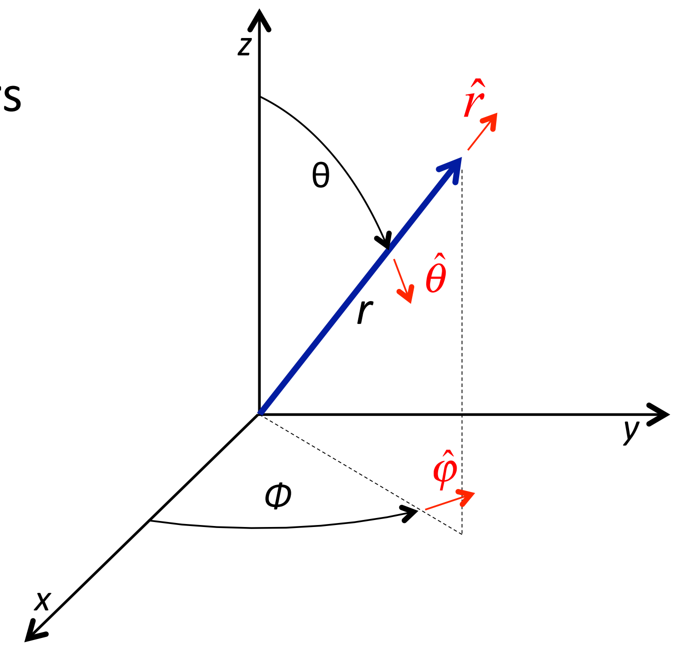
Spherical Coordinates

Sometimes, problems are simpler to understand and solve if we use other coordinate systems.

Spherical coordinates are most useful when there is spherical symmetry (*i.e.*, the problem looks the same in any orientation).

The Cartesian components of the spherical unit vectors are:

$$\hat{r} = (\sin\theta\cos\varphi,\ \ \sin\theta\sin\varphi,\ \ \cos\theta)$$
$$\hat{\theta} = (\cos\theta\cos\varphi,\ \ \cos\theta\sin\varphi,\ -\sin\theta)$$
$$\hat{\varphi} = (-\sin\varphi,\ \ \cos\varphi,\ \ 0)$$



Each tells us the (x,y,z) motion when only that particular component changes. One important feature of spherical (and other curvilinear) coordinates is that the directions of the unit vectors are not constant, so that taking derivatives in those coordinate systems is more complicated.

Divergence in Spherical Coordinates

Contrary to intuition, this expression is not correct:

$$\text{No!}\quad \vec{\nabla}\cdot\vec{E} = \frac{\partial E_r}{\partial r} + \frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_\varphi}{\partial \varphi} \quad \text{No!}$$

It fails, because $\hat{r}$, $\hat{\theta}$, and $\hat{\varphi}$ are functions of position.

Instead, we have:

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta E_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial E_\varphi}{\partial\varphi}$$

This form is useful in problems with spherical symmetry.

Let's calculate the divergence of the Coulomb field, $\vec{E} = a\dfrac{\hat{r}}{r^2}$. Simple:

Beware of this word. It's usually a trap.

$E_\theta = 0$. $E_\Phi = 0$. $E_r = a/r^2$, so $\partial\left(r^2 E_r\right)/\partial r = 0$.

Therefore, $\vec{\nabla}\cdot\vec{E} = 0$ everywhere !! ??

We know this is wrong, because there is charge at the origin. The problem is that ***E*** is not well behaved at the origin – its derivatives are not defined at the singularity.

How can we deal with this?

Singularities and the Dirac δ-function

See Griffiths, 1.5

We have a point charge at the origin. ρ is zero everywhere except there, where it is infinite. That's the singularity.

We know how to deal with it, because the total charge is *Q*. That means that if we integrate $\int_{Volume} \rho\, dV$ over any volume that contains the origin, we obtain *Q*. If the volume does not contain the origin, we obtain 0.

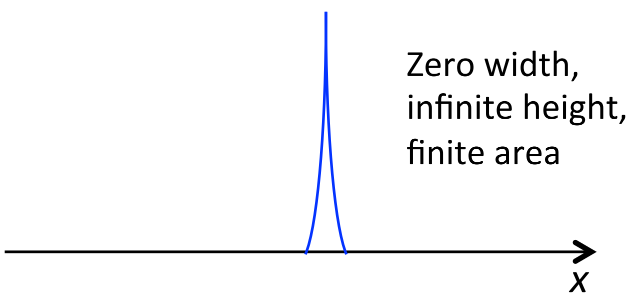
> A function, in this case, $\rho(\boldsymbol{r})$, that is zero everywhere except at one point, but that has a finite (non-zero) integral, is called a Dirac δ-function.

