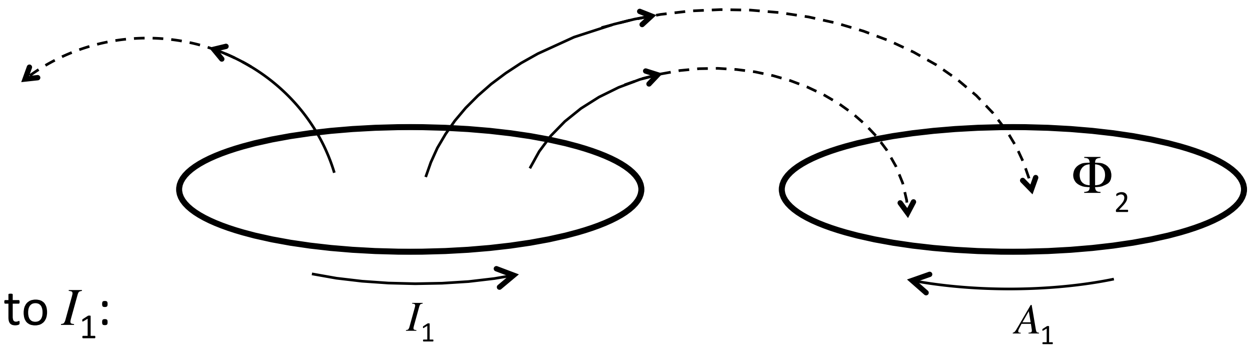
Mutual and Self inductance (7.2.3)

If we have two loops.
Current in one will
produce flux in the other:



Φ_2 is clearly proportional to I_1:

$\Phi_2 = M_{21} I_1$. M_{21} is called the mutual inductance and, like *L*, depends only on the geometry and the possible presence of magnetic materials.

I claim that $M_{21} = M_{12}$.

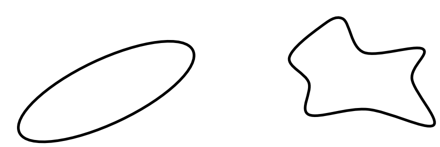
Proof: $\Phi_2 = \int_{\text{surface 2}} \vec{B}_1 \cdot \vec{a}_2 = \int_{\text{surface 2}} (\vec{\nabla} \times \vec{A}_1) \cdot d\vec{a}_2 = \oint_{\text{loop 2}} \vec{A}_1 \cdot d\vec{\ell}_2$

Remember: $\vec{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint_{\text{loop 1}} \frac{d\vec{l}_1}{\mathscr{r}}$

So, $M_{21} = \frac{\Phi_2}{I_1} = \frac{\mu_0}{4\pi} \oint_{\text{loop 1}} \oint_{\text{loop 2}} \frac{d\vec{\ell}_1 \cdot d\vec{\ell}_2}{\mathscr{r}}$

$= M_{12}$ because the integral is symmetric

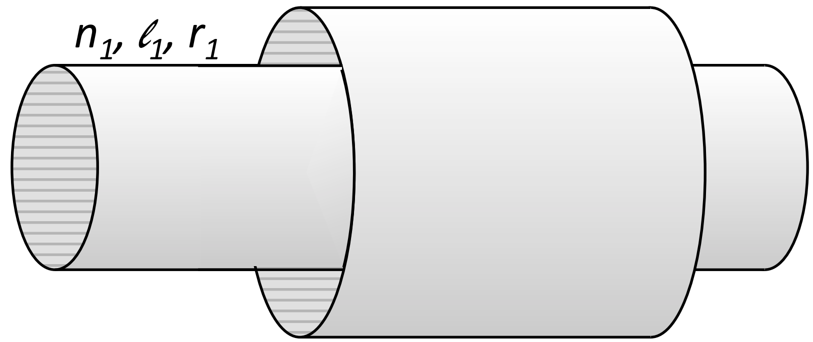
No matter how asymmetrical the geometry is, the effect of current in loop 1 on loop 2 is the same as the effect of loop 2 on loop 1.



Example:

Two solenoids, one inside the other.

r_1 and r_2 are much smaller than ℓ_1 and ℓ_2.



