Exam 1 Review

I'll go through some problems that illustrate the concepts and techniques that you've learned.

- Separation of variables in cylindrical coordinates.
- Apply boundary conditions to a problem.
- Expand a function in terms of Legendre polynomials.
- Show a field that has non-zero curl, even though loop = 0.
- Calculate the charge density, given ***E***.

Cylindrical coordinates: (HW5, problem 3)

Remember that we are ignoring any *z*-dependence, to simplify the math. This reduces it to polar coordinates (s,θ). Laplace's equation is:

$$\nabla^2 V(s,\theta) = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

As with other coordinate systems, try a solution of the form: $V(s,\theta) = S(s)\Theta(\theta)$

So,

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial S\Theta}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 S\Theta}{\partial \theta^2} = 0$$

$$s\Theta\frac{d}{ds}\left(s\frac{dS}{ds}\right) + S\frac{d^2\Theta}{d\theta^2} = 0$$ Pull constants out of derivatives. Multiply by s^2.

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = 0$$ Divide through by $S\Theta$.

Each term must be constant, and the constants must sum to zero:

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) = c \qquad\qquad \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -c$$

The solutions to the Θ equation are:

$$\Theta(\theta) = \sin(k\theta) \text{ or } \cos(k\theta) \text{ for } c > 0. \qquad k \equiv \sqrt{|c|} \qquad \frac{d^2\Theta}{d\theta^2} = -c\Theta$$

$$\Theta(\theta) = e^{\pm k\theta} \text{ for } c < 0.$$

Only the trig functions are allowed, because the exponentials don't satisfy the boundary condition: Θ(2π) = Θ(0) (periodicity around the circle).
Periodicity also requires that *k* be an integer.

When *c* = 0, we need to use a slightly different approach, because sin(0) = 0.
We're looking for functions of θ that have no second derivative, namely *A* + *B*θ.
Periodicity requires *B* =0. So, *c* = 0 implies no θ dependence.

For the radial equation, try power law solutions, $S = s^a$.

$$s\frac{d}{ds}\left(s\frac{dS}{ds}\right)=cS$$

$$s\frac{d}{ds}\left(s\frac{ds^a}{ds}\right)=cs^a$$

$$\frac{d}{ds}\left(s\left(as^{a-1}\right)\right)=cs^{a-1}$$

$$a^2s^{a-1}=cs^{a-1} \quad \Rightarrow \quad a=\pm\sqrt{c}=\pm k$$

Again, when k = 0, we need to look for the missing solution.

$$\frac{d}{ds}\left(s\frac{dS}{ds}\right)=0$$

We want the expression in parentheses to be a constant.
That works if S is constant, but we already have that solution (s^0).
It also works if dS/ds = 1/s, in which case S = ln(s). That's the missing solution.

Finally, we have:

$$V(s,\theta)=\left(A_k\sin(k\theta)+B_k\cos(k\theta)\right)\left(C_ks^k+D_ks^{-k}\right) \qquad \text{for each } k\neq 0$$

$$V(s,\theta)=C_0+D_0\ln(s) \qquad \text{for k=0}$$

In general:

$$V(s,\theta)=C_0+D_0\ln(s)+\sum_{k=1}^{\infty}\left(A_k\sin(k\theta)+B_k\cos(k\theta)\right)\left(C_ks^k+D_ks^{-k}\right)$$

Boundary value problem:

Suppose a cylinder of radius, R, has surface charge density, $\sigma(\theta) = \sigma_0 \sin^3\theta$. Calculate $V(s,\theta)$ inside and out.

Boundary conditions:

- $V = 0$ at $s = \infty$.
  We need to be careful about this, because ln(s) diverges at both s = 0 and s = ∞. If ln(*s*) appears, we need to keep it and choose an appropriate reference point. That happens when there is a nonzero net charge density. In that case it will look like a line charge at large *s*, and *V* will be proportional to ln(*s*).
- *V* is finite at *s* = 0. We don't keep the ln(*s*) term unless there is a line charge at the origin (the equivalent of a point charge in spherical coordinates).
- *V* is continuous at *s* = *R*. Only ***E*** is discontinuous at a surface charge.

First, we need to determine what values of *k* contribute.

Use this trig identity: $\sin^3\theta = \dfrac{3\sin\theta - \sin 3\theta}{4}$. So, *k* = 1 and *k* = 3 contribute.

Look it up!

Consider the continuity of V at $s = R$. $V(R_-) = V(R_+)$

$$\sigma(\theta) = \sigma_0 \frac{3\sin\theta - \sin 3\theta}{4}$$

Because the trig functions are orthogonal, each angular term must separately satisfy the boundary condition that comes from the corresponding σ(θ) term.

Work on each k separately:

$k = 1$: $\sin\theta (A_1 C_1)_{in} R = \sin\theta (A_1 D_1)_{out} R^{-1}$

$$(A_1 C_1)_{in} R^2 = (A_1 D_1)_{out}$$

I've required that V be finite at $s = 0$ ($D_{1\,in} = 0$) and at $s = \infty$ ($C_{1\,out} = 0$).

$k = 3$: $\sin(3\theta)\left((A_3 C_3)_{in} R^3\right) = \sin(3\theta)\left((A_3 D_3)_{out} R^{-3}\right)$

$$(A_3 C_3)_{in} R^6 = (A_3 D_3)_{out}$$

↑ ↑

I grouped coefficients, because only the product is important.

Note that the coefficients (A, B, C, and D) for $s < R$ do not necessarily equal the ones for $s > R$.

Consider the discontinuity of $\boldsymbol{E}$ at $s = R$. $E_{s\text{ out}} - E_{s\text{ in}} = \sigma/\varepsilon_0$ $\quad \sigma(\theta) = \sigma_0 \dfrac{3\sin\theta - \sin 3\theta}{4}$

The radial part of the gradient is dV/ds.

$$E_s = -\frac{dV}{ds} = -\left(A_k \sin(k\theta) + B_k \cos(k\theta)\right)\left(C_k k s^{k-1} - D_k k s^{-(k+1)}\right) \quad \text{for } k \neq 0$$

$$k = 1:\ -\left.\frac{dV}{ds}\right|_{s=R_{\text{out}}} + \left.\frac{dV}{ds}\right|_{s=R_{\text{in}}} = \sin\theta\left(\left(A_1 D_1\right)_{\text{out}} R^{-2} + \left(A_1 C_1\right)_{\text{in}}\right) = \frac{\sigma_0}{\varepsilon_0}\frac{3\sin\theta}{4}$$

$$k = 3:\ -\left.\frac{dV}{ds}\right|_{s=R_{\text{oud}}} + \left.\frac{dV}{ds}\right|_{s=R_{\text{in}}} = \sin(3\theta)\left(3\left(A_2 D_2\right)_{\text{out}} R^{-4}\right) + \sin(3\theta)\left(3\left(A_2 C_2\right)_{\text{in}} R^2\right) = \frac{-\sigma_0 \sin(3\theta)}{4\varepsilon_0}$$

So,
$$\left(A_1 D_1\right)_{\text{out}} R^{-2} + \left(A_1 C_1\right)_{\text{in}} = \frac{3\sigma_0}{4\varepsilon_0}$$

$$\left(A_3 D_3\right)_{\text{out}} R^{-6} + \left(A_3 C_3\right)_{\text{in}} = \frac{-\sigma_0}{12\varepsilon_0 R^2}$$

From before:
$$\left(A_1 D_1\right)_{\text{out}} R^{-2} - \left(A_1 C_1\right)_{\text{in}} = 0$$
$$\left(A_3 D_3\right)_{\text{out}} R^{-6} - \left(A_3 C_3\right)_{in} = 0$$

Two pairs (k = 1 and k = 3) of simultaneous equations in two unknowns.
I'm not going to solve them here.

Boundary value problems can be very tedious!
Don't worry about this for the exam.

Expand a function as a sum of Legendre polynomials

We can't always figure out the expansion coefficients by inspection.

Suppose $V(\theta) = V_0(\theta(\pi-\theta))$, a parabolic dependence, going to zero at $\theta = 0$ and π. We're going to have to do integrals.

Recall: (9/20/13 lecture): $$V_0(\theta) = \sum_{l=0}^{\infty} a_l P_l(\cos\theta)$$

where, $$a_l = \frac{2l+1}{2}\int_0^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta$$

First of all, remember that the P_l are even/odd functions of cos(θ) for even/odd l. Here, our potential is an even function (symmetric about θ = π/2), so we only need to do half of the integrals. Here are the first two:

$$a_0 = \frac{V_0}{2}\int_0^{\pi}\left(\theta(\pi-\theta)\right)\underbrace{1}_{P_0}\sin\theta\, d\theta = 2V_0$$

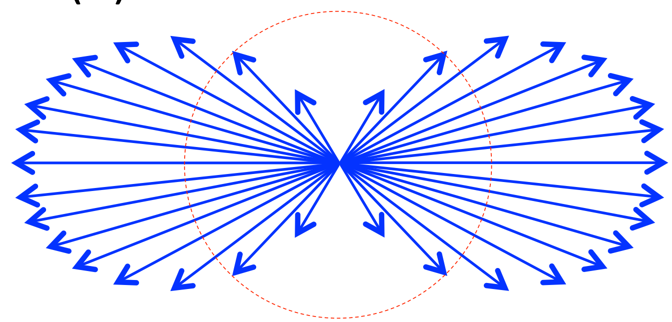
$$a_2 = \frac{3V_0}{2}\int_0^{\pi}\left(\theta(\pi-\theta)\right)\underbrace{\tfrac{1}{2}\left(3\cos^2\theta - 1\right)}_{P_2}\sin\theta\, d\theta = -3V_0$$

Confession:
I let Mathematica do the integrals for me.

