# Conducting fluids (e.g., plasmas)

This is important in stars, the Earth's core, and plasma fusion.

We will prove Alfven's theorem: (see G, problem 7.59)

> In a perfectly conducting fluid ($\sigma \to \infty$), $\frac{d\Phi}{dt} = 0$ through any loop that is moving with the fluid.

That is, the flux is "frozen in" - the field is carried along with the fluid.

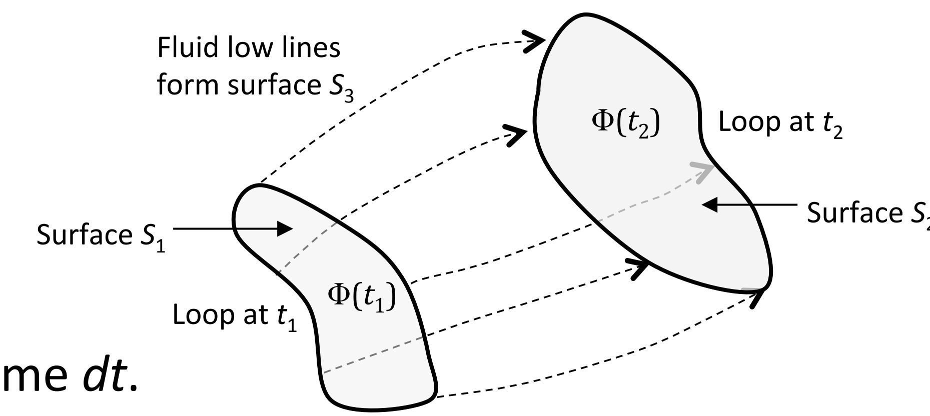
Proof:

Start with $\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B})$ .

To avoid ***J*** = ∞, we need *E* = -***v***×***B***.

Thus, $-\vec{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B})$ .

We'll use this later.




The fluid moves from surface $S_1$ to $S_2$ in time *dt*.

We want to calculate:

$$\Delta\Phi = \int_{S_2} \vec{B}(t + dt) \cdot d\vec{a} - \int_{S_1} \vec{B}(t) \cdot d\vec{a}$$

This is not very convenient, because the times are different.
Fix that by considering infinitesimal *dt*.

$$\int_{S_2} \vec{B}(t+dt)\cdot d\vec{a} = \int_{S_2} \vec{B}(t)\cdot d\vec{a} + dt\int_{S_2} \frac{\partial \vec{B}(t)}{\partial t}\cdot d\vec{a}$$

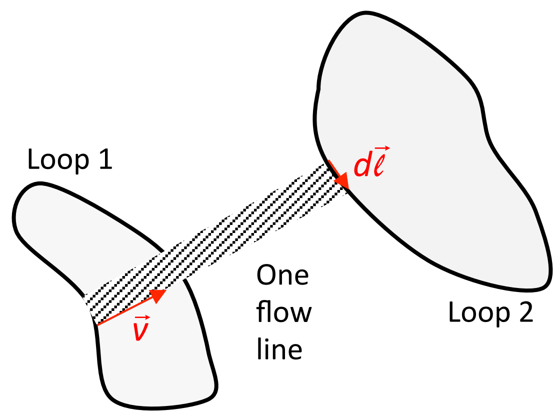
So, $\Delta\Phi = \int_{S_2} \vec{B}(t)\cdot d\vec{a} + dt\int_{S_2} \frac{\partial \vec{B}(t)}{\partial t}\cdot d\vec{a} - \int_{S_1} \vec{B}(t)\cdot d\vec{a}$

Now use Gauss's law at time *t*. $-\int_{S_1} \vec{B}(t)\cdot d\vec{a} + \int_{S_2} \vec{B}(t)\cdot d\vec{a} + \int_{S_3} \vec{B}(t)\cdot d\vec{a} = 0$

The minus sign on the $S_1$ integral results from the fact that its normal (defined in the $\Phi$ calculation) points in.

So, $\Delta\Phi = dt\int_{S_2} \frac{\partial \vec{B}(t)}{\partial t}\cdot d\vec{a} - \int_{S_3} \vec{B}(t)\cdot d\vec{a}$ .

Now look at the figure, which I've redrawn with $\boldsymbol{v}$ and $d\boldsymbol{\ell}$ added. We can see that in the $S_3$ integral, $d\vec{a} = dt(d\vec{\ell}\times\vec{v})$.




