# Maxwell's Equations, Completed (7.3)

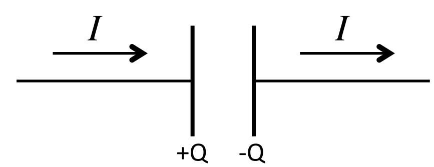
The equations we have so far are inconsistent.
In particular, there is a problem with Ampere's Law (see Oct. 18 lecture):

Apply this vector identity: $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v})=0$ to Ampere's law: $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$

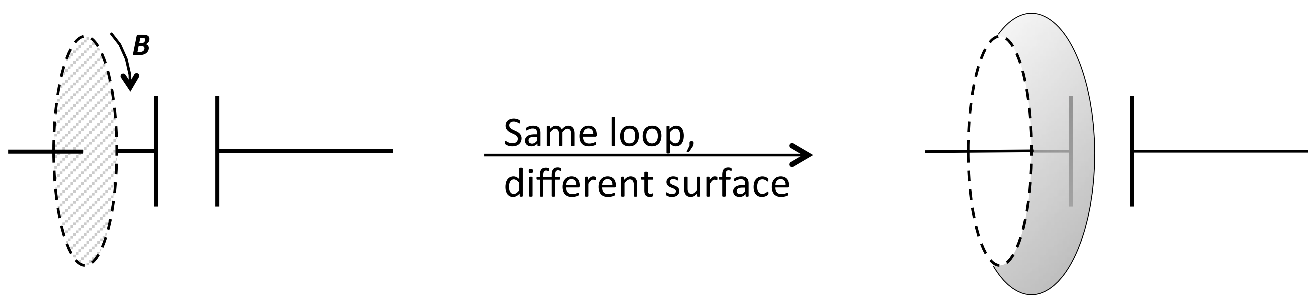
$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{B}\right)=\mu_0\vec{\nabla}\cdot\vec{J}=-\mu_0\frac{\partial\rho}{\partial t}$$

Ampere's law must be modified when we get away from the static situation.

Recall from P212 a capacitor that is being charged up:




Draw an Amperean loop around the wire:




$$\oint_{\text{loop}}\vec{B}\cdot d\vec{\ell}=\mu_0 I$$

↑ Current flux through the surface

$$\oint_{\text{loop}}\vec{B}\cdot d\vec{\ell}=0?$$

↑ Current flux through the surface

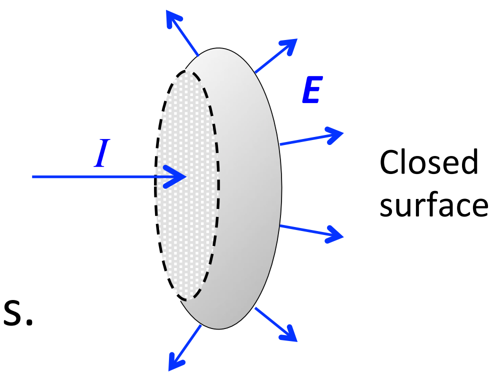
## Can we fix the problem? Or, does Ampere's Law fail?

The continuity equation and Gauss's law save us:

Continuity: $\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \quad \Rightarrow \quad \oint_{\text{surface}} \vec{J}\cdot d\vec{a} = -\int_{\text{volume}} \frac{\partial\rho}{\partial t} = -\frac{dQ_{\text{in}}}{dt}$

Gauss: $Q_{\text{in}} = \varepsilon_0 \int_{\text{surface}} \vec{E}\cdot d\vec{a} \quad \Rightarrow \quad \frac{dQ_{\text{in}}}{dt} = \varepsilon_0 \int_{\text{surface}} \frac{\partial\vec{E}}{\partial t}\cdot d\vec{a}$

Therefore: $\oint_{\text{surface}} \vec{J}\cdot d\vec{a} = -\varepsilon_0 \oint_{\text{surface}} \frac{\partial\vec{E}}{\partial t}\cdot d\vec{a}$




In our capacitor example: Current flows in, and
outward electric flux increases.

There is no reason the current flow and the electric flux need to be spatially separated. In general: $\oint_{\text{surface}} (\vec{J} + \varepsilon_0 \frac{\partial\vec{E}}{\partial t})\cdot d\vec{a} = 0$

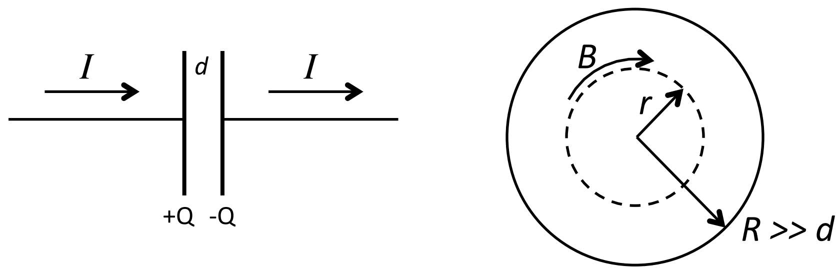
**Therefore, this modification of Ampere's Law works:**

$$\oint_{\text{loop}} \vec{B}\cdot d\vec{\ell} = \mu_0 \int_{\text{surface}} (\vec{J} + \varepsilon_0 \tfrac{\partial\vec{E}}{\partial t})\cdot d\vec{a} \quad \Rightarrow \quad \vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \varepsilon_0 \tfrac{\partial\vec{E}}{\partial t}\right)$$

A changing electric field produces a magnetic field.

Displacement current

## Example: Magnetic field in a capacitor.




***E*** is uniform in the gap: $E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\pi R^2 \varepsilon_o}$. $\frac{dE}{dt} = \frac{I}{\pi R^2 \varepsilon_0}$

There is also a magnetic field in the $\phi$ direction:

$$2\pi r B = \varepsilon_0 \mu_0 (\pi r^2) \frac{dE}{dt} \quad \Rightarrow \quad B = \frac{\mu_0 I}{2\pi} \frac{r}{R^2}$$

The displacement current fills the gap, not just along a line. So, to enclose the entire displacement current, we need $r = R$.

At $r = R$: $B = \frac{\mu_0 I}{2\pi R}$ . This is the same as around a wire, as expected.

## Maxwell's equations:

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\times\vec{B}=+\mu_0\left(\vec{J}+\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

Beware the ± signs.

- A changing electric field produces a magnetic field
- $\vec{\nabla}\times\vec{E}$ has no "current" term, and $\vec{\nabla}\cdot\vec{B}$ has no "charge" term, because there are no magnetic monopoles.

End 11/22/13