Comment:

Consider this vector identity: $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{v}\right)=0$ (G, #9)

Let's apply it to Ampere's law: $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{B}\right)=\mu_0\vec{\nabla}\cdot\vec{J}=-\mu_0\frac{\partial\rho}{\partial t}$$

This implies that Ampere's law must be modified when we get away from the static situation (*i.e.*, when $\frac{\partial\rho}{\partial t}\neq 0$). We'll deal with this later.

Magnetic Vector Potential (5.4)

Here we are now (in the time independent situation):

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{(Gauss)} \qquad \vec{\nabla}\times\vec{E} = 0$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} \quad \text{(Ampere)}$$

This is a complete description of electro- and magneto-statics. (You can go home now.)

Of course, there are concepts that make life much simpler, and problem solving much easier. One of these is conservation of energy: $\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla}V$
The potential energy of a charge is $U = qV$.

The magnetic field does no work, so potential energy isn't useful.
In any case, because the curl is not zero in general, we can't usually define V_B.

Can we take advantage of $\vec{\nabla}\cdot\vec{B} = 0$?
It turns out that we can.

Our ability to define $\vec{E} = -\vec{\nabla}V$ follows from this vector identity: $\vec{\nabla}\times\left(\vec{\nabla}V\right) = 0$

Another vector identity: $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right) = 0$ tells us that if we define $\vec{B} \equiv \vec{\nabla}\times\vec{A}$
we are guaranteed to satisfy the divergence equation.

A is called the magnetic vector potential.

The definition of ***A*** is only useful if we have a way to calculate it from the current distributions, akin to our calculation of *V*:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}}dVol' \qquad\qquad \vec{A}(\vec{r}) = \ ?$$

The connection between ***A*** and ***J*** is given by Ampere's law:

$$\begin{aligned} \vec{\nabla}\times\vec{B} \qquad\qquad &= \mu_0\vec{J} \\ = \vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) &= \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) - \vec{\nabla}^2\vec{A} \end{aligned}$$

Vector identity #11

The boxed equation looks a bit messy. Fortunately, we can simplify it.

Remember our freedom of choice with the scalar potential. The fact that it is defined in terms of its derivative, $\vec{E} = -\vec{\nabla}V$, means that we can add any constant to *V* without affecting the physics. We can pick *V* = 0 anywhere we want.

There is a similar freedom of choice with ***A***.

$\boldsymbol{A}$ is defined by its derivative: $\vec{B} = \vec{\nabla} \times \vec{A}$

This means that we can add to $\boldsymbol{A}$ any vector field, $\boldsymbol{X}$, that has $\vec{\nabla} \times \vec{X} = 0$.

In particular, any $\vec{X} = \vec{\nabla}\lambda$ will work, because $\vec{\nabla} \times \left(\vec{\nabla}\lambda\right) = 0$.

Identity #10

That is, suppose we transform $\boldsymbol{A}$: $\vec{A}' = \vec{A} + \nabla\lambda$.

Then, $\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A}$, and we have the same $\boldsymbol{B}$.

This freedom is called "gauge invariance".
$\boldsymbol{A}$ to $\boldsymbol{A}'$ is called a "gauge transformation".

How can we use this freedom to simplify the problem?

We have: $\mu_0 \vec{J} = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \vec{\nabla}^2 \vec{A}$

We'd like to eliminate this term.

If we can find an $\boldsymbol{A}'$ such that $\vec{\nabla} \cdot \vec{A}' = 0$, then we'll have reduced the problem to Poisson's equation (and, in current-free regions, to Laplace's equation). We want to find a λ that accomplishes this. We want:

$$0 = \vec{\nabla} \cdot \vec{A}' = \vec{\nabla} \cdot \vec{A} + \underbrace{\vec{\nabla} \cdot \left(\vec{\nabla}\lambda\right)}_{= \nabla^2 \lambda} \Rightarrow \nabla^2 \lambda = -\vec{\nabla} \cdot \vec{A} \qquad \text{(Poisson's equation)}$$

This equation always has a solution, but I don't want to solve it.

I don't need to. Once we know that $\boldsymbol{A}'$ exists, we can solve for it directly:

$$\boxed{\mu_0 \vec{J} = -\vec{\nabla}^2 \vec{A}'}$$

Choosing $\vec{\nabla} \cdot \vec{A}' = 0$ is the equivalent of picking $V = 0$ where we want.

$\vec{\nabla}^2\vec{A}'=-\mu_0\vec{J}$ is three Poisson equations, one for each component.
Therefore, the solutions look just like the solutions for the scalar potential:

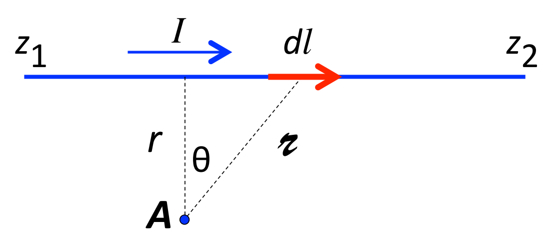
$$\nabla^2 V=-\frac{\rho}{\varepsilon_0} \Rightarrow V=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho}{\mathscr{r}}dVol$$

$$\nabla^2\vec{A}=-\mu_0\vec{J} \Rightarrow \vec{A}=\frac{\mu_o}{4\pi}\int\frac{\vec{J}}{\mathscr{r}}dVol$$

Remember: These solutions are only valid for localized charge and current distributions.

Example: The vector potential produced by a finite wire:

$$\vec{A}=\frac{\mu_0}{4\pi}\int\frac{I\hat{z}}{\mathscr{r}}dz=\frac{\mu_0 I}{4\pi}\hat{z}\int\frac{dz}{\left(z^2+r^2\right)^{1/2}}dz=\frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2+\left(z_2^2+r^2\right)^{1/2}}{z_1+\left(z_1^2+r^2\right)^{1/2}}\right]\hat{z}$$



Does this make sense? Calculate ***B***:

$$\vec{B}=\vec{\nabla}\times\vec{A}=-\frac{\partial A_z}{\partial r}\hat{\varphi}=\frac{\mu_o I}{4\pi r}\left[\frac{z_2}{\left(z_2^2+r^2\right)^{1/2}}-\frac{z_1}{\left(z_1^2+r^2\right)^{1/2}}\right]$$

($\sin\theta_2$, $\sin\theta_1$)

Comments:
- ***A*** is poorly behaved when z_1 or z_2 go to ∞.
- To get this, you must complete the square.

This is the answer we obtained the other day.

End 10/18/13