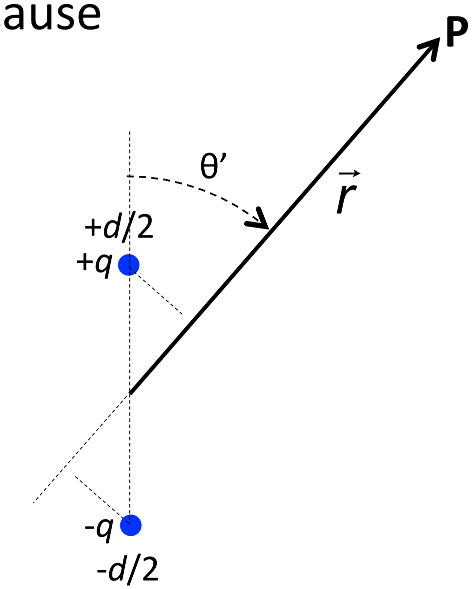
$l = 1$: The integral becomes $\int_{\text{object}} \underbrace{r'^{1}}_{=r'} \underbrace{P_1}_{=\cos\theta}(\cos\theta') \rho(\vec{r}\,') \, d\text{Vol}$

This is the component of the dipole moment along ***r***, because *r*'cosθ is the distance toward or away from **P**.

To see this, go back to the two-point-charge case:

The integral becomes a sum:

$$q\frac{d}{2}\cos\theta' + (-q)\left(\frac{-d}{2}\cos\theta'\right) = qd\cos\theta' = \vec{p}\cdot\hat{r}$$



The Dipole Field (3.4.4)

You did this problem qualitatively in discussion.
Here's the exact derivation. We have (for $\vec{p} \,//\, \hat{z}$):

$$V(r,\theta) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}, \text{ and } \vec{E} = -\vec{\nabla} V$$

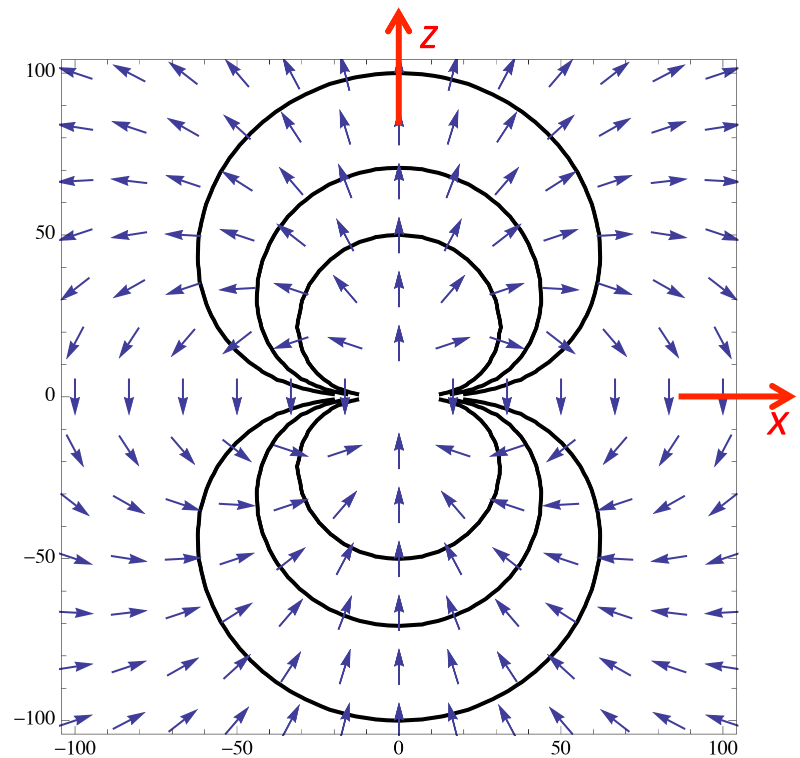
Take the gradient in spherical coordinates:

$$E_r = -\frac{\partial V}{\partial r} = +\frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}\left(> 0\ \forall\theta\right)$$

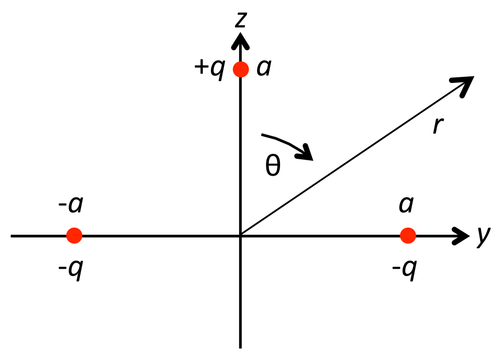
$E_\varphi = 0.$ V has no φ dependence,
because $\vec{p}$ points along $\hat{z}$.

Note that $E \propto r^{-3}$, not r^{-2},
because the total charge is zero.



Equipotentials are not circles:
$z^2 = c\left(x^2 + z^2\right)^3$, because $\cos\theta = \frac{z}{r}$

Example: (G, 3.32)



Calculate ***p*** for this configuration:

Point charges, so the integrals become sums.
Note: $Q_{tot} = -q$.

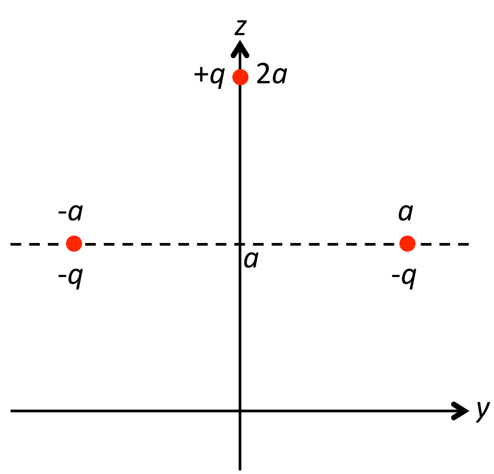
$$p_x = \sum_i x_i q_i = 0$$ Everything is in the *y*-*z* plane.

$$p_y = 0$$ By symmetry.

$$p_z = qa$$

So, the field at large distances (ignoring the higher multipoles) is:

$$\vec{E} = \frac{-q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} + \frac{qa}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$$



However, beware!!
Suppose we choose a different coordinate system:
(shifted by -*a* along z) then $p_z = 0$!!

What are we to make of this?

It is easy to show (See G, pp. 151-152) that shifting the charge coordinates by ***d*** changes ***p***: $\vec{p} \to \vec{p} - Q_{\text{tot}}\vec{d}$.

In general, only the smallest non-vanishing ℓ-moment is independent of the choice of coordinates. The exact solution for ***E*** is, in general an infinite sum of multipole terms. Changing coordinates will change the relative contributions of the various terms (except Q_{tot}, of course).

Another example (continuous charge distribution):
Consider a sphere of radius *R*, inside of which the charge density is $\rho = \rho_0 \frac{z}{R}$.
By symmetry, $\vec{p}$ points along $\hat{z}$. So:

$$p_z = \int z\rho\, dV = \frac{\rho_0}{R}\int z^2\, dVol$$

$$= \frac{\rho_0}{R}\int_0^{2\pi}\int_{-1}^{1}\int_0^{R} r^2\cos^2\theta\, r^2\, dr\, d\cos\theta\, d\varphi$$

$$= \frac{\rho_0}{R}\frac{R^5}{5}\frac{2}{3}2\pi = \frac{R\rho_0}{5}Vol$$

End 9/25/13