Integrals Over Charge Distributions

There were questions in office hours about how to use equations that describe continuous charge distributions, $\rho(r,\theta)$ or $\rho(x,y,z)$ in situations where the distribution is restricted to surface, $\sigma(x,y)$ or $\sigma(\theta)$, line, $\lambda(z)$, or discrete, q_i, distributions.

The calculation of many physical quantities involves the integral over a a function times a charge distribution. For example:

$$Q_{\text{tot}} = \int \rho(x,y,z)\,dx\,dy\,dz \qquad or \qquad Q_{\text{tot}} = 2\pi \int \rho(r,\theta)\,r^2\,dr\,\sin\theta\,d\theta$$

From the ϕ integral

$$p_z = \int z\rho(x,y,z)\,dx\,dy\,dz \qquad or \qquad p_z = 2\pi \int r\cos\theta\,\rho(r,\theta)\,r^2\,dr\,\sin\theta\,d\theta$$

$$\vec{F} = \int \vec{E}(x,y,z)\rho(x,y,z)\,dxdydz \qquad or \qquad \vec{F} = 2\pi \int \vec{E}(r,\theta)\rho(r,\theta)\,r^2\,dr\,\sin\theta\,d\theta$$

Suppose the charge is restricted to a surface of constant r (a spherical shell of radius R). Then, we only need to integrate over θ, replacing r with R everywhere:

$$\vec{F} = 2\pi R^2 \int \vec{E}(R,\theta)\sigma(R,\theta)\sin\theta\,d\theta$$

Similarly for a line charge ($x = x_0$, $y = y_0$): $p_z = \int z\lambda(z)\,dz$

What about a collection of point charges? There's no integral to do!

If we have a collection of point charges, we must do a sum rather than an integral:

$$Q_{\text{tot}} = \int \rho(x,y,z)\,dx\,dy\,dz \quad \rightarrow \quad Q_{\text{tot}} = \sum_i q_i$$

$$p_z = \int z\rho(x,y,z)\,dx\,dy\,dz \quad \rightarrow \quad p_z = \sum_i z_i q_i$$

$$\vec{F} = \int \vec{E}(x,y,z)\rho(x,y,z)\,dx\,dy\,dz \quad \rightarrow \quad \vec{F} = \sum_i \vec{E}_i q_i$$

This seems intuitive, and it can be justified mathematically using Dirac delta functions. We can describe a distribution of point charges this way:

$$\rho(x,y,z) \quad \rightarrow \quad \sum_i q_i \delta(\vec{x}-\vec{x}_i)$$

Remember that the δ-function is defined by its behavior in an integral:

$\int f(x)\delta(x-x_0)dx = f(x_0)$ if x_0 is in the range of the integral.

If you let $f(x)$ be ρ, $z\rho$, or $V\rho$ as in the above integrals, you'll get the results as sums over the point charges. For example:

$$p_z = \int z \sum_i q_i \delta(\vec{x}-\vec{x}_i)\,dx\,dy\,dz$$

Collections of dipoles (4.2.1)

In a macroscopic object, we have jillions of little dipoles, which may or may not be aligned. The macroscopic fields they produce will depend on averages.

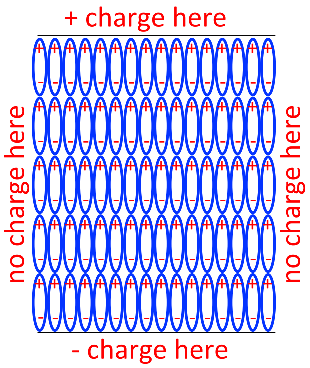
We define: $\boxed{\text{Polarization} \equiv \dfrac{\text{dipole moment}}{\text{volume}}}$. It is a density.

The *E* field of a polarized object

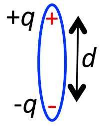
To calculate the field, we need to determine the charge density.
Instead of going through a mathematical derivation, I'll give an intuitive description of what's happening. Griffiths does the math version.

Imagine a volume filled with a bunch of perfectly aligned dipoles:
If the density of dipoles is uniform in the volume,
then the + and – charges cancel.

In this situation, the only unpaired charges are at the top and bottom surfaces.



