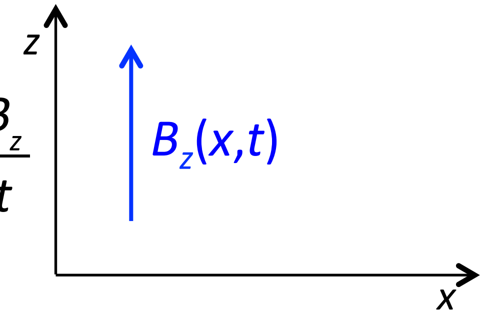
Let's solve a one dimensional example:

$$D\frac{\partial^2 B_z}{\partial x^2} = \frac{\partial B_z}{\partial t}$$



Solve it by separation of variables:
Remember, D is positive.

Let $B_z(x,t) = X(x)T(t)$. $\frac{1}{X}\frac{d^2X}{dx^2} = K = \frac{1}{DT}\frac{dT}{dt}$

The solution is: $X(x) = X_0 e^{\pm\sqrt{K}x}$

$T(t) = T_0 e^{KDt}$

The separation constant, K, must be negative to avoid exponential blow-up. Thus, solutions with a simple exponential time dependence have an oscillatory spatial dependence.

How do we deal with other spatial dependence? Use Fourier decomposition.

We could stick with sin() and cos(), but, since we are already writing the oscillatory behavior as exponentials ($e^{i\theta} = \cos\theta + i\sin\theta$), it's simpler to write the Fourier decomposition that way as well. Given a function, $F(x)$:

The Fourier decomposition is: $\tilde{F}(k) = \int_{-\infty}^{\infty} F(x) e^{-ikx}\, dx$ k is the wave number, $2\pi/\lambda$.

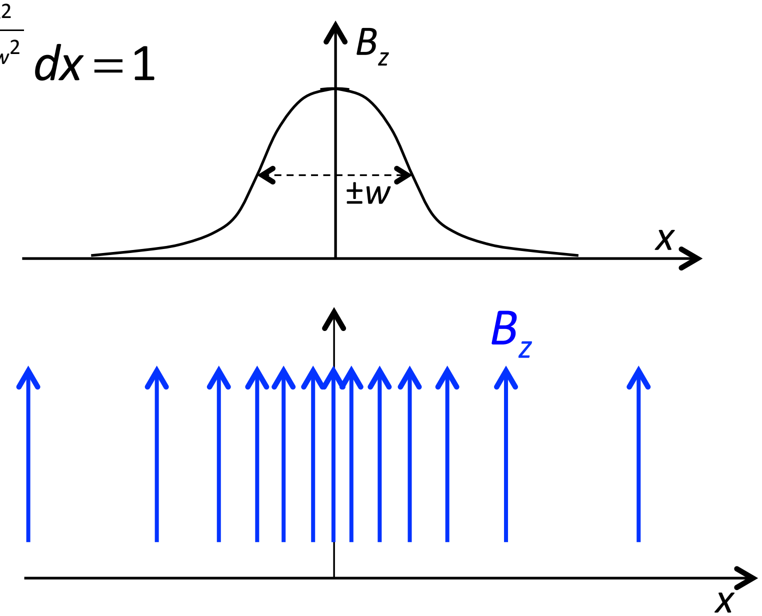
$F(x)$ can be reconstituted: $\tilde{F}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{F}(k) e^{ikx}\, dk$

Example:

Suppose that at $t = 0$, $B_z(x,0) = \frac{\Phi}{w\sqrt{2\pi}} e^{-\frac{x^2}{2w^2}}$

Φ is the total magnetic flux, because $\frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2w^2}} dx = 1$

The magnetic field has a Gaussian shape:



w is the width (more precisely, the second moment). We'll see that as time progresses the field will spread out: *w* will increase..

The procedure:

Fourier decompose $B_z(x,0)$. Each Fourier component has a well defined exponential time dependence. We can determine $B_z(x,t)$ by putting the time evolved components back together at that time.

This is a standard technique for evolving from complex initial conditions.

In our case, at t = 0:

$$\tilde{B}_z(k,0) = \frac{\Phi}{w\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2w^2}} e^{-ikx}\, dx = e^{-\frac{k^2 w^2}{2}} \Phi$$

$-k^2 = K$, the separation constant

This was evaluated by completing the square in the exponent and knowing that

$$\frac{1}{w\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2w^2}(x+ikw)^2}\, dx = 1$$

You need Math 446 or 448 (complex variables) for this.

Each Fourier mode has time dependence $e^{-k^2 Dt}$, so:

$$\tilde{B}(k,t) = \Phi e^{-\frac{k^2 w^2}{2}} e^{-k^2 Dt} = \Phi e^{-\frac{k^2}{2}(w^2+2Dt)} \equiv \Phi e^{-\frac{k^2 w'^2}{2}} \quad \text{where } w'^2 \equiv w^2 + 2Dt$$

This is exactly the same functional form as at t = 0, with a time dependent w'.

Thus: $B_z(x,t) = \frac{\Phi}{w'\sqrt{2\pi}} e^{-\frac{x^2}{2w'^2}}$

The field retains its Gaussian shape, but spreads out: $w' \equiv \sqrt{w^2 + 2Dt}$

For large t ($2Dt \gg w^2$): $w' \approx \sqrt{2Dt}$ The width is proportional to √t.

This is typical diffusion behavior.

Numerical estimates:

The dimensions of K are $1/m^2$.

D are m^2/s.

In a typical metal (copper) $\sigma = 6\times10^7$ $Ohm^{-1}\ m^{-1}$.

$D = 0.013\ m^2/s$.

The $\sqrt{t}$ time dependence means that the spreading is initially rapid, but then slows down dramatically.

For 1 ns: $\sqrt{Dt} \sim 3.6\ \mu m$

For 1 sec: 0.13 m

For 1 year: 100 m

For 10^{10} year: 60,000 km This is less than the size of the Sun.

The Sun's diffusion constant (near the surface) is actually quite a bit larger, $D_{Sun} \sim 200\ km^2/s$, so the time to diffuse 10^6 km is only ~ 10^{10} sec ~ 10^3 years.

The Sun and the Earth both have dynamos that maintain the *B* fields.