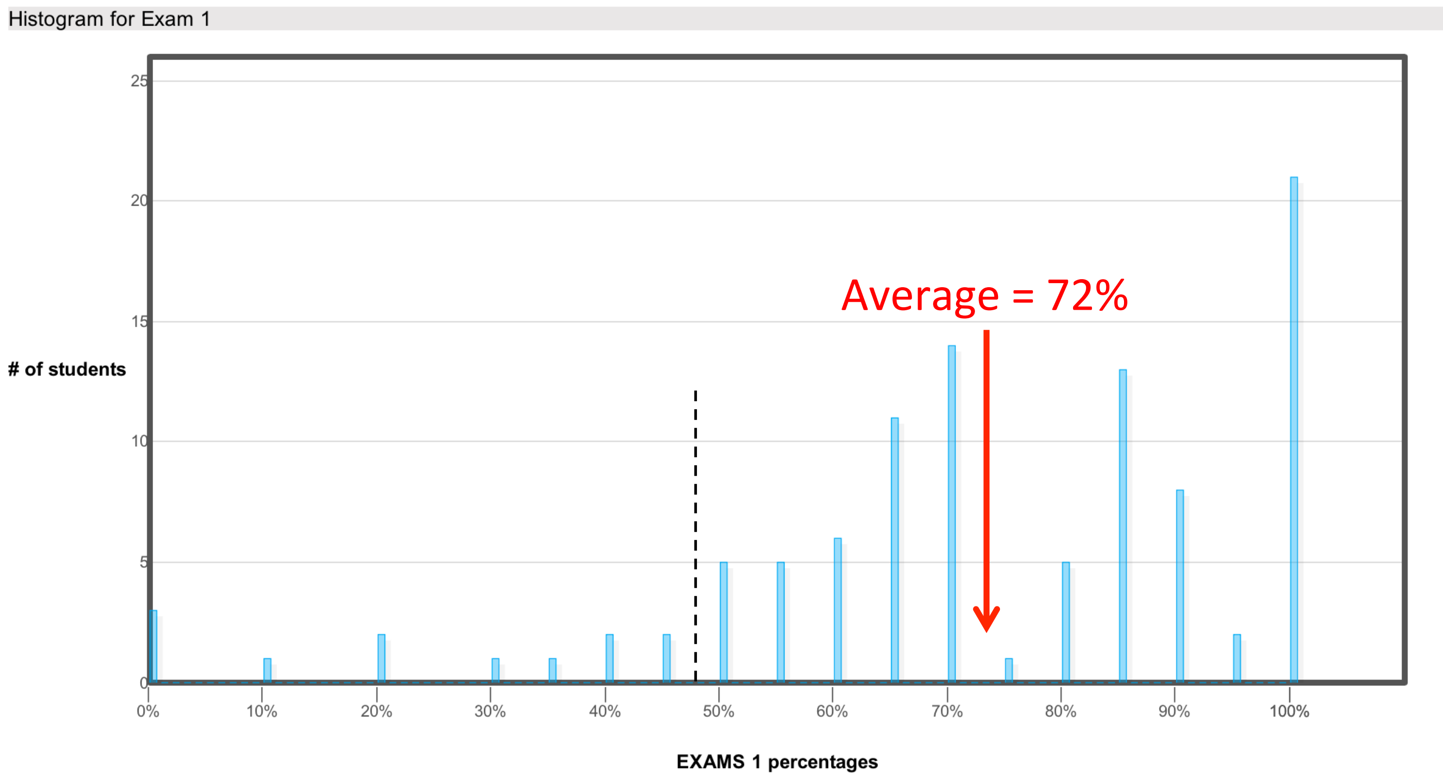
Exam results



The most common conceptual issues:

- Spherical versus cylindrical coordinates.
- How to use orthogonality.

Tonight's discussion will review these topics.

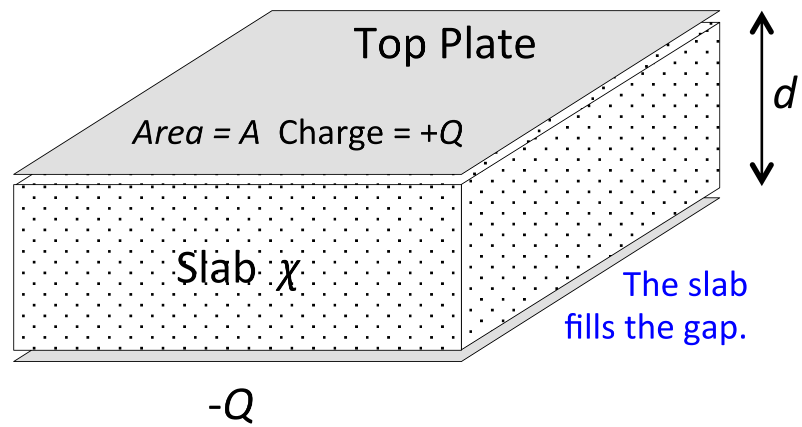
People who scored below about 50% (10/20) should see me.