Comments:

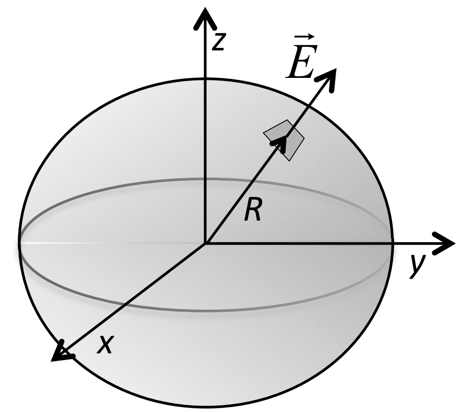
- δ-functions are defined in 1, 2, and 3 dimensions. You'll need them whenever there are 0-D (point), 1-D (line), or 2-D (sheet) singularities.
- They are also important in the development of Green function solutions to differential equations.

Graphically, a δ-function is a "needle":

Disclaimer: Mathematically inclined students will complain that I'm being sloppy. That's true …



An often-confusing point:

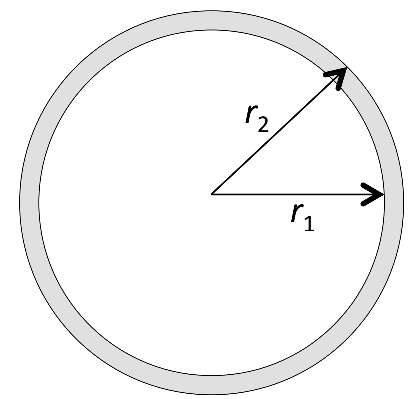


Recall this example:
E due to a uniform spherical surface charge, density = σ.

Let's calculate the pressure on the surface. Due to the repulsive forces, there is an outward pressure.

The force per unit area on a charged surface is ***F*** = σ***E***.
However, $E = \frac{Q}{4\pi\varepsilon_o r^2}$ outside, and $E = 0$ inside. What value of *E* should we use?

The way to deal with this is to eliminate the discontinuity by considering a very thin shell, analyze the problem, and then take the limit as the thickness goes to zero. If we get a plausible result, then we're good; otherwise, we're in trouble.



So, consider a spherical shell between r_1 and r_2, with an arbitrary radial (but spherically symmetric) charge density, $\rho(r)$.

The force on an infinitesimal shell, *dr*, at radius *r* ($r_1 < r < r_2$) is: $dF = E(r) \cdot \left(4\pi r_1^2\right) \rho\left(r\right) dr$

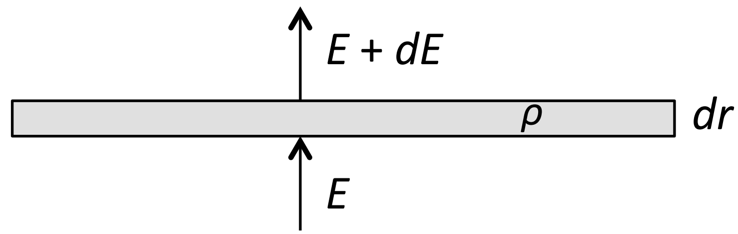
I write r_1 (*i.e.*, a constant), because, for a thin shell, the surface area does not change much between r_1 and r_2.

We have: $dF = E(r)\cdot\left(4\pi r_1^2\right)\rho(r)\,dr$.

To integrate this, (to find the total force) we first need to relate $E(r)$ and ρ(r).

Remember Gauss's law. It tells us that as one passes through a sheet of charge, the electric field changes:

$$dE = \frac{\sigma}{\varepsilon_0} = \frac{\rho\,dr}{\varepsilon_0}$$



If $dr \ll r_1$, we can treat the sheet as a flat surface.

