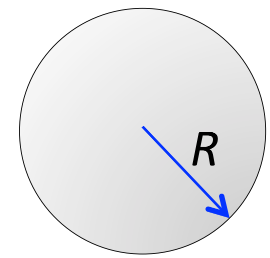
Example: (G, problem 6.18)

A permeable sphere is put in an otherwise uniform ***B*** field.

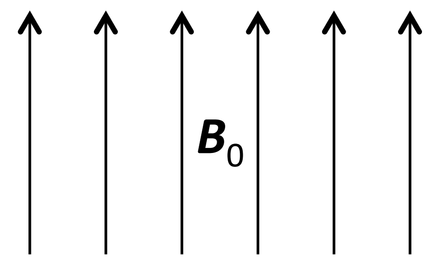


Use the fact that there are no free currents.
This means that $\vec{\nabla}\times\vec{H}=0$ everywhere. So, we can write ***H*** as the gradient of a potential: $\vec{H}=-\vec{\nabla}W$

Thus, *W* satisfies Poisson's equation:

$$\nabla^2 W=-\vec{\nabla}\cdot\vec{H}=+\vec{\nabla}\cdot\vec{M}$$

This follows from $\vec{B}=\mu_0\left(\vec{H}+\vec{M}\right)$
and: $\vec{\nabla}\cdot\vec{B}=0$



$\vec{\nabla}\cdot\vec{M}\neq 0$ only on the surface:

- Outside: ***M*** = 0.
- Inside: ***H*** = ***B/***μ. So $\vec{\nabla}\cdot\vec{M}=0$.

Thus, we have Laplace's equation for *W*,
with boundary conditions at the surface of the sphere.

Boundary conditions:

1) $B \rightarrow B_0 \hat{z}$ as $r \rightarrow \infty$

Uniform field, $\boldsymbol{B}_0$, at infinity.

$$\vec{H} \rightarrow \frac{B_0}{\mu_0}\hat{z} \quad \Rightarrow \quad W \rightarrow -\frac{B_0}{\mu_0} z$$

Just like the electrostatic case.

2) $W_{in}(R,\theta) = W_{out}(R,\theta)$

H is well behaved: *W* must be continuous.

3) $$-\mu \frac{\partial W_{in}}{\partial r}\bigg|_R = -\mu_0 \frac{\partial W_{out}}{\partial r}\bigg|_R$$

$$B_{\perp in} = B_{\perp out}$$

As before, expand W_{in} and W_{out} as sums of Legendre polynomials.

$$W_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

$$W_{out}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^{-(\ell+1)} P_\ell(\cos\theta) - \frac{B_0}{\mu_0} \underbrace{r\cos\theta}_{= z}$$

Apply condition 3: $-\mu\sum_{\ell=0}^{\infty}\ell A_\ell R^{\ell-1}P_\ell(\cos\theta)=B_0\cos\theta+\mu_0\sum_{\ell=0}^{\infty}(\ell+1)B_\ell R^{-(\ell+2)}P_\ell(\cos\theta)$

Apply condition 2: $\ell\neq 1:\quad A_\ell=B_\ell R^{-(2\ell+1)}$

$$\ell=1:\quad A_1=-\tfrac{B_0}{\mu_0}+B_1R^{-3}$$

Combine:
(eliminate B_ℓ)

$$\ell\neq 1:\quad -\mu\ell A_\ell R^{\ell-1}=\mu_0(\ell+1)A_\ell R^{-(\ell+2)}\quad\Rightarrow\quad A_\ell=0$$

$$\ell=1:\quad -\mu A_1=B_0+\mu_0 2A_1R^{-3}\quad\Rightarrow\quad A_1=\tfrac{-3B_0}{2\mu_0+\mu}$$

$$B_1=\left(\tfrac{\mu-\mu_0}{\mu_0(2\mu_0+\mu)}\right)R^3B_0$$

So:

$$W_{\text{in}}(r,\theta)=-\tfrac{3B_0}{2\mu_0+\mu}r\cos\theta$$

$$\vec{H}_{\text{in}}=-\vec{\nabla}W_{\text{in}}=\tfrac{3B_0}{2\mu_0+\mu}\hat{z}$$

$$\vec{B}_{\text{in}}=\mu\vec{H}_{\text{in}}=\tfrac{3\mu}{2\mu_0+\mu}\vec{B}_0=\tfrac{3(1+\chi_m)}{3+\chi_m}\vec{B}_0$$

Note: As χ_m goes to 0, $\boldsymbol{B}_{\text{in}}$ approaches $\boldsymbol{B}_0$.
B_1 is the field produced by the induced polarization of the sphere ($\propto R^3$).

