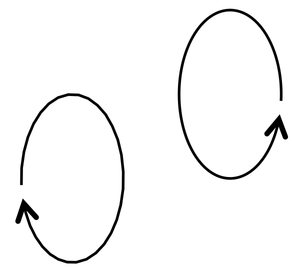
Currents and *B* Fields (5.2)

For now, we'll consider situations with steady currents.
This means not only that ***J*** is constant,
but also that ρ is constant.



The latter condition implies that $\vec{\nabla}\cdot\vec{J}=0$ everywhere.
This can only work if current flows in loops:
If the lines end anywhere, then charge will pile up at the ends: ρ won't be constant.

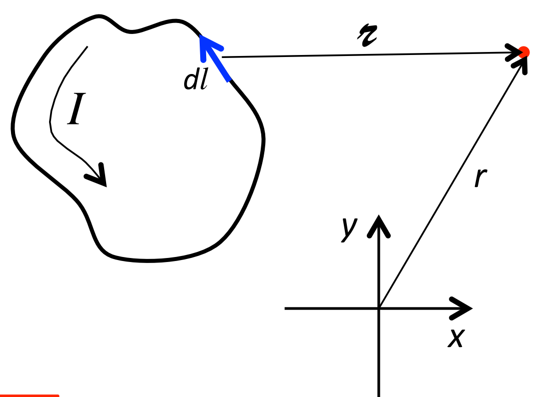
There is no such thing as a point steady current, unlike a point charge.

In this situation, we have (empirically) the Biot-Savart Law:

$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\oint\frac{\vec{I}\times\hat{\mathscr{r}}}{\mathscr{r}^2}dl=\frac{\mu_0 I}{4\pi}\oint\frac{d\vec{l}\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

One can generalize this to surface
or volume current densities.

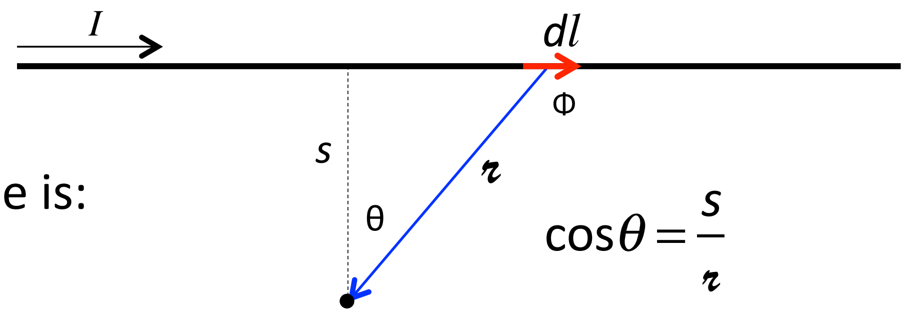
- Replace $\vec{I}$ with $\vec{K}$ or $\vec{J}$.
- Replace $d\vec{l}$ with $d\vec{a}$ or $dVol$.



Bio-Savart is the magnetic equivalent of Coulomb's law.

Example:

Consider an infinitely long straight wire:



$d\vec{l} \times \hat{\mathscr{r}}$ points into the page. Its magnitude is:

$$\left|d\vec{l} \times \hat{\mathscr{r}}\right| = dl\sin\varphi = dl\cos\theta = dl\left(s/\mathscr{r}\right) \qquad \cos\theta = \frac{s}{\mathscr{r}}$$

Also, $\frac{1}{\mathscr{r}^2} = \frac{\cos^2\theta}{s^2}$

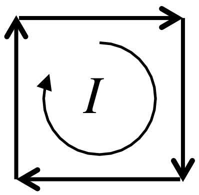
Here is the integral we must do: $\vec{B} = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{+\infty}\frac{\cos^3\theta}{s^2}dl$

Write *dl* in terms of *d*θ: $dl = s\,d\left(\tan\theta\right) = s\sec^2\theta\, d\theta = \frac{s\,d\theta}{\cos^2\theta}$

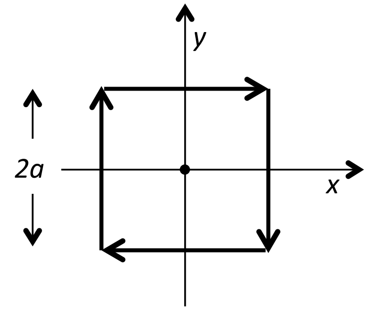
Thus: $\vec{B} = \frac{\mu_0 I}{4\pi s}\int_{\theta_{min}}^{\theta_{max}}\cos\theta\, d\theta = \frac{\mu_0 I}{4\pi s}\left[\sin\theta\right]_{\theta_{min}}^{\theta_{max}}$

