Maxwell's Equations in Matter (7.3.5)

Remember that ***B*** and ***E*** are sourced by all the current and charge, while the curl of ***H*** and the divergence of ***D*** respond to the free currents and charges.

We just wrote Maxwell's equations using ***B*** and ***E*** only, but that is not always the most convenient notation. When there are polarizable materials, we want to take into account χ_M and χ_E. (or, ***P*** and ***M***).

What do Maxwell's equations look like in this case?

A complication – there is another way to produce bound current,

Because the bound charges move around if $\frac{\partial \vec{P}}{\partial t} \neq 0$.

Recall that $\rho_b = -\vec{\nabla} \cdot \vec{P}$. So, with time dependence,

$$\frac{\partial \rho_b}{\partial t} = -\frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{P}\right) = -\vec{\nabla} \cdot \left(\frac{\partial \vec{P}}{\partial t}\right) \equiv -\vec{\nabla} \cdot J_p$$

J$_p$ is the current that results from $\frac{\partial \vec{P}}{\partial t}$.

$\vec{J}_p = \frac{\partial \vec{P}}{\partial t}$ is the contribution to bound current from time dependence.

There may be $\vec{\nabla} \times \vec{M}$ at the same time.

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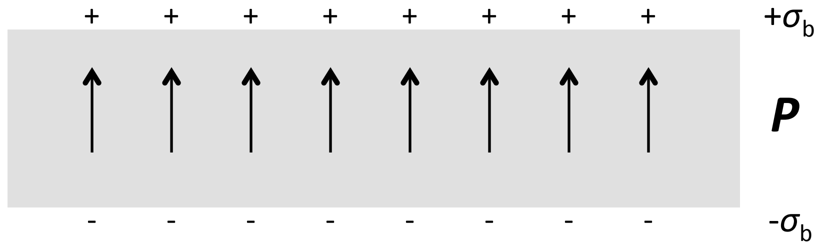
The physical picture:

$$\sigma_b = -\vec{P}\cdot\hat{n}$$



If ***P*** is changing, a bound current must be flowing to charge up the surfaces.

We must include $\boldsymbol{J}_p$ in the total current flow: $\vec{J} = \vec{J}_f + \vec{\nabla}\times\vec{M} + \dfrac{\partial\vec{P}}{\partial t}$

We can now rewrite Maxwell's equations:

$$1:\ \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\left(\rho_f - \vec{\nabla}\cdot\vec{P}\right) \Rightarrow \vec{\nabla}\cdot\vec{D} = \rho_f$$ (not changed by time dependence)

$$2:\ \vec{\nabla}\cdot\vec{B} = 0$$

$$3:\ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

No ρ or ***J***, so no changes here.

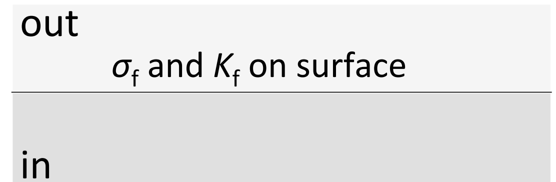
$$4:\ \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\left(\vec{J}_f + \vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t}\right) + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

Use: $\vec{B} = \mu_0\left(\vec{H}+\vec{M}\right)$ and $\vec{D} = \vec{P} + \varepsilon_0\vec{E}$, to obtain $\vec{\nabla}\times\vec{H} = \vec{J}_f + \dfrac{\partial\vec{D}}{\partial t}$

This is useful if you know: $\boldsymbol{P} = \varepsilon_0\chi_E\boldsymbol{E}$ and $\boldsymbol{M} = \chi_M\boldsymbol{H}$.

Boundary Conditions in Matter (7.3.6)

Each of Maxwell's equations gives a boundary condition at an interface:



$$\vec{\nabla}\cdot\vec{D}=\rho_f \quad \Rightarrow \quad D_\perp(\text{out})=D_\perp(\text{in}) \quad \text{if } \sigma_f=0.$$

$$\vec{\nabla}\cdot\vec{B}=0 \quad \Rightarrow \quad B_\perp(\text{out})=B_\perp(\text{in})$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \quad \Rightarrow \quad E_\parallel(\text{out})=E_\parallel(\text{in}) \quad \text{because } \frac{\partial\vec{B}}{\partial t} \text{ is finite.}$$

$$\vec{\nabla}\times\vec{H}=\vec{J}_f+\frac{\partial\vec{D}}{\partial t} \quad \Rightarrow \quad H_\parallel(\text{out})=H_\parallel(\text{in}) \quad \text{if } K_f=0.$$

Neither $\frac{\partial\vec{E}}{\partial t}$, nor $\frac{\partial\vec{B}}{\partial t}$ affects surface boundary conditions.

Example: Meissner effect and magnetic levitation by a superconductor

There was a homework problem last year (HW14) to show that the current must reside on the surface of a superconductor. This is a consequence of the fact that ***B*** = 0 inside a superconductor (the Meissner effect). It's a QM effect, so I can't derive it here. However, there is a phenomenological description that is very interesting.

We have learned that the vector potential, A, is not uniquely determined:

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \Rightarrow \quad \vec{A}' = \vec{A} + \vec{\nabla}\lambda \text{ also works. This is called "gauge invariance".}$$

However, Fritz and Heinz London discovered (1935) that the Meissner effect requires that ***A*** be fixed to a specific value:

$$\vec{A} = -\frac{m}{q^2 n}\vec{J}$$

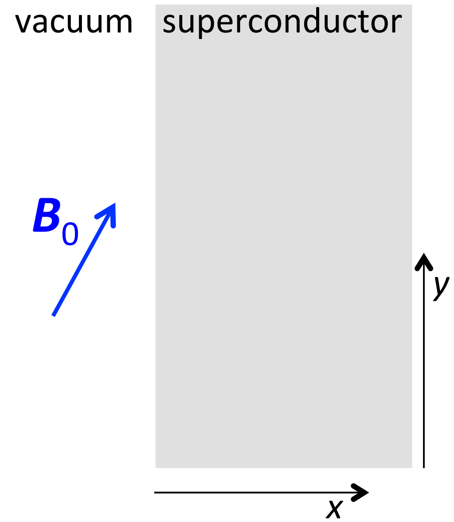
m is the mass of the current carriers	(~ $2m_e$!!)
q is the charge of the current carriers	(= 2e!)
n is the number density of current carriers	(~ $n_e/2$).

This loss of gauge invariance is another example of "spontaneous symmetry breaking" that was mentioned in the discussion of ferromagnetism.

Let's see how this gives the Meissner effect.

Meissner effect:

Consider a magnetic field outside the superconductor:



Inside:

$$\vec{\nabla}\times\vec{J} = -\frac{q^2 n}{m}\vec{\nabla}\times\vec{A} = -\frac{q^2 n}{m}\vec{B} \qquad \text{London's condition}$$

$$\mu_0\left(\vec{\nabla}\times\vec{J}\right) = \vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right) \qquad \text{Maxwell}$$

$$= \vec{\nabla}\left(\underbrace{\nabla\cdot\vec{B}}_{=0}\right) - \nabla^2\vec{B} \qquad \text{Math identity}$$

So: $\nabla^2\vec{B} = +k\vec{B}$ $\qquad k = \frac{\mu_0 q^2 n}{m}$

$\vec{B} = \vec{B}_0 e^{\pm\sqrt{k}x}$ $\qquad$ There is no *y* or *z* dependence.

$\frac{1}{\sqrt{k}}$ is called the penetration depth, λ_L.

λ_L is typically a few tens of nanometers.

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