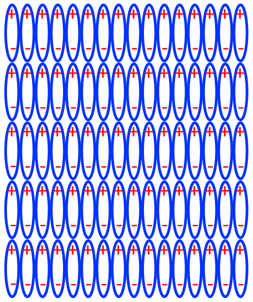
Let's calculate the bound surface charge density:



Imagine that each dipole has charges ±*q* separated by length *d* ($p = qd$).
Then, the polarization, P, is:

$$P = p\rho_d = qd\rho_d \leftarrow \text{volume density of dipoles}$$

However, $\rho_d = \dfrac{\sigma_d}{d}$ ← surface density of dipoles

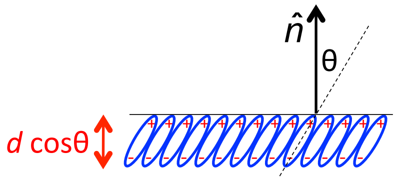


Therefore, $\sigma_b = q\sigma_d = qd\rho_d = P$

(Bound) surface charge density

You can verify that the units of *P* are surface charge density (C/m^2).

If the surface is not perpendicular to the dipoles:



then σ is smaller by cosθ, because $\rho_d = \dfrac{\sigma_d}{d\cos\theta}$.

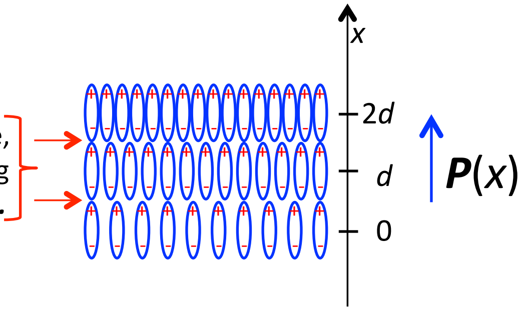
So, we can write: $\boxed{\sigma_b = \vec{P}\cdot\hat{n}}$

We call it "bound charge", because it's not free to move around.

Suppose ***P*** is not uniform:

Suppose there is a density variation along *x* (the polarization direction):

There will be a volume bound charge density, because the dipoles don't cancel:



Net negative charge here, if *P* is increasing in the direction of ***P***.

$$\rho_b = \frac{-q\sigma_d(x=d)+q\sigma_d(x=0)}{d}$$

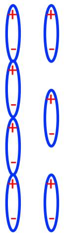
$$= -\frac{qd\rho_d(d)-qd\rho_d(0)}{d}$$

$$= -\frac{\vec{P}(d)-\vec{P}(0)}{d} = -\frac{dP_x}{dx}$$

- Dipole densities, σ_d and ρ_d
- $\sigma_\text{d} = d\rho_\text{d}$
- Polarization: $P = qd\rho_\text{d}$

The *y* and *z* directions are similar (if $\text{P}_{y \text{ or } z}$ is changing along *y* or *z*).
Thus: $\rho_b = -\vec{\nabla}\cdot\vec{P}$

"Sideways" derivatives, e.g. dP_x/dy, don't contribute.



Griffiths derives this result more rigorously (pp. 166-168) but less intuitively.

Dielectrics (4.3.1)

ρ_f is the "free" charge

One might wonder why it's useful to talk about bound charge.

After all, "charge is charge", and (for example) $\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0}$, where $\rho=\rho_b+\rho_f$.

This is a useful distinction to draw, because we usually have control over ρ_f, but ρ_b results from the response of the material to the applied external field.

Let's see where this leads:

$$\begin{aligned}\varepsilon_0\vec{\nabla}\cdot\vec{E}&=\rho_f+\rho_b\\&=\rho_f-\vec{\nabla}\cdot\vec{P}\end{aligned}\quad\Rightarrow\quad\vec{\nabla}\cdot\underbrace{\left(\varepsilon_0\vec{E}+\vec{P}\right)}_{\equiv\vec{D}}=\rho_f$$

D is called
(for historical reasons)
the electric displacement.

The divergence of ***D*** depends only on the free charge density.

So, we have: $\boxed{\vec{\nabla}\cdot\vec{D}=\rho_f\qquad\vec{\nabla}\cdot\vec{P}=-\rho_b\qquad\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0}}$

You'll have to live with the odd placement of ± signs, and of ε_o. They are features of SI units.

We'll see shortly that $\vec{\nabla}\times\vec{D}$ is not always zero.
We can't always use Laplace's eq. to calculate ***D***.

Comments:

- ***D*** is the field we can control by manipulating the free charge.
- ***E*** is the total electric field. It depends on ***D*** and the response, ***P***, of the material.
- The polarization reduces ***E*** from what it would be in vacuum (***P*** = 0).
- Caveat: $\varepsilon_0 \mathbf{E} = \mathbf{D} - \mathbf{P}$. So, ***E*** has different units (N/C or V/m) than ***D*** or ***P*** (C/m^2).

Example:

A material with a linear response. Assume: $\vec{P} = \varepsilon_0 \chi \vec{E}$.
χ is a positive, dimensionless number, called the electric susceptibility.