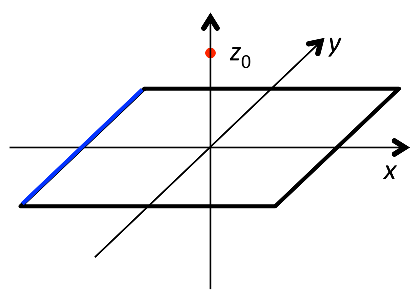
=2 for an ∞ wire

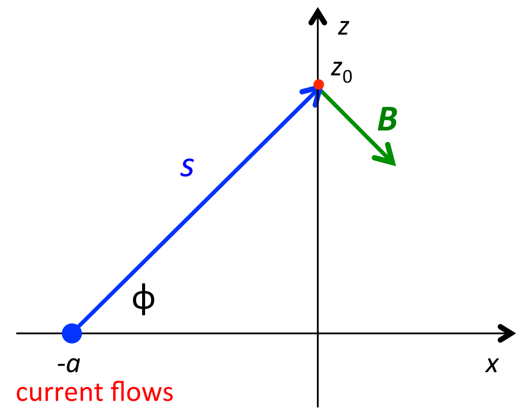
Note: This is valid for a finite wire as long as it is part of a complete loop (*e.g.*, a square):



Example: The magnetic field on the axis, at (0, 0, z_0) of a square loop:







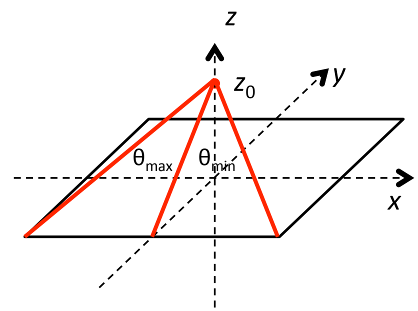
By symmetry, B must point along the z-axis.

All four segments produce the same B_z. Look at one.

$$B = \frac{\mu_0 I}{4\pi s}\left[\sin\theta\right]_{\theta_{min}}^{\theta_{max}}$$

$$s = \left(a^2 + z_0^2\right)^{1/2} \quad \text{and} \quad \cos\varphi = \frac{a}{s}$$

$$\sin\theta_{max} = \frac{a}{\left(a^2+s^2\right)^{1/2}} = \frac{a}{\left(2a^2+z_0^2\right)^{1/2}}$$



$$\text{So, } B_z = -B\cos\varphi = -\frac{\mu_0 I}{4\pi}\frac{1}{\left(a^2+z_0^2\right)^{1/2}}\frac{2a}{\left(2a^2+z_0^2\right)^{1/2}}\frac{a}{\left(a^2+z_0^2\right)^{1/2}}$$

$$= -\frac{\mu_0 I a^2}{2\pi}\frac{4 \text{ (four sides)}}{\left(a^2+z_0^2\right)\left(2a^2+z_0^2\right)^{1/2}}$$

For large z_0, $B \propto \frac{1}{z_0^3}$.

This is a magnetic dipole.

There are no monopoles.

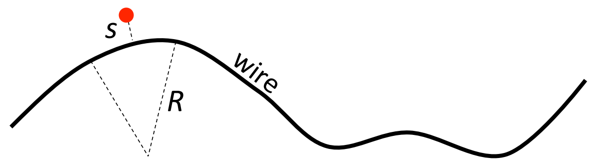
Ampere's Law (5.3)

Let's consider the curl and divergence of *B*.

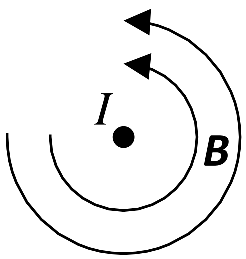
The B field near a current-carrying wire is: $\vec{B} = \frac{\mu_o I}{2\pi s}\hat{\varphi}$

Cylindrical coordinates:
This is for I along the +z axis.
Use the right-hand rule.

Note: This holds for any wire,
if s << the radius of curvature:



Look at it end-on:
I is out of the picture.