A problem with the curl:

You may recall an example where $\vec{\nabla}\times\vec{E}=0$, but $\oint\vec{E}\cdot d\vec{l}\neq 0$.
There was a singularity at the origin Now, let's look at the converse.

Suppose I tell you that (in spherical coordinates) $\vec{E}=\sin^2(\theta)r\hat{r}$.
Is this a legal field? (Note: There is no singularity.)

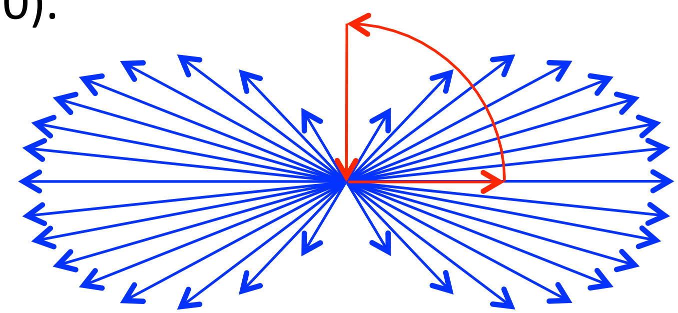


E has only an r component, so the loop integral shown is guaranteed to be zero.

However, let's calculate the curl.
E_r has only theta dependence, so there is only one non-zero term:

$$\left(\vec{\nabla}\times\vec{E}\right)_{\varphi}=\frac{1}{r}\frac{\partial E_r}{\partial\theta}=2\sin\theta\cos\theta=\sin 2\theta\neq 0!!$$

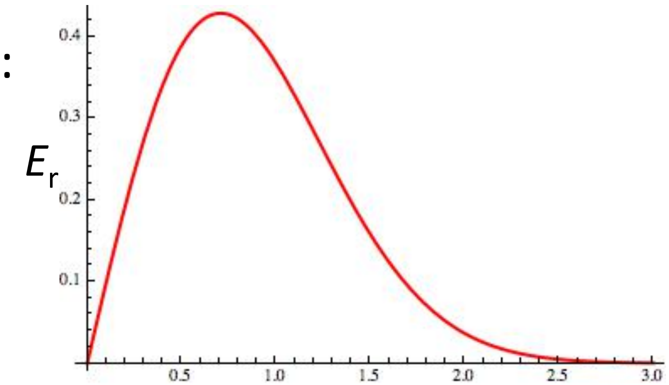
The answer is that to be legal, the loop integral must be zero for every loop.
Look at this loop (go out at θ = π/2 and return at θ = 0).



The moral:
Trust the curl (with the caveat about singularities).
Loop integrals might fool you.

Given *E*, calculate the charge density:

What charge density will produce this electric field:



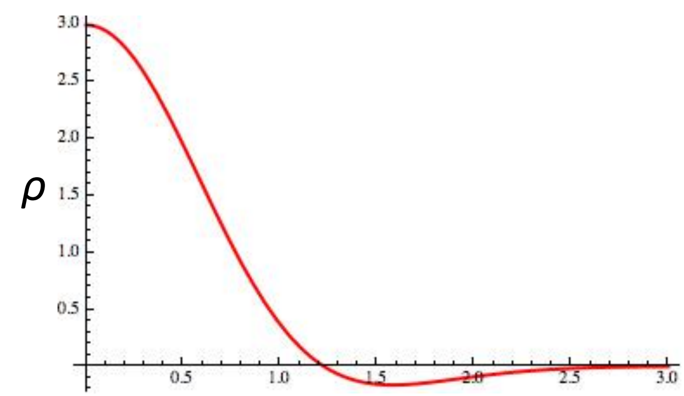
$$\vec{E}(x,y,z) = E_0 e^{-r^2} r\hat{r} = E_0 e^{-(x^2+y^2+z^2)}(x,y,z)$$

spherical   Cartesian

Use Gauss's law: $\vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0}$

You can do it in spherical coordinates: $\vec{\nabla}\cdot\vec{E} = \dfrac{1}{r^2}\dfrac{\partial(r^2 E_r)}{\partial r}$ The only nonvanishing term

Or Cartesian: $\vec{\nabla}\cdot\vec{E} = \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} + \dfrac{\partial E_z}{\partial z}$



In either case, $\rho(r) = \varepsilon_0 E_0 e^{-r^2}(3-2r^2)$

Note: $Q_{tot} = 0$.

Because this problem has spherical symmetry,
you could also do it by drawing spherical Gaussian surfaces.