Then, the electric displacement is: $\vec{D} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi \vec{E} = \varepsilon_0 (1+\chi) \vec{E}$

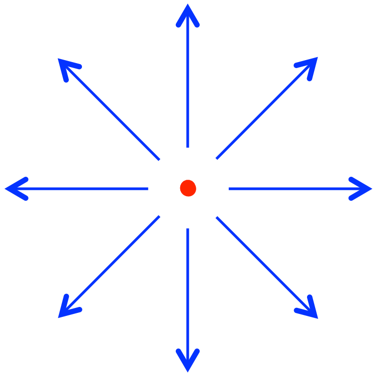
Definitions:

- $\varepsilon_r \equiv 1 + \chi$ is called the dielectric constant of the material.
- $\varepsilon \equiv \varepsilon_0 (1+\chi) = \varepsilon_0 \varepsilon_r$ is called the permittivity. $\mathbf{D} = \varepsilon \mathbf{E}$.

Example:

A large (infinite) piece of linear material ($\chi \neq 0$) with an embedded charge, *Q*.

a: What is E everywhere?



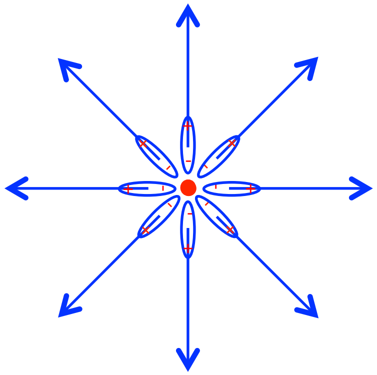
Q determines ***D***: $\vec{D} = \dfrac{Q}{4\pi r^2}\hat{r}$. Note: No ε_0.

If this were vacuum ($\chi = 0$), then ***E*** = ***D***/ε_0, as expected.

For $\chi \neq 0$: ***E*** = ***D***/ε = ***D***/$(1+\chi)\varepsilon_0$, smaller due to polarization.

This is a general feature of dielectrics: ***E*** is reduced.

b: Where is the bound charge?



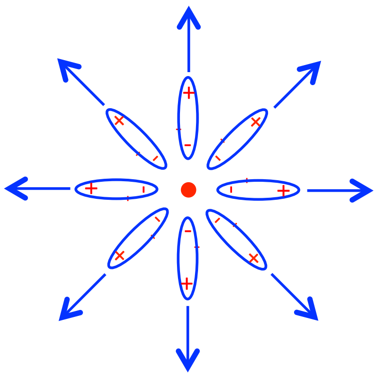
$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\varepsilon_0\chi\vec{\nabla}\cdot\vec{E} \qquad \vec{P} = \varepsilon_0\chi\vec{E}$$

$$= -\frac{\chi}{1+\chi}\vec{\nabla}\cdot\vec{D} \qquad \vec{D} = \varepsilon_0(1+\chi)\vec{E}$$

$$= -\left(\frac{\chi}{1+\chi}\right)\rho_f \qquad \rho_f \text{ is } Q \text{ (the point charge)}$$

There is a bound point charge, $Q_b = -\left(\dfrac{\chi}{1+\chi}\right)Q$ at the same place as *Q*.

This is why a dielectric reduces ***E***.
The polarization generates a negative charge that "screens" the free charge.
As χ becomes large, $Q_b \rightarrow -Q$, and $\vec{E} \rightarrow 0$.

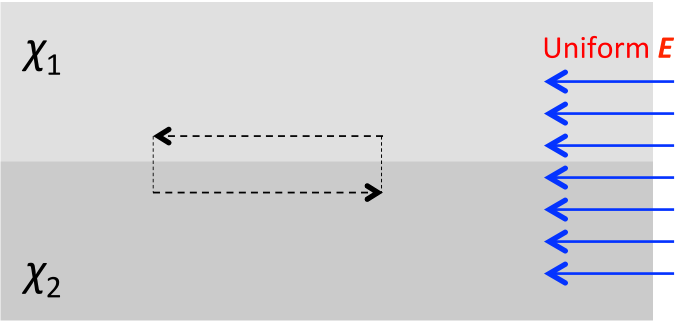


Beware: If χ varies with position, then $\vec{\nabla} \times \vec{P}$ and $\vec{\nabla} \times \vec{P}$ might $\neq 0$.

Consider the interface between two materials:

$$\vec{P}_1 = \varepsilon_0 \chi_1 \vec{E} \qquad \neq \vec{P}_2$$

$$\vec{D}_1 = \varepsilon_0 \left(1 + \chi_1\right) \vec{E} \quad \neq \vec{D}_2$$



The integrals of ***P*** and ***D*** around loops that span the interface are not zero.

This means that you can't always use Laplace's equation to calculate ***D*** or ***P***.

End 10/4/13