Because ***B*** is inversely proportional to *s*, the integral around a loop of radius *s* is $\oint \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi s}\oint dl = \mu_0 I$ independent of *s*.

- In fact, this is true for any loop (any shape) that goes once around the wire.
- Any loop that does not go around the wire has $\oint \vec{B}\cdot d\vec{l} = 0$.

This is an important result, so we'll prove it.

Proof that $\oint \vec{B}\cdot d\vec{l} = \mu_0 I$ for any loop around a wire.

Do it in cylindrical coordinates. (s,Φ,z). $\vec{B} = \left(0, \dfrac{\mu_0 I}{2\pi s}, 0\right)$, and $d\vec{l} = \left(ds, s d\varphi, dz\right)$

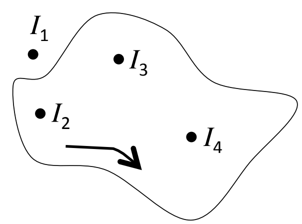
Thus, $\vec{B}\cdot d\vec{l} = \dfrac{\mu_0 I}{2\pi} d\varphi$.

Neither radial (s) nor longitudinal (z) motion contributes to the integral.

And $\int_{\varphi_1}^{\varphi_2} \vec{B}\cdot d\vec{l} = \dfrac{\mu_0 I}{2\pi}\left(\varphi_2 - \varphi_1\right)$.

- A complete loop gives $\mu_0 I$.
- Multiple loops (N times around) gives $N\mu_0 I$.
- Going the other way around ($d\varphi$ becomes $-d\varphi$) changes the sign.

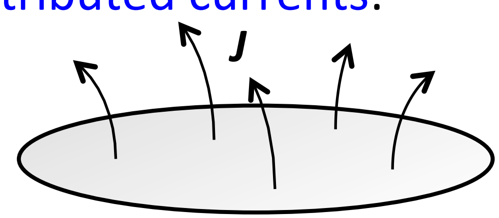
We can generalize this result using superposition: $\oint \vec{B}\cdot d\vec{l} = \mu_0 \sum_i I_i$
Only the wires that pass through the loop contribute.



Clearly, we can let the discrete wires become distributed currents:

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \int_{\text{surface}} \vec{J}\cdot\hat{n}\, da$$

The current flux through the loop.



Now, remember our friend, Stokes' theorem: $\oint \vec{B}\cdot d\vec{l} = \int_{\text{surface}} \left(\vec{\nabla}\times\vec{B}\right)\cdot\hat{n}da$

So, we have: $\mu_0 \int_{\substack{\text{any}\\ \text{surface}}} \vec{J}\cdot\hat{n}da = \int_{\substack{\text{any}\\ \text{surface}}} \left(\vec{\nabla}\times\vec{B}\right)\cdot\hat{n}da$

As we did with the electric field (because it works for any surface through any loop), this can be written as a local (differential) equation:

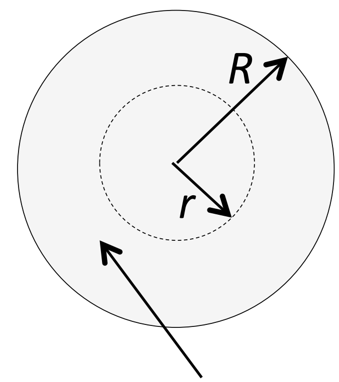
$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}$$

This is Ampere's Law.
It is one of Maxwell's equations

In a current-free region, $\vec{\nabla}\times\vec{B} = 0$.

Example: (G, 5.13a)

Consider a circular wire of radius *R* that carries total current I.
Suppose the current density is uniform in the wire:



Uniform J inside.

This is the actual situation for DC currents in wires.

Calculate ***B*** inside and outside the wire.
Symmetry tells us that ***B*** has only a ϕ component.

$J = \frac{I}{\pi R^2}$. Units are Amp/m^2.

For $r < R$: $\oint B\,dl = \mu_0 I_{\text{enclosed}}$

$$2\pi r B = \mu_0 \cdot J \cdot \text{Area} = \mu_0 \frac{I}{\pi R^2} \pi r^2$$

$$B = \frac{\mu_0 I}{2\pi R}\left(\frac{r}{R}\right)$$

For $r > R$: $I_{\text{enclosed}} = I$, and $B = \frac{\mu_0 I}{2\pi r}$

The same as from a line current

Cylindrical symmetry makes current problems simple, just as spherical symmetry does for charges.

End 10/14/13