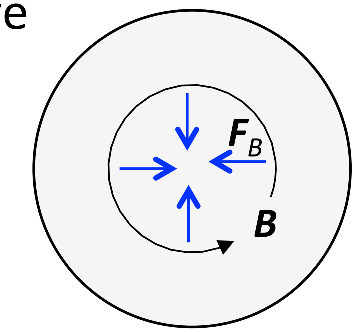
# Is DC current really uniform in a wire?

I'll justify the assumption that $J = \frac{I}{\pi R^2}$, so that (for $r < R$), $B = \frac{\mu_0 I}{2\pi R}\left(\frac{r}{R}\right) = \frac{\mu_o}{2} Jr$

Consider a copper wire ($R$ = $10^{-3}$ m) carrying 1 Ampere.
The free charge density (mobile electrons) is $\rho_f \sim 10^{10}$ C/m$^3$ (one per atom).

Because $B \neq 0$, there will be a Lorentz force on the moving electrons. This force must be balanced by a radial electrical field – a negative charge density near the center. That is, the electrons (and the current density) must move slightly radially inward. I will assume that the motion is small, so that $J$ remains uniform.




$J$ points out of the picture.
Electron motion is in.

The Lorentz force per unit volume is: $F_B = JB = \frac{\mu_0}{2} J^2 r$

Balance this with a radial electrical field: $E = \frac{\rho_b r}{2\varepsilon_0}$ assuming that $\rho_b$ is uniform

The electrical force per unit volume is: $F_E = \rho_b E = \frac{\rho_b^2 r}{2\varepsilon_0}$

$\rho_b$ is the "unbalanced" charge that results from an increase of $\rho_f$ near the center.

Remember: The copper atoms can't move.

Balance the forces: $\frac{\rho_b^2 r}{2\varepsilon_0} = \frac{\mu_0}{2} J^2 r \Rightarrow \rho_b^2 = \underbrace{\varepsilon_0 \mu_0}_{=\frac{1}{c^2}} J^2 \Rightarrow \rho_b = \frac{J}{c} = \rho_f \frac{v}{c}$

We need to know how fast the electrons are moving.

$$\rho_f v A = I \Rightarrow v = \frac{I}{\rho_f A} \sim 10^{-4} \frac{m}{s}$$

The electrons are moving very slowly, because $\rho_f$ is so large.

Thus, $\rho_b = \rho_f \frac{10^{-4}}{10^8} = 10^{-12} \rho_f$

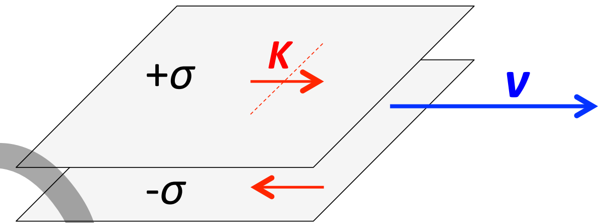
Only a tiny readjustment of the charge is needed. So, $J$ remains uniform.

Comment: The factor of 1/c will appear over and over when we compare electrical forces to magnetic forces.

## Example: (G, 5.16)

Another problem that shows how *c* appears. (and also surface current, *K*)

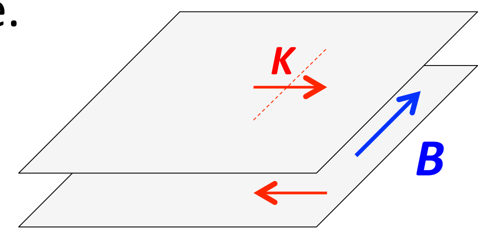
A parallel plate capacitor with uniform surface charge, $\pm\sigma$, is moving with speed, v, as shown.




a) What is the magnetic field inside and outside the gap?

The surface current, K, is $\sigma v$. Look at the dashed red line.
The units of *K* are (C/s)/m.

Draw an Amperian loop that spans one plate so that K passes through it.
From Ampere's law, Each plate produces $B = \mu_0 K/2 = \mu_0 \sigma v/2$, with the appropriate ± signs for above and below the plates (and the ± charges).
The fields add in the gap ($B = \mu_0 \sigma v$) and cancel outside.