Capacitors

This is an important application of dielectrics.

One purpose of a capacitor is to hold charge. One limitation is the "breakdown voltage", so we'd like to make $C \equiv Q/V$ as large as possible: large Q, small V.

Let's look at a parallel-plate capacitor:
On each plate, $\sigma_f = \pm Q/A$.



Note: Q is free charge.
We put it on the plate.

$$D = \sigma_f$$

$$E = \frac{\sigma_f}{\varepsilon}$$

Permittivity: $\varepsilon = (1+\chi)\,\varepsilon_0$

$$V = Ed = \frac{\sigma_f d}{\varepsilon} = \frac{Qd}{\varepsilon A}$$

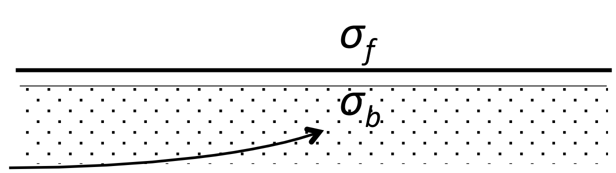
$$C = \frac{Q}{V} = \varepsilon\frac{A}{d} = \varepsilon_0\frac{A}{d}(1+\chi)$$

If $\chi = 0$:

$$E = \frac{\sigma_f}{\varepsilon_0}$$

$$V = \frac{Qd}{\varepsilon_0 A}$$

$$C = \varepsilon_0\frac{A}{d}$$



V is reduced, because there is bound surface charge.
C is increased by a factor of $(1+\chi) = \varepsilon_r$ (the dielectric constant).

Boundary Conditions (4.4.2)

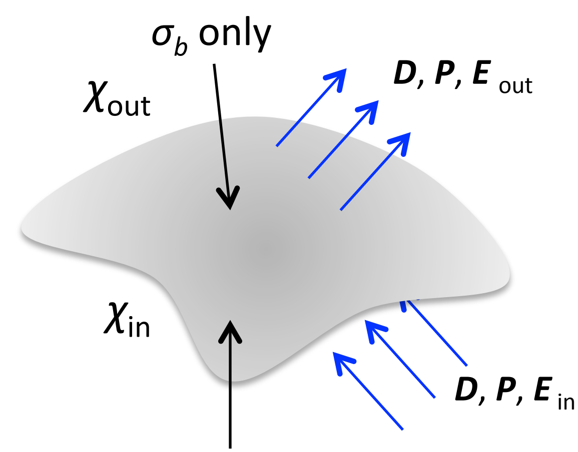
We saw that for a point free charge embedded in a dielectric, the bound charge resides at the free charge. This is a general result for uniform dielectrics (χ not a function of position).

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot\left(\frac{\chi}{1+\chi}\vec{D}\right) = -\left(\frac{\chi}{1+\chi}\right)\rho_f$$

This simplifies problem solving. If there is no embedded free charge (*e.g.*, there is only an externally applied field), $\rho_b = 0$. There is only bound charge at the surface.

What are the boundary conditions?

- At the surface, $\vec{\nabla}\cdot\vec{D} = 0$, so $\vec{D}_{out}\cdot\hat{n} = \vec{D}_{in}\cdot\hat{n}$.
  The normal component of ***D*** is continuous.
  Notes:
  - This is not true of E or P, because they are sensitive to σ_b.
  - $\vec{\nabla}\times\vec{D}$ might $\neq 0$, so the tangential component is not necessarily continuous.



Surface of the dielectric

- $\vec{\nabla}\times\vec{E}$ does equal zero, so the tangential component of ***E*** is continuous.

We usually want to calculate ***E*** and V

so rewrite the ***D*** boundary condition:

$$\vec{D}_{\perp\,\text{out}} = \vec{D}_{\perp\,\text{in}}$$

$$\varepsilon_{\text{out}} \vec{E}_{\perp\,\text{out}} = \varepsilon_{\text{in}} \vec{E}_{\perp\,\text{in}}$$

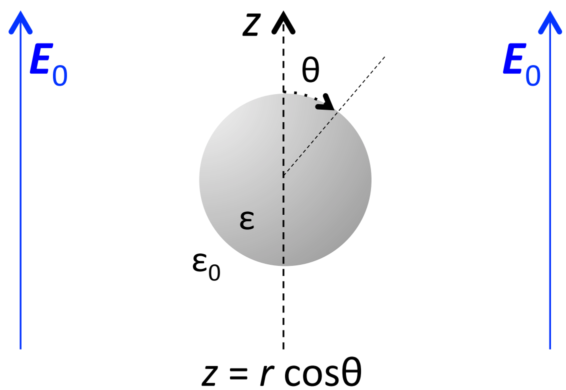
$$\varepsilon_{\text{out}} \left.\frac{\partial V}{\partial n}\right|_{\text{out}} = \varepsilon_{\text{in}} \left.\frac{\partial V}{\partial n}\right|_{\text{in}}$$

This means:

Problems with uniform dielectrics (constant χ) in external fields (no embedded charge) reduce to Laplace's equation with this boundary condition.