Thus, $dF = E\left(4\pi r_1^2\right)\varepsilon_0\,dE$

Now we can do the integral:

$$F = 4\pi\varepsilon_0 r_1^2 \int_{E(r_1)}^{E(r_2)} E\,dE = \left[4\pi\varepsilon_0 r_1^2\right]\left[\tfrac{1}{2}\left(E^2(r_2) - E^2(r_1)\right)\right]$$

$$= 4\pi\varepsilon_0 r_1^2\left(E(r_2) - E(r_1)\right)\tfrac{E(r_2)+E(r_1)}{2}$$

$$= 4\pi r_1^2\sigma\,\tfrac{E(r_2)+E(r_1)}{2} = Q_{tot}\,\tfrac{E(r_2)+E(r_1)}{2}$$

$$E(r_2) - E(r_1) = \frac{\sigma}{\varepsilon_0}$$

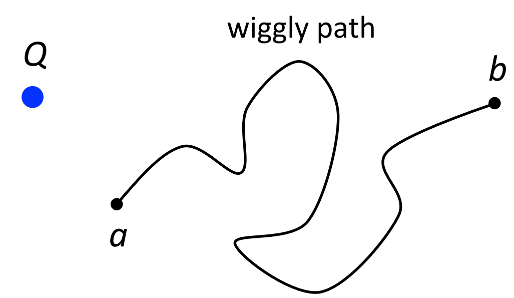
So, the right answer is to use the average of the fields on the inside and outside. This is a reliable result as the shell approaches zero thickness, because it does not depend on the thickness (as long as the shell is reasonably thin).

The Electric Potential: Curl of *E*

Consider our trusty point charge: $\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^2}$

Let's calculate the work done on another charge as it moves from *a* to *b*.

We need to integrate: $W = \frac{qQ}{4\pi\varepsilon_0}\int_a^b \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot d\vec{l}$



This integral is most easily done in spherical coordinates.
It simplifies from 3-D to 1-D, because $\hat{\mathscr{r}} = (1,0,0)$.
Therefore we only need $dl_{\mathscr{r}} = d\mathscr{r}$.

For completeness:
dl_ϑ = r dθ
dl_Φ = r sinθ dϕ

Thus, $W = \frac{qQ}{4\pi\varepsilon_0}\int_{\mathscr{r}_a}^{\mathscr{r}_b}\frac{d\mathscr{r}}{\mathscr{r}^2} = \frac{qQ}{\varepsilon_0}\left(\frac{1}{\mathscr{r}_a} - \frac{1}{\mathscr{r}_b}\right).$

This is nice: $V = \frac{Q}{4\pi\varepsilon_0 r}$. But that's not my main point.

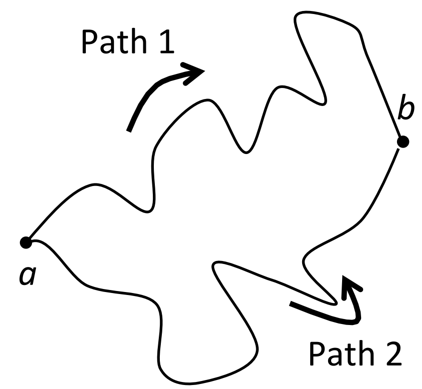
The important point is that W does not depend on the path taken!

On to the curl!

- Superposition tells us:
  path independence holds for any distribution of charge.

- Path independence tells us:
  the loop integral of ***E*** is zero for any loop: $\oint_{loop} \vec{E} \cdot d\vec{l} = 0$

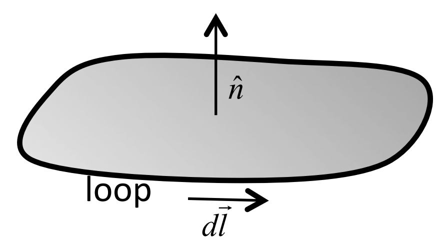
  Consider two paths between *a* and *b*. The two path integrals are equal. Thus, the loop integral is zero, because one path will be traversed backwards.



- Now, put in a little math: Stokes' theorem (discovered by Kelvin).

$$\oint_{loop} \vec{E} \cdot d\vec{l} = \int_{surface} \left(\vec{\nabla} \times \vec{E}\right) \cdot \vec{n}\, dA$$ ⇐ No physics here!

The loop is the boundary of the surface. Every surface has a boundary, and the loop integral on every boundary is zero. Therefore, we have $\int_{surface} \left(\vec{\nabla} \times \vec{E}\right) \cdot \vec{n}\, dA = 0$ for every surface.



Therefore, $\vec{\nabla} \times \vec{E} = 0$ everywhere.

This is another part of Maxwell's equations.

Remember: This is only true for static fields.

End 8/29/14