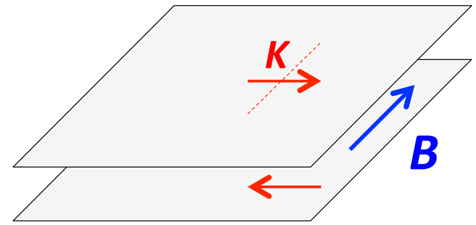

See G, example 5.8 for the solution to the single-plate problem.

b) What is the magnetic force on each plate?

The force on each plate is K times the field due to the other plate:

$$\vec{F}_B = \vec{K} \times \vec{B} = (\sigma v)(\mu_0 \sigma v / 2) = \tfrac{\mu_0}{2}(\sigma v)^2$$

The force points up on the top plate and down on the bottom plate (*i.e.*, they repel).




Note: $F_B$ is the force per unit area on the plates.

c) For what speed, *v*, would the magnetic force balance the electrical force?

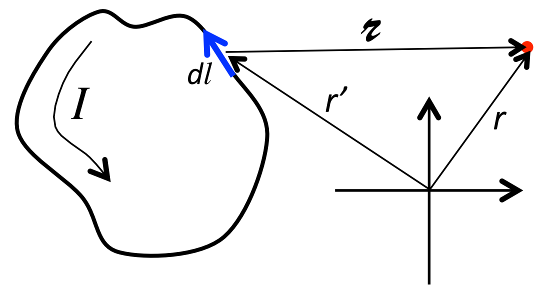
The electric force (per unit area) is $\sigma$ times the electric field, $\sigma/2\varepsilon_0$, due to the other plate: $F_E = \frac{1}{2\varepsilon_0}\sigma^2$. It is attractive.

Setting $F_B = F_E$ yields $v^2 = \frac{1}{\mu_0 \varepsilon_o} = c^2$.

The speed must equal the speed of light for the forces to balance.

# The Divergence of *B*

Remember Biot-Savart for a current loop:

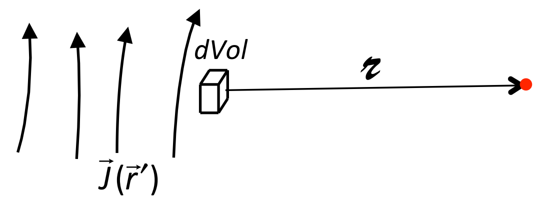



$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

When we have a current density, this must be rewritten:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{Vol'} \frac{\vec{J}(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} dVol'$$

The ′ reminds us that we're integrating over ***r*′**, the location of the current.




Let's calculate $\vec{\nabla} \cdot \vec{B}$:

REMEMBER: $\vec{\nabla}$ is the derivative with respect to the observation point (*r*), not the source location (*r*′).

$$\vec{\nabla} \cdot \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{Vol'} \vec{\nabla} \cdot \left( \frac{\vec{J}(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} \right) dVol'$$

We can take $\vec{\nabla}$ inside the integral.

Let's look at this.

$$\vec{\nabla}\cdot\left(\frac{\vec{J}(\vec{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)=\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\cdot\left(\vec{\nabla}\times\vec{J}(\vec{r}')\right)-\vec{J}(\vec{r}')\cdot\left(\vec{\nabla}\times\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)$$

= 0 because $\vec{\nabla}$ doesn't operate on $r'$.

= 0 because $\hat{\boldsymbol{\mathscr{r}}}$ is radial and isotropic - a "Coulomb" field.

Math identity #6 inside Griffiths' front cover

$$=0$$

So, **the divergence of *B* is always zero.** ← **No magnetic monopoles.**

The B field lines may not have ends (magnetic charges). They must form loops.

Note that in a current-free region:

$$\vec{\nabla}\cdot\vec{B}=0$$
$$\vec{\nabla}\times\vec{B}=0$$

Just like ***E*** in a charge-free region.

This means that we could define a scalar potential, $V_B$, such that $\vec{B}=-\vec{\nabla}V_B$ and $\nabla^2 V_B=0$. I won't pursue this approach.

End 10/16/13