Constant Q:

$$W = \frac{1}{2}\frac{Q^2}{C} \Rightarrow \frac{dW}{dx} = -\frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx}$$

Then: $F_x = -\frac{dW}{dx} = +\frac{\varepsilon_0 \chi w}{2d} V_0^2$

The slab is pulled in (x increases) because W (the energy of the charge configuration) decreases as C increases.

$$C = \frac{\varepsilon_0 w}{d}\left(1 + \chi x\right)$$

$$\frac{dC}{dx} = \frac{\varepsilon_0 w}{d}\chi$$

$$W = \frac{1}{2}CV_0^2 = \frac{1}{2}\frac{Q^2}{C}$$

Constant V:

$$F_x = -\frac{1}{2}V_0^2\frac{dC}{dx} = -\frac{\varepsilon_0 \chi w}{2d}V_0^2$$

$V = \frac{Q}{C}$

Is this correct? NO!!

$F_x dx$ is not the only work being done.

Why?

(The battery also does work (it pushes charge onto the plates), so $F_x \neq -dW/dx$.

The moral (see Griffiths): The two forces must be equal, because at any given instant, the force is determined by $q\mathbf{E}$, the location of the charges and the electric field configuration. (Also: Keep track of all the players.)

Magnetism

Lorentz force (5.1)

Let's look at the motion of a charged particle in a ***B*** field.
You know from P212 that $\boxed{\vec{F} = q\vec{v} \times \vec{B}}$. This is the Lorentz Force Law.

Magnetic force is independent of the electric force (except that it's the same *q*).
The total force is the sum: $\vec{F}_{\text{tot}} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$.

One important feature of the magnetic force is that it does no work:

$$P_{\text{ower}} = \vec{F} \cdot \vec{v} = q\left(\vec{v} \times \vec{B}\right) \cdot \vec{v} = 0 \text{ always!} \quad \Rightarrow \quad |v| = \text{constant}$$

Let's look at the motion of a charged particle (q, m) in a uniform ***B*** field.
You already know (I hope) that it's a circle. Consider ***B*** = (0, 0, B_z).

$$\vec{F} = m\vec{a}: \quad \left.\begin{array}{l} m\dfrac{dv_x}{dt} = +qv_y B_z \\[2ex] m\dfrac{dv_y}{dt} = -qv_x B_z \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \dfrac{d^2 v_x}{dt^2} = -\left(\dfrac{qB_z}{m}\right)^2 v_x \\[2ex] \dfrac{d^2 v_y}{dt^2} = -\left(\dfrac{qB_z}{m}\right)^2 v_y \end{array}\right.$$

$$m\frac{dv_z}{dt} = 0 \quad \Rightarrow \quad v_z = \text{constant}$$

The solutions are exponentials: $x = X_0 e^{kt}$, $y = Y_0 e^{kt}$, where $k^2 = -\left(\frac{qB_z}{m}\right)^2 \Rightarrow k = \pm i\left(\frac{qB_z}{m}\right)$

Because k is imaginary, the solutions are actually sin() and cos(). $e^{i\theta} = \cos\theta + i\sin\theta$

The representation of trig functions as exponentials is ubiquitous.
It vastly simplifies the algebra, and makes many trig identities self-evident.

How do we know the x-y trajectory is a circle?

Substitute into one of the diff. eqs. (I picked the first one):

$$v_x = X_0 k e^{kt}, \quad \frac{dv_x}{dt} = X_0 k^2 e^{kt}, \quad v_y = Y_0 k e^{kt}$$

$$\Rightarrow -mX_0\left(\frac{qB_z}{m}\right)^2 = \pm iqY_0 B_z\left(\frac{qB_z}{m}\right) \Rightarrow X_0 = \pm iY_0$$

The x and y motions have equal amplitude and are 90° out of phase.

Comment:

$\omega_c \equiv \frac{qB}{m}$

is called the cyclotron frequency.

For a proton in a 1 T field:

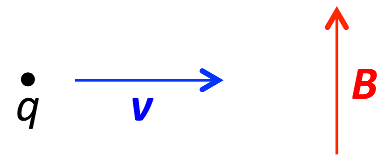
$$\omega_c = \frac{1.6\times10^{-19}\,\text{C}\times 1\,\text{T}}{1.7\times10^{-27}\,\text{kg}} = 9.4\times10^{7}\ \text{Rad/s}$$

$= 15$ MHz

Radio frequency.