Magnetic Shielding

It is often desirable to create a field-free region (***B*** as small as possible):

- To measure small magnetic effects.
- Some apparatus does not work well in magnetic fields.

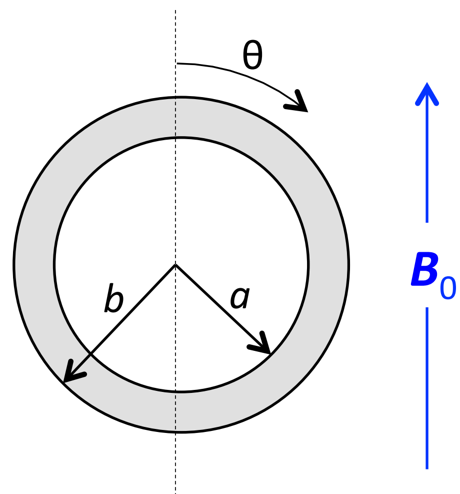
It is easy to create an electric field-free region, because the interior of any conductor (including any empty regions) works.

It turns out to be easy for ***B*** as well: Make a box out of large μ ($\gg \mu_0$) material. Iron works in many situations.

Consider this situation:

A hollow sphere of permeable material.
We want to calculate ***B*** for $r < a$.

As before, use the potential, W.



There are three regions:

$$r > b: \quad W = -\frac{B_0}{\mu_0} r\cos\theta + \sum_{\ell=0}^{\infty} \alpha_\ell r^{-(\ell+1)} P_\ell(\cos\theta)$$

$$a < r < b: \quad W = \sum_{\ell=0}^{\infty} (\beta_\ell r^\ell + \gamma_\ell r^{-(\ell+1)}) P_\ell(\cos\theta)$$

$$r < a: \quad W = \sum_{\ell=0}^{\infty} \delta_\ell r^\ell P_\ell(\cos\theta)$$

Both r dependences are OK here.

We have to apply the BC's:

- $W_{\text{in}} = W_{\text{out}}$
- $\mu_{\text{in}} \frac{\partial W_{\text{in}}}{\partial r} = \mu_{\text{out}} \frac{\partial W_{\text{out}}}{\partial r}$

at both surfaces ($r = a$ and $r = b$). This gives four equations in four unknowns (α_ℓ, β_ℓ, γ_ℓ, δ_ℓ) for each ℓ.

I use Greek letters to avoid confusion with *a*, *b*, *etc*.

As with the solid sphere, only $\ell = 1$ gives non-zero solutions. The equations are:

$$\begin{aligned}
\alpha_1 \quad -b^3\beta_1 \quad -\gamma_1 \quad\quad &= b^3 \tfrac{B_0}{\mu_0} \\
2\mu_0\alpha_1 \quad +\mu b^3\beta_1 \quad -2\mu\gamma_1 \quad\quad &= -b^3 B_0 \\
a^3\beta_1 \quad +\gamma_1 \quad -a^3\delta_1 &= 0 \\
\mu a^3\beta_1 \quad -2\mu\gamma_1 \quad -\mu_0 a^3\delta_1 &= 0
\end{aligned}$$

This is easily solved. (use Mathematica)

For a highly permeable ($\mu >> \mu_0$) material:

$$\alpha_1 = b^3 \frac{B_0}{\mu_0}$$

The induced dipole moment of the sphere.

$$B_{\text{in}} = -\mu_0\delta_1 = \frac{3}{2}\left(\frac{\mu_o}{\mu}\right)\left(\frac{1}{1-\left(\frac{a}{b}\right)^3}\right) B_0$$

The field for $r < a$.

The field looks like this: