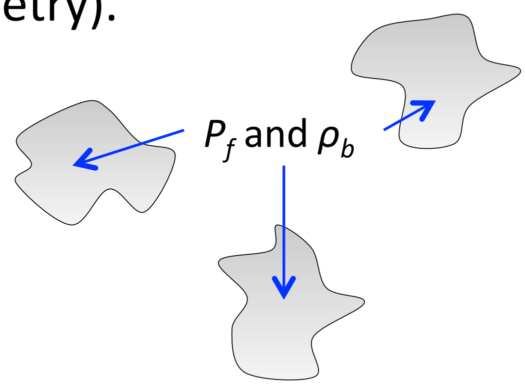
Energy with Dielectrics (4.4.3)

Earlier, we calculated the energy required to assemble a collection of charges. The answer will be different in the presence of dielectric material, because there is potential energy associated with dipoles in electric fields.

It takes energy to polarize the material (to "stretch the springs", as G puts it), so the energy will be larger.

Note: We will only move the free charges around. The location of the bound charges is determines by the dielectric properties (and geometry).

Consider a partially assembled system, described by $\rho_f(\mathbf{r})$:
These objects may be conductors, dielectrics, or whatever.



Now, let's add a little more free charge, $\Delta\rho_f(\mathbf{r})$.
How much energy did it take?
I assume that we know $V(\mathbf{r})$at this moment.

Then: $\Delta W = \int \Delta\rho_f(\vec{r})V(\vec{r})d^3r$

Because $\Delta\rho$ is small, any effects due to ΔV are quadratically small.

Now, use the fact that $\vec{\nabla}\cdot\vec{D}=\rho_f$.

Then $\Delta\rho_f=\vec{\nabla}\cdot\Delta\vec{D}$, and $\Delta W=\int V\left(\vec{\nabla}\cdot\Delta\vec{D}\right)dVol$.

Use our favorite vector identity (#5): $\nabla\cdot\left(f\vec{v}\right)=f\vec{\nabla}\cdot\vec{v}+\vec{v}\cdot\vec{\nabla}f$

or $f\vec{\nabla}\cdot\vec{v}=\nabla\cdot\left(f\vec{v}\right)-\vec{v}\cdot\vec{\nabla}f$

Here, $\vec{v}$ is $\Delta\vec{D}$ and f is V. So: $\Delta W=\int\nabla\cdot\left(V\Delta\vec{D}\right)dVol-\int\left(\Delta\vec{D}\cdot\underbrace{\vec{\nabla}V}_{-E}\right)dVol$

$$=\int\left(\Delta\vec{D}\cdot\vec{E}\right)dVol$$

=0, if the fields vanish at ∞
(divergence theorem)

For a linear dielectric: $\Delta\boldsymbol{D}=\varepsilon\Delta\boldsymbol{E}$. Therefore: $\Delta\vec{D}\cdot\vec{E}=\varepsilon\Delta\vec{E}\cdot\vec{E}=\frac{\varepsilon}{2}\Delta(E^2)=\frac{1}{2}\Delta(\vec{D}\cdot\vec{E})$

Components of $\Delta\boldsymbol{E}$ that are perpendicular to $\boldsymbol{E}$ do not contribute to ΔE^2.

Thus: $\Delta W=\frac{1}{2}\Delta\int\vec{D}\cdot\vec{E}\,dVol$

Or: $\boxed{W=\frac{1}{2}\int\vec{D}\cdot\vec{E}\,dVol}$

As expected, for a given $\boldsymbol{E}$ (which means a given V) W is larger when $\varepsilon>\varepsilon_0$ ($\chi\neq 0$). This increases the usefulness of capacitors as energy storage devices.