I'm going to turn the $S_3$ surface integral into a Loop 2 line integral.

This works, because:

- For infinitesimal *dt*, the flow lines are straight.
- ***v****dt* goes from loop 1 to loop 2.

$$\int_{S_3} \vec{B}\cdot d\vec{a} = dt \oint_{loop2} \vec{B}\cdot(d\vec{\ell}\times\vec{v}) = dt \oint_{loop2} d\vec{\ell}\cdot(\vec{v}\times\vec{B}) = dt\int_{S_2}(\vec{\nabla}\times(\vec{v}\times\vec{B}))\cdot d\vec{a}$$

We can evaluate ***B*** at Loop 2 rather than on surface 3, because the difference is infinitesimal, and it's already multiplied by an infinitesimal, *dt*.

Finally, we have: $\frac{d\Phi}{dt} = \int_{S_2}\left[\frac{\partial\vec{B}}{\partial t} - \vec{\nabla}\times(\vec{v}\times\vec{B})\right]\cdot d\vec{a} = 0$

This result is an example of a convective derivative.
We must be careful to distinguish between partial and total derivatives.
For example, consider a function, *F*(*x*,*t*).

$\frac{\partial F}{\partial t}$ describes the rate of change of *F* at a fixed position.
What does a moving observer (e.g., one moving with a fluid) see?
For him, the total time derivative is: $\frac{dF}{dt} = \frac{\partial F}{\partial t} + (\vec{v}\cdot\vec{\nabla})F$

It depends on both the time and space dependence.

# Time evolution of ***B*** when *σ* is finite: (not in the book)

Suppose we have a conducting medium, ***J*** = *σ**E***, (metal or plasma) with a *B* field in it. How does ***B*** evolve?

Use Maxwell's equations:

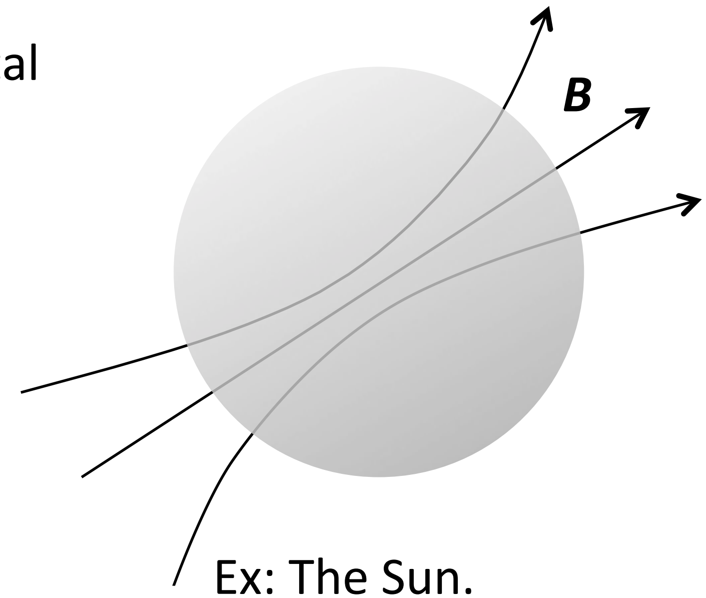
$$-\vec{\nabla}\times\vec{E}=\frac{\partial\vec{B}}{\partial t}=-\frac{1}{\sigma}\left(\vec{\nabla}\times\vec{J}\right) \Rightarrow \vec{\nabla}\times\vec{J}=-\sigma\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B}=\mu_0\left(\vec{J}+\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)=\mu_0\left(\vec{J}+\frac{\varepsilon_0}{\sigma}\frac{\partial\vec{J}}{\partial t}\right)$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right)=-\nabla^2\vec{B}=\mu_0\left(\vec{\nabla}\times\vec{J}\right)+\frac{\mu_0\varepsilon_0}{\sigma}\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{J}\right)$$

*σ* is large, so ignore this term.

So, $$\frac{1}{\mu_0\sigma}\nabla^2\vec{B}=\frac{\partial\vec{B}}{\partial t}$$




Ex: The Sun.
Solenoidal currents, due to the Sun's rotation, will tend to produce "tubes" of magnetic flux.

This is called the diffusion equation. Second space derivative, first time derivative.

The diffusion constant is: $D\equiv\frac{1}{\mu_0\sigma}$.

End 12/6/13