A Paradox:

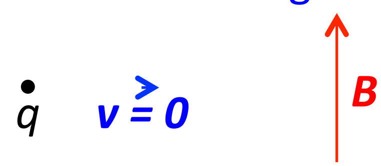
Consider this:



There will be a force: $\vec{F} = q\vec{V} \times \vec{B}$

However, suppose we look at this problem in a moving reference frame

(one moving with the charge).



What is the force now?

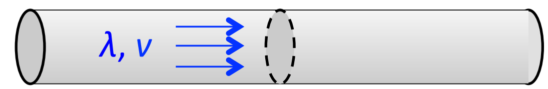
It can’t be zero, because both observers see *q* accelerate.

We’ll resolve this paradox next semester. It requires special relativity.
In fact, it was EM paradoxes (not the Michelson-Morley experiment) that led Einstein to develop SR.

The resolution: ***E*** ≠ 0 in the moving reference frame.

Currents and Forces (5.1.3)

Consider a current-carrying wire:



The wire is electrically neutral, because there is –λ of stationary charge (in the atoms).

The charge density (per unit length of wire) of current carriers is λ, so the current is: $I = \lambda v$.

So, the force per unit length on the wire is:

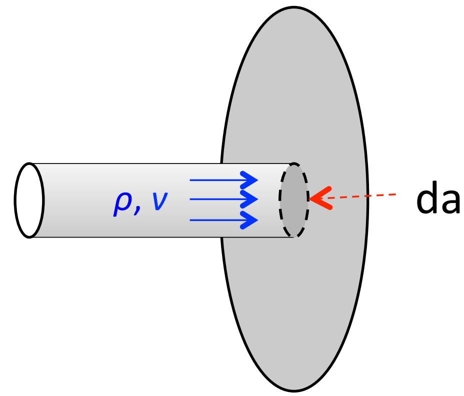
$$d\vec{F} = (\lambda dl)\vec{v} \times \vec{B}$$
$$= (\lambda v) d\vec{l} \times \vec{B} \quad \text{because } \vec{v} \parallel d\vec{l}.$$
$$= I d\vec{l} \times \vec{B}$$

Thus, $\vec{F}_{\text{tot}} = I \int d\vec{l} \times \vec{B}$

Note that for a loop in a uniform ***B***: $\vec{F}_{\text{tot}} = -I\vec{B} \times \underbrace{\oint d\vec{l}}_{=0} = 0$

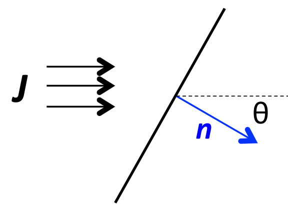
We can derive similar force equations for surface (2-D) and volume (3-D) currents, denoted by ***K*** and ***J*** respectively. (G, pp. 211-212)

Conservation of Charge



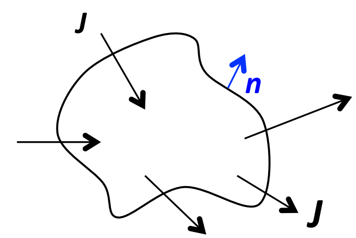
A 3-D current density, ***J***, is determined by the 3-D charge density and velocity.
***J* = *ρ*v, is the current per unit area** (C/s·m^2) through a surface perpendicular to the flow.

If the surface is not perpendicular to ***J***, then there is a cos(θ) effect:



Current flux (per unit area) = $\vec{J}\cdot\hat{n}$
This is just like field flux.

Let's look at a closed surface (enclosing a volume of interest).
The total current flowing through this surface is $\oint_{surface} \vec{J}\cdot\hat{n}\,da$.



Apply the divergence theorem to this:

$$\oint_S \vec{J}\cdot d\vec{a} = \int_{Vol} \vec{\nabla}\cdot\vec{J}\,dVol$$

$$= -\frac{dQ_{in}}{dt} = -\frac{d}{dt}\int_{Vol} \rho\,dVol$$

Conservation of charge implies:
Charge flowing out means charge inside is decreasing.

As we did with the ***E*** field manipulation, observe that this equation:

$$\int_{Vol} \vec{\nabla} \cdot \vec{J}\, dVol = -\int_{Vol} \frac{d\rho}{dt} dVol$$

is true for any arbitrary volume. Therefore, the integrands must be equal everywhere:

$$\boxed{\vec{\nabla} \cdot \vec{J} = -\frac{d\rho}{dt}}$$

This is called the continuity equation. It is the local statement of conservation of charge ("local" means it relates ***J*** and ρ at a point, not integrated over a volume).

You'll see a very similar equation in QM (P486), where the conservation of probability is expressed in terms of the divergence of the probability current.