Forces on Dielectrics

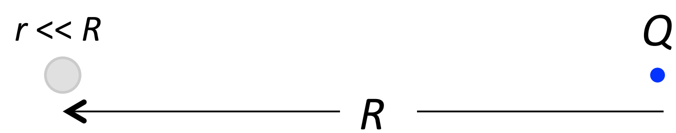
We saw last week that an electric dipole feels a force when placed in a non-uniform field: $\vec{F} = \left(\vec{p}\cdot\vec{\nabla}\right)\vec{E}$ Our energy consideration lets us calculate forces on dielectrics. The origin is similar:

- The material gains a dipole moment.
- The energy will decrease if the dielectric moves to a high field region:

$$\boxed{W = \tfrac{1}{2}\int \vec{D}\cdot\vec{E}\, dVol}$$

I say decrease, because in this problem we are holding the free charge fixed, not the voltage. That is, ***D*** remains constant.

Example: A small dielectric sphere near a point charge:
We already solved this:



$$\vec{p} = \left(\tfrac{4}{3}\pi r^3\right)\left(\frac{3\chi}{3+\chi}\right)\vec{E}$$

$$\vec{F} = \left(\vec{p}\cdot\vec{\nabla}\right)\vec{E} = \left(\tfrac{4}{3}\pi r^3\right)\left(\frac{3\chi}{3+\chi}\right)\left(\frac{Q}{4\pi\varepsilon_0}\right)^2\left(\frac{2}{R^5}\right)\left(-\hat{r}\right)$$

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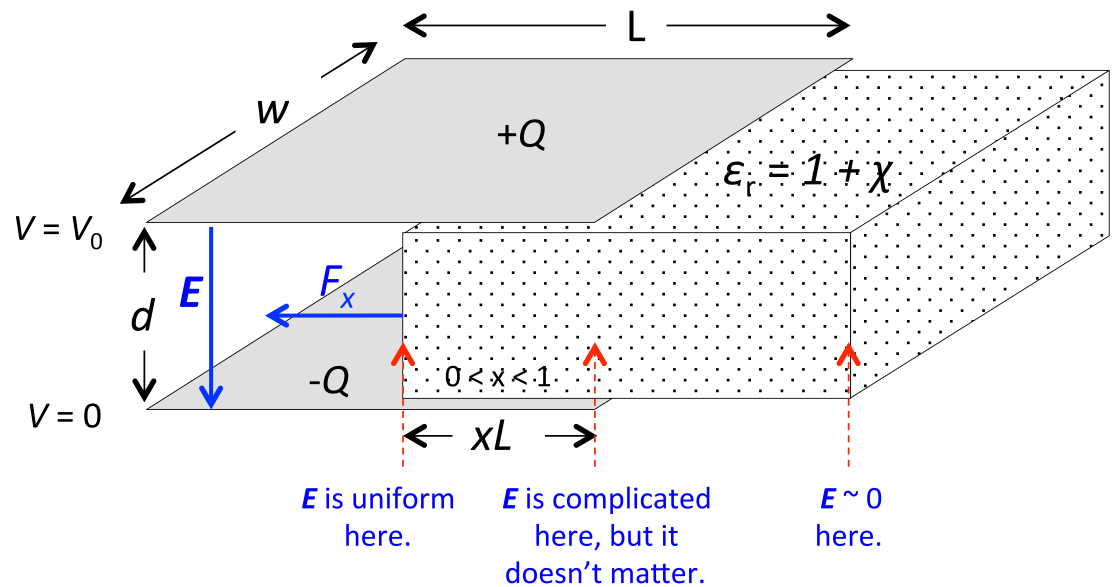
Why $1/R^5$?

Example: (G, 4.4.4) (Illustrating a trap.)

Force on a dielectric partially inserted in a capacitor:

One could use $F = \int qE$, but this problem is *much easier* to do using energy, because we don't know ***E*** in the "fringing region".



Use $F_x = -\dfrac{dW}{dx}$ Why?

$$W = \frac{1}{2}CV_0^2 = \frac{1}{2}\frac{Q^2}{C}$$

where $C = \dfrac{\varepsilon_0 A}{d}(\chi x + 1)$ ← A HW problem.

We'll consider two situations:

- Assume Q is constant.
- Assume V is constant.

Note: E is determined by V_0. We won't write it explicitly.

In the second case, a battery supplies charge to keep $V = V_0$.

(Why is this necessary?)

End 10/9/13