Example: (G, Ex 4.7)

Dielectric sphere in an otherwise uniform ***E*** field.
"Otherwise uniform" means $\vec{E} \to \vec{E}_0$ at large distances.



Boundary conditions:

1) V is continuous at $r = R$.
2) $\varepsilon \left.\frac{\partial V_{\text{in}}}{\partial r}\right|_R = \varepsilon_0 \left.\frac{\partial V_{\text{out}}}{\partial r}\right|_R$
3) $V_{\text{out}} \to -E_0 r\cos\theta \quad \text{for } r \gg R$

So, for all $r > R$, V_{out} must have this form:

$$V_{\text{out}}(r,\theta) = -E_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

← All these terms go to zero as r goes to ∞.

Condition 1: (continuity of V at $r = R$) tells us: $\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = -E_0 R\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta)$

V_{in} at $r = R$ V_{out} at $r = R$

So, for every $l \neq 1$: $A_l R^l = \frac{B_l}{R^{l+1}}$

For $l = 1$: $A_1 R^1 = -E_0 R + \frac{B_1}{R^2}$

Condition 2: (continuity of D_r) is: $\frac{\varepsilon}{\varepsilon_0} \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -E_0 \cos\theta - \sum_{l=0}^{\infty} \frac{(l+1)B_l}{R^{l+2}} P_l(\cos\theta)$

For $l \neq 1$: $\varepsilon_r l A_l R^{l-1} = -\frac{(l+1)B_l}{R^{l+2}}$

For $l = 1$: $\varepsilon_r A_1 = -E_0 - \frac{2B_1}{R^3}$

I took the r derivative of V and moved ε_0 to the left side.

$\varepsilon/\varepsilon_0 = \varepsilon_r$, the dielectric constant.

The $l \neq 1$ equations have no non-zero solutions, because $\frac{B_l}{R^{l+1}} = -\left(\frac{l+1}{l}\right)\left(\frac{1}{\varepsilon_r}\right)\left(\frac{B_l}{R^{l+1}}\right)$

For $l = 1$: $A_1 = -\frac{3E_0}{\varepsilon_r + 2}$, and $B_1 = \frac{\varepsilon_r - 1}{\varepsilon_r + 2} R^3 E_0$

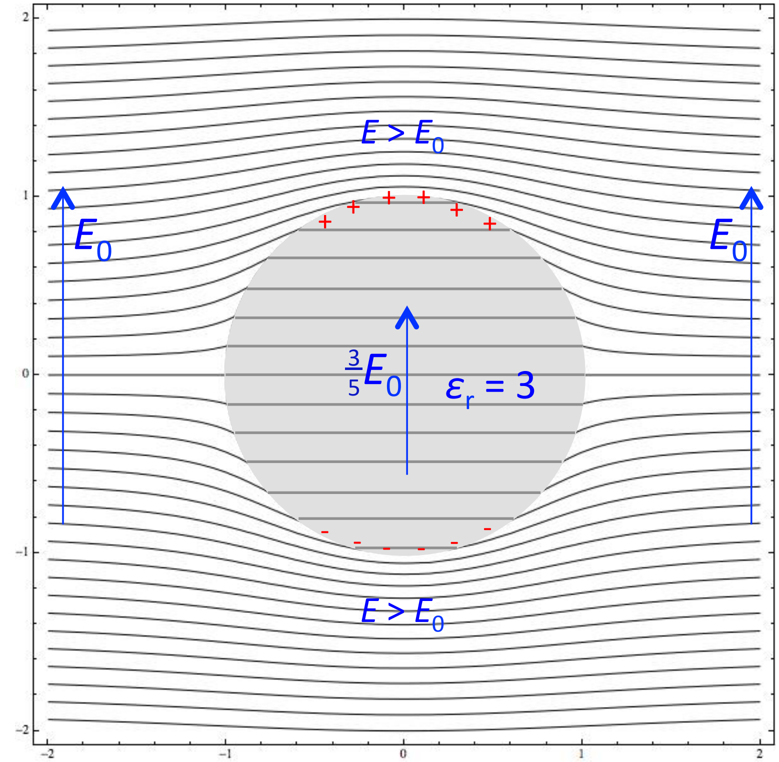
Note the behavior as ε_r approaches 1.

What does it look like?

$$V_{\text{out}} = -E_0 r\cos\theta + \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) E_0 R^3 \frac{\cos\theta}{r^2}$$

dipole potential

$$V_{\text{in}} = -\left(\frac{3}{\varepsilon_r + 2}\right) E_0 r\cos\theta$$



Unlike Griffiths, I am plotting contours of constant V.

- Uniform (reduced) field inside.
- Increased field outside, near the poles. (why?)
- Uniform field, plus a dipole, outside. $p = 4\pi\varepsilon_0\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) E_0 R^3 = \left(\frac{4}{3}\pi R^3\right)\left(\frac{3\chi}{3+\chi}\right) E_0.$
- The surface of the sphere is not an equipotential.

End 10/7/13