For I_1: $B_1 = \mu_0 n_1 I_1 \quad \Rightarrow \quad \Phi_2 = \underbrace{(\mu_0 n_1 I_1)}_{B_1}\underbrace{(\pi r_1^2)}_{A_1}\underbrace{(n_2 \ell_2)}_{N_2}$

For I_2: $B_2 = \mu_0 n_2 I_2 \quad \Rightarrow \quad \Phi_1 = \underbrace{(\mu_0 n_2 I_2)}_{B_2}\underbrace{(\pi r_1^2)}_{A_1}\underbrace{(n_1 \ell_2)}_{??}$

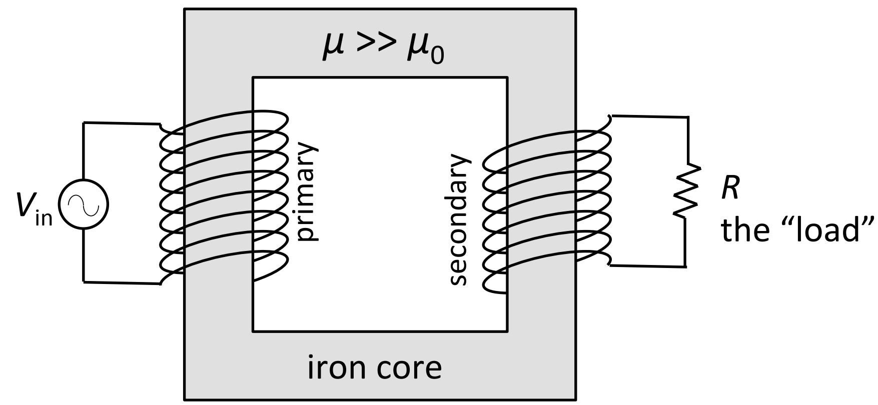
There are two oddities here:

- Why do we use the cross section, A_1, of solenoid #1 in the Φ_1 expression?
  The reason is that the ***B*** field in solenoid 2 at $r > r_1$, does not contribute to Φ_1.
- Where did that odd looking term, $n_1\ell_2$, come from?
  The ***B*** field produced by solenoid 2 only extends a length ℓ_2, not the entire ℓ_1.

So, $M = (\mu_0 n_2)(\pi r_1^2)(n_1 \ell_2)$

Example: Transformer (G, problem 7.54)

Transformers are used to increase or decrease AC voltages. Their usefulness is the reason we use AC power distribution (advocated by Tesla) rather than DC (advocated by Edison).



The iron core servers two purposes:

- B is much larger for is given I – large inductance.
- The field is trapped in the iron: $\frac{\Phi_p}{N_p} = \frac{\Phi_s}{N_s}$ ← Number of turns in the primary and secondary

Let's calculate the inductances:

Suppose that $\Phi_p = L_p I_p$ (L_p is known).

Then, $\Phi_s = \frac{N_s}{N_p}\Phi_p = \frac{N_s}{N_p}L_p I_p = MI_p$

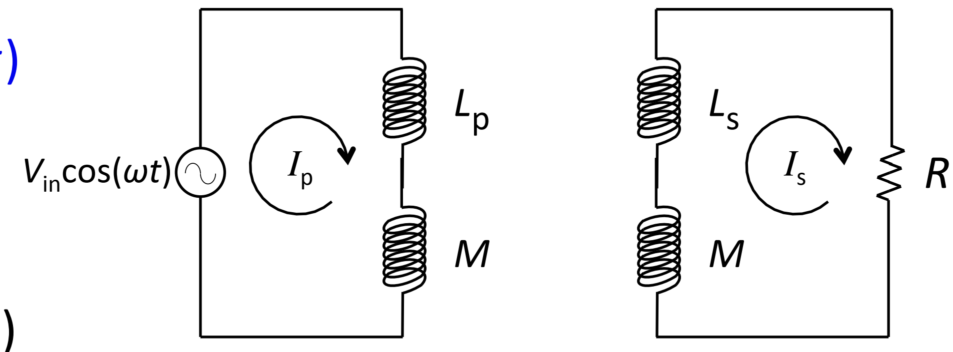
Similarly, $\Phi_p = \frac{N_p}{N_s}L_s I_s = MI_s$

$\Rightarrow$

$$L_s = \left(\frac{N_s}{N_p}\right)^2 L_p$$

$$M^2 = \left(\frac{N_s}{N_p}\right)^2 L_p^2 = L_p L_s$$

Now, consider what happens when we drive the transformer: $V_{\text{in}}(t) = V_{\text{in}}\cos(\omega t)$



We have two circuits:

So:

$$L_p \frac{dI_p}{dt} + M\frac{dI_s}{dt} = V_{\text{in}}\cos(\omega t)$$

$$L_s \frac{dI_s}{dt} + M\frac{dI_p}{dt} = -I_s R$$

$$L_p L_s \frac{dI_p}{dt} + M L_s \frac{dI_s}{dt} = V_{\text{in}} L_s \cos(\omega t)$$

$$M^2 \cancel{\frac{dI_p}{dt}} - M^2 \cancel{\frac{dI_p}{dt}} - M I_s R = V_{\text{in}} L_s \cos(\omega t)$$

$$I_s(t) = -\frac{L_s}{MR} V_{\text{in}}\cos(\omega t)$$ ← No phase shift!

$$\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right| = \frac{|I_s R|}{V_{\text{in}}} = \frac{L_s}{M} = \frac{N_s}{N_p}$$ ⇐ The voltage ratio.

End 11/18/13